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RIGIDITY PROPERTIES OF $\mathbb{Z}^d$-ACTIONS ON TORI AND SOLENOIDS

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ABSTRACT. We show that Haar measure is a unique measure on a torus or more generally a solenoid X invariant under a not virtually cyclic totally irreducible $\mathbb{Z}^d$-action by automorphisms of X such that at least one element of the action acts with positive entropy. We also give a corresponding theorem in the non-irreducible case. These results have applications regarding measurable factors and joinings of these algebraic $\mathbb{Z}^d$-actions.

1. INTRODUCTION AND MAIN RESULTS

The map $T_p : x \mapsto px$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has many closed invariant sets and many invariant measures. Furstenberg initiated the study of jointly invariant sets in his seminal paper [8]. A set $A \subset \mathbb{T}$ is called *jointly invariant* under T_p and T_q if $T_p(A) \subset A$ and $T_q(A) \subset A$. Furstenberg proved that if p and q are multiplicatively independent integers, then any closed jointly invariant set is either finite or all of $\mathbb{T}$.

Furstenberg also raised the question of what the jointly invariant measures are, that is, which probability measures μ on $\mathbb{T}$ satisfy $(T_p)_*\mu = (T_q)_*\mu = \mu$. The obvious ones are Lebesgue measure, atomic measures supported on finite invariant sets, and (non-ergodic) convex combinations of these.

Here we give a partial answer to this question in the following more general setting of $\mathbb{Z}^d$-actions on solenoids.

In the following a *solenoid* X is a compact, connected, abelian group whose Pontryagin dual $\widehat{X}$ can be embedded into a finite-dimensional vector space over $\mathbb{Q}$. The simplest example is a finite-dimensional torus. A $\mathbb{Z}^d$-action α by automorphisms of a solenoid X is called *irreducible* if there is no proper infinite closed subgroup which is invariant under α, and *totally irreducible* if there is no finite index subgroup $\Lambda \subset \mathbb{Z}^d$ and no proper infinite closed subgroup $Y \subset X$ which is invariant under the induced action α_Λ. A $\mathbb{Z}^d$-action is *virtually cyclic* if there exists $\mathbf{n} \in \mathbb{Z}^d$ such that for every element $\mathbf{m} \in \Lambda$ of a finite index subgroup $\Lambda \subset \mathbb{Z}^d$ there exists some $k \in \mathbb{Z}$ with $\alpha^{\mathbf{m}} = \alpha^{k\mathbf{n}}$.

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Theorem 1.1. *Let α be a totally irreducible, not virtually cyclic $\mathbb{Z}^d$-action by automorphisms of a solenoid X. Let μ be an α-ergodic measure. Then either $\mu = \lambda$ is the Haar measure of X, or the entropy $\mathsf{h}_\mu(\alpha^{\mathbf{n}}) = 0$ vanishes for all $\mathbf{n} \in \mathbb{Z}^d$.*

We summarize the history of this problem. The topological generalization of Furstenberg's result to higher dimensions was given by Berend [1, 2]: An action on a torus or solenoid has no proper, infinite, closed, and invariant subsets if and only if it is totally irreducible, not virtually cyclic, and contains a hyperbolic element. One direction of this theorem is easy to see: if either of these properties fails, one can construct a proper, infinite, closed, invariant subset. For example, if a $\mathbb{Z}^d$-action on a torus does not contain a hyperbolic element, then it can be shown that there exists a common eigenspace $W \cong \mathbb{C}$ of the matrices defining the action so that the corresponding eigenvalues ξ satisfy $|\xi| = 1$. Therefore, the unit ball B in W gives an infinite closed invariant subset. Notice that we do not assume any hyperbolicity in Theorem 1.1.

The first partial result for the measure problem on $\mathbb{T}$ was given by Lyons [23] under a strong additional assumption. Rudolph [25] weakened this assumption considerably, and proved the following theorem.

