ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 5, Pages 55–65 (April 30, 1999)
S 1079-6762(99)00061-X

TANGENTIAL HILBERT PROBLEM FOR PERTURBATIONS OF HYPERELLIPTIC HAMILTONIAN SYSTEMS

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(Communicated by Jeff Xia)

Abstract. The tangential Hilbert 16th problem is to place an upper bound for the number of isolated ovals of algebraic level curves $\{H(x,y) = \text{const}\}$ over which the integral of a polynomial 1-form $P(x,y)\,dx + Q(x,y)\,dy$ (the Abelian integral) may vanish, the answer to be given in terms of the degrees $n = \deg H$ and $d = \max(\deg P, \deg Q)$.

We describe an algorithm producing this upper bound in the form of a primitive recursive (in fact, elementary) function of n and d for the particular case of hyperelliptic polynomials $H(x,y) = y^2 + U(x)$ under the additional assumption that all critical values of U are real. This is the first general result on zeros of Abelian integrals that is completely constructive (i.e., contains no existential assertions of any kind).

The paper is a research announcement preceding the forthcoming complete exposition. The main ingredients of the proof are explained and the differential algebraic generalization (that is the core result) is given.

1. Tangential Hilbert problem and bounds for the number of limit cycles in perturbed Hamiltonian systems

1.1. Complete Abelian integrals and the tangential Hilbert Sixteenth problem.

Integrals of polynomial 1-forms over closed ovals of real algebraic curves, called (complete) *Abelian integrals*, naturally arise in many problems of geometry and analysis, but probably the most important is the link to the bifurcation of limit cycles of planar vector fields and the Hilbert Sixteenth problem. Recall that the question originally posed by Hilbert in 1900 was on the maximal number of limit cycles a polynomial vector field of degree d on the plane may have. This problem is still open even in the local version, for systems ε-close to integrable or Hamiltonian ones. However, there is a certain hope that the "linearized", or *tangential* Hilbert 16th problem can be more treatable.

Consider a polynomial perturbation of a Hamiltonian polynomial vector field

$$\dot{x} = -\frac{\partial H}{\partial y} - \varepsilon Q(x,y), \qquad \dot{y} = \frac{\partial H}{\partial x} + \varepsilon P(x,y). \tag{1.1}$$

An oval γ of the level curve $H(x,y) = h$ which is a closed (but nonisolated) periodic trajectory for $\varepsilon = 0$, may generate a limit cycle for small nonzero values of ε only

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Received by the editors October 23, 1998.

1991 *Mathematics Subject Classification.* Primary 14K20, 34C05, 58F21; Secondary 34A20, 30C15.

if the accumulated energy dissipation is zero in the first approximation, i.e., when

$$0 = \oint_\gamma P(x,y)\,dx + Q(x,y)\,dy, \qquad \gamma \subseteq \{H(x,y) = h\}. \tag{1.2}$$

The expression on the right hand side of (1.2) is a *complete Abelian integral*, and assuming the polynomials H, P, Q fixed, it is a function $I = I(h)$ of the value $h \in \mathbb{R}$, in general multivalued if the corresponding level curve contains several real ovals. The value $I(h)$ is the first variation of the Poincaré return map for the system (1.1) with respect to the parameter ε, computed in the chart h at $\varepsilon = 0$.

Thus the linearized or *tangential* Hilbert problem arises (see [1, problems by V. Arnold] for a recent reference): *for any collection of polynomials $H, P, Q \in \mathbb{R}[x,y]$ of degree $\leqslant d$ give an upper bound for the number of real ovals γ over which the integral* (1.2) *vanishes, but not identically* (in the latter case the perturbation (1.1) is *conservative* in the first approximation, and higher variations must be considered). The bound should be given in terms of d only, in other words, it must be uniform over all combinations of polynomials of admissible degrees.

1.2. **Hyperelliptic case.** A very important particular case is the *hyperelliptic* one, when $H(x,y) = \frac{1}{2}y^2 + U(x)$, $U \in \mathbb{R}[x]$, $\deg U = d \geqslant 5$: in this case the level curves are hyperelliptic (rational for $d = 1, 2$, elliptic for $d = 3, 4$). The polynomial U in this case will be always referred to as the *potential*, since such Hamiltonian systems correspond to the Newtonian system $\ddot{x} = -\frac{\partial U}{\partial x}$ describing a free particle in the potential field in one degree of freedom. The integral (1.2) in this case is called a *hyperelliptic integral*. The tangential Hilbert problem restricted to the hyperelliptic case, was studied in many papers including [4, 23].