Theorem 1.2 ([25, Thm. 4.9])**.** *Let $p, q \geq 2$ be relatively prime positive integers, and let μ be a T_p, T_q-ergodic measure on $\mathbb{T}$. Then either $\mu = \lambda$ is the Lebesgue measure, or the entropy of T_p and T_q is zero.*

Johnson [11] lifted the relative primality assumption, showing it is enough to assume that p and q are multiplicatively independent. By the ergodic decomposition every invariant measure ν can be written as a convex combination of a family of ergodic measures μ_τ, $\tau \in \mathfrak{T}$. If ν has positive entropy, the same must apply for some μ_τ. So Theorem 1.2 also shows that every positive entropy measure is a convex combination of the Lebesgue measure and a zero entropy measure. Thus the only restricting assumption here is positive entropy. Feldman [7], Parry [24], and Host [9] have found different proofs of this theorem, but positive entropy remains a crucial assumption.

Katok and Spatzier [16, 17] obtained the first analogous results for actions on higher-dimensional tori and homogeneous spaces. However, their method required either an additional ergodicity assumption on the measure (satisfied for example if every one parameter subgroup of the suspension acts ergodically), or that the action is totally non-symplectic (TNS). A careful and readable account of these results has been written by Kalinin and Katok [13], which also fixed some minor inaccuracies. Theorem 1.1 gives a full generalization of the result of Rudolph and Johnson to actions on higher-dimensional solenoids.

Without total irreducibility the Haar measure of the group is no longer the only measure with positive entropy. Thus the general theorem below is (necessarily) longer in its formulation than Theorem 1.1. It strengthens e.g. [13, Thm. 3.1], which has a similar conclusion but stronger assumptions.

Theorem 1.3. *Let α be a $\mathbb{Z}^d$-action ($d \geq 2$) by automorphisms of a solenoid X. Suppose α has no virtually cyclic factors, and let μ be an α-ergodic measure on X. Then there exists a subgroup $\Lambda \subset \mathbb{Z}^d$ of finite index and a decomposition $\mu = \frac{1}{M}(\mu_1 + \cdots + \mu_M)$ of μ into mutually singular measures with the following properties for every $i = 1, \ldots, M$.*

(1) *The measure μ_i is α_Λ-ergodic, where α_Λ is the restriction of α to Λ.*

(2) *There exists an α_Λ-invariant closed subgroup G_i such that μ_i is invariant under translation with elements in G_i, i.e. $\mu_i(A) = \mu_i(A+g)$ for all $g \in G_i$ and every measurable set A.*
(3) *For $\mathbf{n} \in \mathbb{Z}^d$, $(\alpha^{\mathbf{n}})_* \mu_i = \mu_j$ for some j and $\alpha^{\mathbf{n}}(G_i) = G_j$.*
(4) *The measure μ_i induces a measure on the factor X/G_i, also denoted by μ_i, with $\mathsf{h}_{\mu_i}(\alpha^{\mathbf{n}}_{X/G_i}) = 0$ for any $\mathbf{n} \in \Lambda$. (Here α_{X/G_i} denotes the action induced on X/G_i.)*

We note that even in the topological category, where Berend gave definitive results regarding the totally irreducible case, the situation for the reducible case is far from understood.

The proofs of Theorems 1.1 and 1.3 follow the outline of Rudolph's proof of Theorem 1.2. One of the main ingredients there was the observation that $\mathsf{h}_\mu(T_p)/\log p = \mathsf{h}_\mu(T_q)/\log q$ (and a relativized version of this equality). This follows from the particularly simple geometry of this system where both T_p and T_q expand the one-dimensional space $\mathbb{T}$ with fixed factors. There is no simple geometrical reason why such an equality should be true for more complicated $\mathbb{Z}^d$-actions on solenoids, and indeed is easily seen to fail in the reducible case. However, somewhat surprisingly, such an equality *is* true for irreducible $\mathbb{Z}^d$ actions, even though this is true from subtle number theoretical reasons (see Theorem 5.1 below). It is interesting to note that along the way we get new and nontrivial information about measures invariant under a single, even hyperbolic, solenoidal automorphism.

We apply Theorem 1.3 to obtain new information about the measurable structure, with respect to the Haar measure, of irreducible algebraic $\mathbb{Z}^d$-actions on tori and solenoids. Our first application characterizes the measurable factors of α, and generalizes the isomorphism rigidity results by A. Katok, S. Katok, and Schmidt [15].

Theorem 1.4. *Let α be an irreducible, not virtually cyclic $\mathbb{Z}^d$-action on a solenoid X, and let $\mathcal{A}$ be an α-invariant σ-algebra. Then either $\mathcal{A} = \{\emptyset, X\}$ (modulo λ), or there is a finite group G which acts on X by affine transformations and*

$$\mathcal{A} = \{A \in \mathcal{B}_X : gA = A \text{ for all } g \in G\} \text{ (modulo } \lambda).$$

In other words, every infinite measurable factor of α is a quotient of X by the action of a finite affine group. The simplest examples of such groups are finite translation groups. However, more complicated examples are also possible; for example, let $w \in X$ be any α-fixed point. Then the action of $G = \{\mathrm{Id}, -\mathrm{Id} + w\}$ on X commutes with α.

The proof of Theorem 1.4 uses the relatively independent joining of the Haar measure with itself over the factor $\mathcal{A}$, which gives an invariant measure on $X \times X$ analyzable by Theorem 1.3. This is similar to the proof of isomorphism rigidity in [15], which followed a suggestion by Thouvenot.

Finally, we characterize disjointness in the case of irreducible actions, which generalizes the corresponding results for TNS actions by Kalinin and Katok [14], and by Kalinin and Spatzier [12].

Theorem 1.5. *Suppose α_1 and α_2 are irreducible, not virtually cyclic $\mathbb{Z}^d$-actions on solenoids X_1 and X_2. Then either they are disjoint, or there exists a finite index subgroup $\Lambda \subset \mathbb{Z}^d$ such that the subactions $\alpha_{1,\Lambda}$ and $\alpha_{2,\Lambda}$ have a common algebraic factor.*

In this announcement we give an essentially complete proof of Theorem 5.1 regarding the relationship between entropies of individual elements of an irreducible action. First we explain in §3 how the various Lyapunov exponents contribute to the entropy. A bound on each of these contributions is given in Theorem 4.1, which is a theorem about measures invariant under a *single* automorphism. Then in §5 we conclude the proof of the entropy identity using a key lemma from [3] regarding the product structure of certain conditional measures. A sketch of how Theorem 1.1 is proved using Theorem 5.1 is given in §6. Full details of the proofs of all theorems announced in this note will be given in [5].

2. Arithmetic automorphisms and irreducible actions

Throughout this note, the term *local field* will denote a locally compact field of characteristic zero; these include $\mathbb{R}$ and $\mathbb{C}$ as well as finite extensions $\mathbb{K}$ of the field of p-adic numbers $\mathbb{Q}_p$. Let $\mathbb{K}$ be any local field, and let $\lambda_{\mathbb{K}}$ be the Haar measure on $\mathbb{K}$. Let $\delta(\mathbb{K}) = 1$ for $\mathbb{K} \neq \mathbb{C}$ and $\delta(\mathbb{C}) = 2$. For $a \in \mathbb{K}$ the norm $|a|_{\mathbb{K}}$ is defined as the real number satisfying

$$\lambda_{\mathbb{K}}(aC) = |a|_{\mathbb{K}}^{\delta(\mathbb{K})} \lambda_{\mathbb{K}}(C) \tag{2.1}$$

for any measurable set $C \subset \mathbb{K}$. Then $|\cdot|_{\mathbb{K}}$ satisfies the triangle inequality for all $\mathbb{K}$.