1.3. **Background.** For low degree Hamiltonians (3 or 4) there are numerous results on the number of zeros for special choices of H, many of them sharp, which will not be discussed here: we note only that the elliptic case corresponding to $H(x,y) = \frac{1}{2}y^2 + \frac{1}{3}x^3 - x$ was completely investigated by G. Petrov [17], while in the case of an arbitrary cubic H a linear bound $5d + 15$ for the number of zeros of $I(h)$ was obtained by E. Horozov and I. Iliev [5]. However, the general results that would be valid for Hamiltonians of arbitrarily high degrees, are much scarcer and substantially less explicit. Perhaps the only known completely explicit general result is an upper bound for *multiplicity* of an isolated zero of an Abelian integral. This bound (polynomial in d) is due to P. Mardešić [13], who proved it using the approach suggested by Yu. Ilyashenko [6].

A. Khovanskiĭ in [10] and A. Varchenko in [24] proved that *for any fixed d the number of isolated zeros of Abelian integrals is uniformly bounded over all Hamiltonians and forms of degree $\leqslant d$.* The assertion of the Khovanskiĭ–Varchenko theorem is purely existential: it gives absolutely no information on how the bound may depend on d.

A simpler problem arises if we fix H and consider integrals of 1-forms ω of increasing degrees $d = \deg\omega = \max(\deg P, \deg Q)$, looking for an asymptotic bound for the number of zeros as $d \to \infty$. In this direction a considerable progress was recently achieved: assuming the Hamiltonian H be sufficiently generic, Yu. Ilyashenko and S. Yakovenko obtained a double exponential in d upper bound for the number of isolated zeros at a positive distance from the critical values of H [7]. Almost immediately D. Novikov and S. Yakovenko improved this result, reducing the bound to a single exponent and making it uniform over all real regular values: the number of real isolated roots of the Abelian integral does not exceed $\exp(Cd)$, where $C = C_{\text{NY}}(H) < +\infty$ is a finite constant depending only on the Hamiltonian [14]. The description of $C_{\text{NY}}(H)$ can be done in geometric terms [7], but the bound blows up to infinity as H approaches the boundary of the open set of Morse polynomials.

Finally A. Khovanskiĭ and G. Petrov proved[1] that the number of isolated zeros may grow at most as $B(n)d + C_{\text{KP}}(n)$, where $B(n)$ is an explicit expression double exponential in $n = \deg H$, while $C_{\text{KP}}(n)$ is a finite constant that depends on n. However, this dependence is purely existential, as before.

Summarizing this brief synopsis, we conclude that today no completely effective upper bound is known that would serve Hamiltonians of an arbitrarily high degree.

1.4. **Solution of the tangential Hilbert problem for hyperelliptic Hamiltonians.** In this announcement we claim a constructive upper bound for the number of zeros of hyperelliptic integrals under the additional assumption that all critical values of the potential $U(x)$ are real. There are several broadly used classes of constructive functions, among them *effective* (algorithmically computable), *primitive recursive* (defined by a finite number of inductive rules) and *elementary* functions, see [12]. Our main theorem asserts the *strongest form of computability* of the upper bound as a function of two natural values n, d.

Theorem 1. *For any real polynomial $U(x) \in \mathbb{R}[x]$ of degree $n+1$ and any differential form $\omega = P\,dx + Q\,dy$ of degree d, the number of real ovals $\gamma \subset \{y^2 + U(x) = h\}$ yielding an isolated zero of the integral $\oint_\gamma \omega$, is bounded by a primitive recursive (in fact, elementary) function $B(n, d)$ of two integer variables d and n, provided that all critical values of U are real.*

A closer inspection of the algorithm proving Theorem 1 suggests that the function $B(n, d)$ grows no faster than a certain *tower* function (iterated exponent) of height 5 or perhaps 6. In any case, this bound is too excessive to believe that it might be realistic: this is the main reason why we never tried to write it explicitly.