The following follows easily from [26, Thm. 29.2 and Sect. 7] (see also [4], [6]).

Proposition 2.1. *Let α be an irreducible algebraic $\mathbb{Z}^d$-action on a connected group. Then there exists a finite product $\mathbb{A} = \mathbb{K}_1 \times \cdots \times \mathbb{K}_m$ of local fields $\mathbb{K}_j$, a $\mathbb{Z}^d$-action $\alpha_{\mathbb{A}}$ by automorphisms of $\mathbb{A}$ whose restrictions to $\mathbb{K}_j$ are linear, $(\alpha_{\mathbb{A}}^{\mathbf{n}}(x))_j = \zeta_{j,\mathbf{n}} x_j$, and a $\alpha_{\mathbb{A}}$-invariant cocompact discrete subgroup Γ of $\mathbb{A}$, such that α is conjugate to the induced action of $\alpha_{\mathbb{A}}$ on $\mathbb{A}/\Gamma$.*

Furthermore, we have

$$\prod_{j=1}^{m} |\zeta_{j,\mathbf{n}}|_j^{\delta(\mathbb{K}_j)} = 1 \text{ for all } \mathbf{n} \in \mathbb{Z}^d, \tag{2.2}$$

and one can choose Γ such that

$$\prod_{j=1}^{m} |a_j|_{\mathbb{K}_j}^{\delta(\mathbb{K}_j)} \geq 1 \text{ for every } \mathbf{a} \in \Gamma. \tag{2.3}$$

We note that the local fields $\mathbb{K}_j$ above are all the Archimedean and some non-Archimedean completions $\mathbb{K}_j$ of a number field $\mathbb{k}$ which depends on the action; (2.2) and (2.3) follow from the elementary properties of number fields and their completions.

A $\mathbb{Z}^d$-action α by automorphisms of a solenoid X is *arithmetic* if the conclusions of Proposition 2.1 hold. An automorphism of X is arithmetic if it is part of an arithmetic $\mathbb{Z}^d$-action for $d \geq 1$.

We shall identify $\mathbb{K}_j$ with the corresponding subspace in $\mathbb{A}$, and refer to these as the eigenspaces. We use the norms $|\cdot|_j = |\cdot|_{\mathbb{K}_j}$ to induce a norm $\|x\| = \max_i |x_i|_i$ on $\mathbb{A}$, and furthermore a metric $d_X(\cdot,\cdot)$ on X. We also write $\delta_j = \delta(\mathbb{K}_j)$. A ball of radius r around $x \in X$ ($a \in \mathbb{A}$) will be denoted by $B_r(x)$ ($B_r(a)$), if we wish to emphasize the space $B_r^X(x)$ ($B_r^{\mathbb{A}}(a)$), and by B_r^X ($B_r^{\mathbb{A}}$) if the center is zero.

3. Entropy, invariant foliations, and conditional measures

In this and the following section we consider a single arithmetic automorphism. In other words, let T be an automorphism of $X = \mathbb{A}/\Gamma$, where Γ satisfies (2.3), and T is induced by a $T_{\mathbb{A}}$ with $(T_{\mathbb{A}}(x))_j = \zeta_j x_j$ for $j = 1, \ldots, m$. It turns out to be useful to study the following more general situation: let S be an arbitrary homeomorphism of a compact space Y with metric $d_Y(\cdot,\cdot)$ and let $\widetilde{T} = S \times T$ be the product map on $\widetilde{X} = Y \times X$. Define $d_{\widetilde{X}}((y,x),(y',x')) = d_Y(y,y') + d_X(x,x')$. We let $\mathcal{B}_Y$ denote the Borel σ-algebras of Y identified with a sub-σ-algebra of $\mathcal{B}_{\widetilde{X}}$ in the obvious way, and we wish to study the relative entropy $\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}|\mathcal{B}_Y)$.

The eigenspace $\mathbb{K}_j$ is expanded by T if $|\zeta_j|_j > 1$. Let $V \subset \mathbb{A}$ be a sum of expanded eigenspaces, and let $W^+ = W^+(T)$ be the sum of all expanded eigenspaces. Then V induces a foliation of $Y \times X$ by letting the leaf through x be $F_V(\widetilde{x}) = \widetilde{x} + V$ (for $\widetilde{x} = (y, a+\Gamma)$ we set $\widetilde{x} + v = (y, a+v+\Gamma)$).

In the following we need the connection between entropy and conditional measures [18], [19], and conditional measures on foliations [16, Sect. 4], [21, Sect. 3]. The former we have to adapt slightly to our problem, and for the latter we will use the notation of [21, Sect. 3].

Definition 3.1. Let V be as above. A σ-algebra $\mathcal{A}$ of Borel subsets of $\widetilde{X}$ is *subordinate to* V if $\mathcal{A}$ is countably generated, for every $\widetilde{x} \in \widetilde{X}$ the atom $[\widetilde{x}]_{\mathcal{A}}$ of $\widetilde{x}$ with respect to $\mathcal{A}$ is contained in the leaf $\widetilde{x} + V$, and for a.e. $\widetilde{x}$

$$\widetilde{x} + B_\epsilon^V \subseteq [\widetilde{x}]_{\mathcal{A}} \subseteq \widetilde{x} + B_\rho^V \text{ for some } \epsilon > 0 \text{ and } \rho > 0.$$

A σ-algebra $\mathcal{A}$ is *increasing* (with respect to $\widetilde{T}$) if $\widetilde{T}\mathcal{A} \subset \mathcal{A}$.

The conditional measures for the foliation F_V can be characterized in terms of σ-algebras subordinate to V; see [21, Thm. 3.6]. Let $\mathcal{M}_\infty(V)$ denote the space of locally finite Borel measures on V equipped with the weakest topology for which $\mu \mapsto \int f d\mu$ is continuous for every $f \in C_c(V)$. For any $v \in V$ let $+_v$ denote the map $w \mapsto w + v$. For two measures ν, ν' we write $\nu \propto \nu'$ if there exists $c > 0$ with $\nu = c\nu'$.

Proposition 3.2. *There exists a Borel measurable map $\widetilde{x} \mapsto \widetilde{\mu}_{\widetilde{x},V}$ from $\widetilde{X} \to \mathcal{M}_\infty(V)$ with the following properties:*

1. *There is a set N_0 of zero measure so that for every $\widetilde{x} \in \widetilde{X}$ and $v \in V$ for which $\widetilde{x}, \widetilde{x} + v \notin N_0$, $\widetilde{\mu}_{\widetilde{x},V} \propto (+_v)_* \widetilde{\mu}_{\widetilde{x}+v,V}$.*
2. *For a.e. $\widetilde{x}$ and $r > 0$, $\widetilde{\mu}_{\widetilde{x},V}(B_r^V) > 0$.*
3. *If $\mathcal{C}$ is a σ-algebra subordinate to V with conditional measures $\widetilde{\mu}_{\widetilde{x}}^{\mathcal{C}}$, then there is a Borel measurable function $c_V(\widetilde{x}, \mathcal{C}) > 0$ so that for a.e. $\widetilde{x}$, for all Borel $B \subset V$ with $\widetilde{x} + B \subset [\widetilde{x}]_{\mathcal{C}}$, $\widetilde{\mu}_{\widetilde{x},V}(B) = c_V(\widetilde{x},\mathcal{C})\widetilde{\mu}_{\widetilde{x}}^{\mathcal{C}}(x + B)$.*

These properties characterize $\widetilde{\mu}_{\widetilde{x},V}$ up to a multiplicative constant a.e. In order to get rid of the the remaining ambiguity we require that $\widetilde{\mu}_{\widetilde{x},V}(B_1^V) = 1$ for all x.