2. H-FIELDS AND THEIR POLYNOMIAL-LIKE PROPERTY

The proof of Theorem 1 goes by induction on n, but the induction step requires introducing more general classes of functions than ordinary hyperelliptic integrals. In other words, Theorem 1 is obtained as a corollary to a more general Theorem 2 concerning *complex* zeros of analytic functions from certain *Picard–Vessiot extensions* [8] of the field of rational functions $\mathbb{C}(t)$ by one or several hyperelliptic integrals.

2.1. **Analytic continuation.** Abelian integrals (1.2) admit analytic continuation as multivalued functions of a complex argument t, ramified over a finite set of points $\Sigma = \{t_1, \dots, t_\mu\} \subset \mathbb{C}$ and eventually $t_0 = \infty$. Generically (and always in the hyperelliptic case), Σ consists of *critical values* of H, and the monodromy group of this extension does not depend on the integrand ω. This implies that an arbitrary Abelian integral can be represented as a linear combination of a finite number of integrals of some 1-forms ω_k with coefficients from $\mathbb{C}(t)$ [25]. These

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[1] The proof of this result was published in [17] for a hyperelliptic polynomial $H = \frac{1}{2}y^2 + U(x)$ under the assumption that all critical points of the potential are real, but it can be generalized for all Morse Hamiltonians by several simple though nonobvious reductions.

forms can be explicitly described and the degrees of the coefficients bounded [3], but in the hyperelliptic case the situation becomes completely transparent and all computations explicit.

2.2. **Basic hyperelliptic integrals and Picard–Vessiot field extensions.** Let $U(x) = x^{n+1} + \cdots$ be a *monic* (i.e., with leading coefficient 1) potential of degree $n+1$, and $\omega_k = x^{k-1} y\,dx$, $k = 1, \dots, n$, differential 1-forms that constitute the basis of cohomology of each nonsingular hyperelliptic *complex* level curve $\phi_t = \{\frac{1}{2}y^2 + U(x) = t\} \subset \mathbb{C}^2$, $t \in \mathbb{C}$. Define the (complete collection of) *basic hyperelliptic integrals* $J_{kj}(t)$ as integrals of the forms ω_k over *vanishing cycles* $\delta_j(t) \in H_1(\phi_t, \mathbb{Z})$, see [2], "growing" from the critical values t_j:

$$J_{kj}(t) = \oint_{\delta_j} \omega_k, \qquad \delta_j = \delta_j(t) \subseteq \phi_t,\ \operatorname{diam} \delta_j(t)\big|_{t \to t_j} \to 0. \tag{2.1}$$

Together they constitute a nondegenerate $n \times n$-matrix $\mathcal{J} = \mathcal{J}(t)$, analytically depending on $t \in \mathbb{C}$ outside the critical locus Σ. This matrix function satisfies a *Picard–Fuchs system* [2] of first order linear differential equations (3.1) with rational coefficients, and the result of analytic continuation of $\mathcal{J}(t)$ along any loop in $\mathbb{C} \smallsetminus \Sigma$ is described by the *Picard–Lefschetz formulas* [2].

Lemma 1 (cf. [3, 25]). *An arbitrary hyperelliptic Abelian integral belongs to the field* $\Bbbk_U = \mathbb{C}(t)(J_{11}, \dots, J_{nn})$, *the extension of the rational functions field* $\mathbb{C}(t)$ *by the basic hyperelliptic integrals* $J_{kj} = J_{kj}(t)$, $j, k = 1, \dots, n$, *defined as in* (2.1).

The field $\Bbbk_U$ completely determined by the potential U is the field of multivalued analytic functions of complex argument t, analytically continuable along any path in $\mathbb{C} \smallsetminus \Sigma$ (this construction will be further generalized in Definition 1 below). Alternatively one can describe $\Bbbk_U$ in purely algebraic terms as a differential field, the Picard–Vessiot extension of $\mathbb{C}(t)$ by solutions of a linear system (3.1), adding all entries of any fundamental matrix solution to the latter. However, the system of generators $\mathcal{J} = \{J_{kj}\}$ is distinguished for many reasons.