Let $\mathcal{P}$ be a finite partition of $\widetilde{X}$, which we identify with the corresponding finite algebra of sets. For any $\epsilon > 0$ let $\partial_\epsilon^V \mathcal{P} = \left\{ \widetilde{x} \in \widetilde{X} : \widetilde{x} + B_\epsilon^V \not\subset [\widetilde{x}]_{\mathcal{P}} \right\}$.

Lemma 3.3. *For any probability measure $\widetilde{\mu}$ on $\widetilde{X}$, there exists a finite partition $\mathcal{P}$ of $\widetilde{X}$ into arbitrarily small sets such that for some fixed C, for every $\epsilon > 0$*

$$\mu(\partial_\epsilon^V \mathcal{P}) < C\epsilon. \tag{3.1}$$

For any σ-algebra $\mathcal{A}$ and $k_0 < k_1 \leq \infty$ set

$$\mathcal{A}^{k_0} = \widetilde{T}^{k_0}\mathcal{A} \quad \text{and} \quad \mathcal{A}^{[k_0,k_1)} = \bigvee_{i=k_0}^{k_1-1} \widetilde{T}^i\mathcal{A}.$$

It can be shown that if $\mathcal{P}$ is a finite partition as above, then there is a countably generated σ-algebra $\mathcal{C}_V \supset \mathcal{P}^{[0,\infty)}$ which is both increasing and V-subordinate satisfying $[x]_{\mathcal{C}_V} = [x]_{\mathcal{P}^{[0,\infty)}} \cap (x+V)$. It follows that $\mathcal{C}_V = \widetilde{T}\mathcal{C}_V \vee \mathcal{P}$. For example, if T is expansive and $V = W^+(T)$, one can take $\mathcal{C}_V = \mathcal{P}^{[0,\infty)}$.

For $\mathcal{C}_V$ as above, we define $\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}, V)$, the *entropy contribution of* V, to be $\mathsf{H}_{\widetilde{\mu}}(\mathcal{C}_V|\widetilde{T}\mathcal{C}_V)$. The following proposition shows it is independent of the choice of $\mathcal{P}$ and $\mathcal{C}_V$:

Proposition 3.4. *Let V be a sum of expanded eigenspaces, and let $\mathcal{C}_V$ be as above. Then*

$$\operatorname{vol}(\widetilde{T}, V, \widetilde{x}) = -\lim_{N\to\infty} \frac{1}{N} \log \widetilde{\mu}_{\widetilde{x},V}\left(T^{-N}(B_1^V(0))\right)$$

exists a.e. and $\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}, V) = \int \operatorname{vol}(\widetilde{T}, V, \widetilde{x})\,\mathrm{d}\widetilde{\mu}(\widetilde{x})$. Furthermore, if $V = W^+$, this contribution equals the relative entropy of $\widetilde{T}$ given $\mathcal{B}_Y$, i.e. $\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}|\mathcal{B}_Y) = \mathsf{h}_{\widetilde{\mu}}(\widetilde{T}, W^+)$.

4. A bound on the entropy contribution

In this section we prove the following theorem.

Theorem 4.1. *Let T be an arithmetic automorphism of $X = \mathbb{A}/\Gamma$, and suppose T is induced by $T_A : \mathbb{A} = \mathbb{K}_1 \times \cdots \times \mathbb{K}_m \to \mathbb{A}$ with $(T_{\mathbb{A}}(x))_j = \zeta_j x_j$ for $1 \leq j \leq m$. Let S be a homeomorphism of a compact metric space Y, write $\widetilde{T} = S \times T$ for the product map on $\widetilde{X} = Y \times X$, and let $\widetilde{\mu}$ be a $\widetilde{T}$-invariant measure on X. Then the entropy contribution of a sum $V = \sum_{j\in I_V} \mathbb{K}_j$ of expanding eigenspaces of T is bounded by*

$$\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}, V) \leq \frac{\sum_{j\in I_V} \delta_j \log|\zeta_j|_j}{\sum_{j\in I_{W^+}} \delta_j \log|\zeta_j|_j} \mathsf{h}_{\widetilde{\mu}}(\widetilde{T}|\mathcal{B}_Y), \tag{4.1}$$

where $W^+ = \sum_{j\in I_{W^+}} \mathbb{K}_j$ is the sum over all expanding eigenspaces.