The number $n = \deg U - 1$ will be referred to as the *gender* of the field $\Bbbk_U$.

To define unambiguously arithmetic operations with multivalued functions, we choose a base point t_* and a collection of simple nonintersecting (except at t_*) paths connecting t_j with t_*. The integrals $J_{kj}(t)$, originally defined only as germs at $t = t_j$ [2], can be continued along these paths and define a collection of germs of analytic functions at $t = t_*$ denoted again by J_{kj}. Then the field operations in $\Bbbk_U$ can be identified with arithmetic operations on germs. Analytic continuation along loops attached to t_*, constitute the group $\operatorname{Mon}(\Bbbk_U)$ of *monodromy* (differential) *automorphisms* of the field.

Each element of $\Bbbk_U$ can be written as a ratio of two *H-polynomials*, each of the form $p = \sum_{k+|\alpha| \leqslant d} c_{k\alpha} t^k \mathcal{J}^\alpha \in \mathbb{C}[t, \mathcal{J}] = \mathbb{C}[t, J_{11}, \dots, J_{nn}]$, where $\alpha = \|\alpha_{kj}\| \in \mathbb{Z}_+^{n^2}$ is a multiindex, $\mathcal{J}^\alpha = \prod_{j,k=1}^{n} J_{kj}^{\alpha_{kj}}$, and d the *degree* of the H-polynomial p. The degree of an arbitrary H-function $p/q \in \Bbbk_U$ is $\max(\deg p, \deg q)$, as usual, and it is preserved by monodromy transformations by virtue of Picard–Lefschetz formulas.

2.3. **General definition of H-fields.** For our purposes (mainly for Lemma 3 below) we need a more general object, extension of $\mathbb{C}(t)$ by adding hyperelliptic integrals associated with *several* different potentials.

Definition 1. An *H-field*[2] $\Bbbk_{U_1,\dots,U_\nu}$ associated with a collection of ν Morse polynomial potentials $U_1,\dots,U_\nu \in \mathbb{R}[x]$ of degrees $n_1+1,\dots,n_\nu+1$, is the extension of the field $\mathbb{C}(t)$ by the complete collection of $n_1^2+\cdots+n_\nu^2$ hyperelliptic integrals associated with each potential $U_1,\dots,U_\nu$.

The *gender* of the H-field $\Bbbk_{U_1,\dots,U_\nu}$ is the sum $n = n_1+\cdots+n_\nu$. The critical locus $\Sigma \subset \mathbb{C}$, generically consisting of n points, is the union $\{t_1,\dots,t_n\}$ of critical values of all respective potentials U_s.

As in the case of a single potential, one can fix settings (the base point, system of paths, etc.) so that each element of the H-field will be associated with a unique algebraic expression and the degree of elements is well defined.

2.4. **Theorem on zeros for H-fields.** We prove that under the additional assumption that $\Sigma \subset \mathbb{R}$, i.e., that all critical values of all potentials are real, the H-fields possess the property that makes them similar to the field of rational functions: the number of *complex* isolated zeros of any H-function admits an upper bound in terms of its degree and gender.

If $\Sigma \subset \mathbb{R}$, then we can assume that the critical points are ordered, $-\infty < t_1 < \cdots < t_n < +\infty$. They subdivide the real axis into $n+1$ finite or semiinfinite intervals ℓ_j. Theorem 2 places an upper bound for the number of real (i.e., on ℓ_j) and complex (in the upper or lower half-planes) zeros of any H-function: a simply connected domain when zeros are counted, needs to be specified because of the multivaluedness of H-functions.

Theorem 2 (main). *There exists a primitive recursive* (*in fact, elementary*) *function $B(n,d)$ such that the number of complex isolated zeros of any H-function $f \in \Bbbk_{U_1,\dots,U_\nu}$ of gender n and degree d in the upper or lower half-planes $\{\pm \operatorname{Im} t > 0\}$ and on each real interval ℓ_j can be at most $B(n,d)$, provided that all critical values of all potentials are real.*

Reduction from Theorem 2 to Theorem 1 is provided by Lemma 1. We believe that the assumption $\Sigma \subset \mathbb{R}$ is technical, but for the moment it cannot be dropped.

The function $B(n,d)$ is determined by the algorithm given in the proof. In principle and if necessary, its growth rate for large n,d can be estimated by a closer inspection of the algorithm. Note that the bound is uniform over all combinations of potentials generating H-fields with the same gender, and over all values of coefficients of H-functions of a given degree.

3. The structure and main ingredients of the proof

3.1. **Preliminary normalization.** For any given gender n the H-fields of this gender are parameterized by a combinatorial invariant (a partition of n describing how many different potentials of each degree were used) and, as soon as the partition is fixed, by the strings of coefficients of all potentials $U_s(x)$. The latter can be to a certain extent resized: using affine transformations $x \mapsto \lambda_s x + \lambda'_s$ with $\lambda_s, \lambda_s \in \mathbb{C}$, it is possible to normalize the string of collections of each potential U_s independently. Besides, one can make a change of the "independent variable" $t \mapsto \mu t + \mu'$ with

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[2] We would like to use the expression "hyperelliptic field" instead of the abbreviation "H-field", but the former term is already in use (though not very common). On the other hand, it would be certainly inadmissible to call elements of an H-field "hyperelliptic functions", since the latter name is firmly attached to functions of a different class. Thus we had to decide between at least three-word-long term and an abbreviation.

$\mu, \mu' \in \mathbb{C}$, common for all potentials. Using these transformations, one can achieve the following: (a) the coefficients of all potentials are explicitly bounded; (b) the roots of all potentials are in the unit disk; (c) the overall critical locus Σ is centered at 0, so that $\sum_{j=1}^{n} t_j = 0$, and does not shrink too much or stretch to infinity: $\max_{i \neq j} |t_i - t_j| = 1$. Clearly, these transformations cannot affect any bound on the number of zeros.

We refer to such H-fields as (properly) *resized*, introducing at the same time the notion of *resized* H-polynomials, H-functions, etc.

3.2. **Picard–Fuchs system for hyperelliptic integrals and variation of argument along arcs distant from the singular locus.** The multivalued matrix-valued function $\mathcal{J}(t)$ formed by basic hyperelliptic integrals (2.1) satisfies a linear system of ordinary differential equations of the form

$$(t + A)\dot{\mathcal{J}}(t) = B\mathcal{J}(t), \qquad A, B \in \operatorname{Mat}_{n\times n}(\mathbb{C}), \tag{3.1}$$

where A, B are constant matrices depending only on the potential U. The general form of (3.1) was established in [4] from geometric considerations, and in [19] the system (3.1) was derived by elementary arguments allowing for explicit description of the matrices in terms of the potential.[3] In particular, $\det(t + A) = \prod_{j=1}^{n}(t - t_j)$ (the product is taken over all critical values of the potential U), and the norms $\|A\|$, $\|B\|$ are bounded in terms of n if the potential is properly resized.

For the case of a general H-field associated with a collection of potentials $\{U_s\}$ a similar system can be written for each potential and, after passing to symmetric products, for the collection of all H-monomials of degree $\leqslant d$ for any particular d:

$$\Delta(t) \cdot \tfrac{d}{dt}\,(t^k \mathcal{J}^\alpha) = \sum\nolimits_{|\beta| \leqslant d} \mathcal{A}_{k,\alpha,\beta}(t)\, \mathcal{J}^\beta, \qquad \forall\ k + |\alpha| \leqslant d, \tag{3.2}$$

where $\mathcal{A}_{k,\alpha,\beta} \in \mathbb{C}[t]$ are polynomials of explicitly bounded (in terms of n and d) degrees with bounded coefficients, and $\Delta(t) = \prod_{t \in \Sigma}(t - t_j)$ is the product taken over the union of *all* critical values of *all* potentials U_s.

In other words, any H-polynomial of a known degree d and gender n can be written as a linear combination of coordinate functions restricted to a certain trajectory of the polynomial (more precisely, rational) vector field (3.2) in the affine space of appropriate dimension (depending on d and n). The main result of [15, 16] applied to the system (3.2) in combination with [26, Corollary 2.7] yields the following property of resized H-fields.

Lemma 2. *Variation of argument of any resized H-function of degree d and gender n along any arc $\gamma \subset \mathbb{C} \smallsetminus \Sigma$ admits an explicit upper bound in terms of n, d, and geometry of the arc γ (its length $|\gamma|$ and the distance from γ to Σ, measured by $\inf_{t\in\gamma} |\Delta(t)| > 0$).*

Variation of argument of any such function along any sufficiently small circular arc around any singular point $t_j \in \Sigma$ or $t_0 = \infty$ is bounded from above by an explicit expression involving only d and n.

The bound provided by Lemma 2 is already given (assuming γ for simplicity at the distance 1 from Σ) by a tower function of height 4 in the variables n and d [15]. This explains why any bound based on using the lemma, must be *very* large.

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[3] *Note added in proof*: the demonstration from the Thesis [19] recently appeared in the book [22, pp. 83–84].

3.3. **Clusterization.** The general principle established in Lemma 2, immediately implies an upper bound for the number of isolated zeros on any compact simply connected subset of $\mathbb{C}\smallsetminus\Sigma$, by virtue of the argument principle. To extend this result for zeros arbitrarily close to ramification points, additional efforts are required.

However, the assumption that the H-field is already resized, makes it possible to break the critical locus into at least two parts with a sufficient spacing between them and also distant from infinity, which is another ramification point. Covering each part by a convex simply connected domain D_j, called *cluster*, with the boundary at a controlled distance from all other singularities, splits the problem of zeros into that for each cluster separately, for a neighborhood D_∞ of infinity and for the complement $C = \mathbb{C} \smallsetminus (D_\infty \cup D_1 \cup D_2)$. The bound for zeros in C follows from Lemma 2 and the argument principle (being multiply connected, C should be further split into simply connected pieces without singularities inside). Thus it is the problem for a single separate cluster that has to be considered.

Zeros near infinity (inside the cluster D_∞) can be counted combining the main result of [20] with that from [15]: this works in fact for any cluster with only one singularity inside. The arguments briefly described below, show how the ideas of [20] can be generalized to cover the case of several ramification points. Very roughly, one has to find a system of functions that after restriction to a cluster would have the same monodromy as the hyperelliptic integrals, but be in some sense simpler. The solution is to consider a full collection of hyperelliptic integrals associated with an appropriate potential of inferior degree (i.e., smaller gender): then one can proceed by induction using the construction from [17]. The latter is briefly explained in the following section.

3.4. **Argument principle after Petrov.** Suppose we have an H-function f with ramification locus Σ on the real axis. Consider a symmetric (with respect to $\mathbb{R}$) domain $\Omega \subset \mathbb{C} \smallsetminus \Sigma$ formed by cutting the cluster (disk) $D = \overline{\Omega}$ along two rays emanating from two adjacent singular points in the opposite directions; see Figure 1. Since Ω is simply connected, f extends as a single-valued function analytic in Ω. To majorize the number of zeros of f in this domain, we apply the argument principle. Assuming H-field to be resized and the exterior arc γ_* distant from Σ, the variation of argument along the arcs γ_j, γ_* is explicitly bounded by Lemma 2. It remains to majorize the variation of argument of f along the upper and lower edges of the real intervals $\sigma_j^\pm$ between the singular points t_j and t_{j+1}.

Let σ be one such edge. It was an observation made by G. Petrov in [18], that *variation of argument of an analytic function f along a connected curve is at most π times the number of roots of* $\operatorname{Im} f$ *on that curve plus* 1, since between any two consecutive roots of $\operatorname{Im} f$ there, variation of argument of f can be at most π. In other words, to majorize the number of zeros of f in Ω, it is sufficient to majorize the number of zeros of $\operatorname{Im}_\sigma f = \operatorname{Im}(f\big|_\sigma)$ on every interval σ that is a part of $\partial\Omega$.

To apply this argument recursively, one has to restore the settings and extend the imaginary parts analytically from their respective edges of definition to obtain germs at t_*. One can easily show that *for an H-function from the field* $\Bbbk = \Bbbk_{U_1,\dots,U_\nu}$ *all these extensions will again belong to the same field*, i.e. there are well-defined maps $\operatorname{Im}_\sigma \Bbbk \to \Bbbk$ for all $\sigma = \sigma_j^\pm$. This follows, e.g., from the fact that the matrices A, B occurring in the system (3.1), are real.