Notice that this estimate is sharp for a product measure $\widetilde{\mu} = \nu \times \lambda$ with λ being the Haar measure on X. A special case of this theorem appeared in [20, Theorem 2.4] (see also [22] for related discussion). Without loss of generality we assume that $I_V = \{1, \ldots, e\}$ and $I_{W^+} = \{1, \ldots, f\}$ with $e \leq f$.

Lemma 4.2. *Let $\kappa = \left(\sum_{j=1}^e \delta_j \log|\zeta_j|_j\right) / \left(\sum_{j=1}^f \delta_j \log|\zeta_j|_j\right)$ be the fraction appearing in Theorem 4.1, and $s > 0$ sufficiently small. Let $\mathcal{P}$ be a finite partition satisfying* (3.1) *so that the diameter of every atom of $\mathcal{P}$ is at most s, and let $\mathcal{C}_V$ be as in §3. For any $N \geq 1$, let E_N be the set of $\widetilde{x} \in \widetilde{X}$ for which the atom of $\widetilde{x}$ with respect to $\mathcal{C}_V = \mathcal{P}^{[0,N)} \vee \mathcal{C}_V^N$ is* **not** *equal to the atom of $\widetilde{x}$ with respect to $\mathcal{A} = \mathcal{P}^{[0,M)} \vee \mathcal{C}_V^N$ for $M = \lceil \kappa N \rceil$. Then $\widetilde{\mu}(E_N) < C\exp(-\rho N)$ for some $C, \rho > 0$.*

In the proof of Lemma 4.2 we will study the atoms $[\widetilde{x}]_{\mathcal{A}}$ more closely. By definition $\mathcal{A}$ is subordinate to V so $[\widetilde{x}]_{\mathcal{A}}$ is a 'bounded subset' of $\widetilde{x} + V$, but a priori this bound is not known.

Proof. Let $r = \inf_{a \in \Gamma \setminus \{0\}} \|a\|$, and take $s < 1/4$ to be small enough so that

$$\|T_{\mathbb{A}}^{-1} v\| + s < r \qquad \text{for every } \|v\| \leq s. \tag{4.2}$$

Suppose $\widetilde{x}, \widetilde{x}' \in \widetilde{X}$ are in the same atom with respect to $\mathcal{A}$. Then by definition of $\mathcal{A}$, $d_{\widetilde{X}}(\widetilde{T}^{-i}\widetilde{x}, \widetilde{T}^{-i}\widetilde{x}') \leq s$ for $i = 0, \dots, M-1$. Since $\mathcal{B}_Y \subset \mathcal{A}$, we can take $w \in \mathbb{A}$ with $\widetilde{x}' = \widetilde{x} + w$. Then $\|T_{\mathbb{A}}^{-i} w - a_i\| \leq s$ for some $a_i \in \Gamma$ and $i = 0, \dots, M-1$. By an appropriate choice of w we can assume that $a_0 = 0$. Applying (4.2) to W we see that $\|a_1\| \leq \|T_{\mathbb{A}}^{-1} w\| + \|a_1 - T_{\mathbb{A}}^{-1} w\| < r$ and so $a_1 = 0$. Continuing like this, we see that $a_i = 0$ and $\|T_{\mathbb{A}}^{-i} w\| \leq s$ for for $i = 0, \dots, M-1$. We conclude that $\widetilde{x}' = \widetilde{x} + w$ for some $w \in B_s^{\mathbb{A}}(0)$ with

$$|\zeta_j^{-M} w_j|_j \leq s \text{ for } j = f+1, \dots, m. \tag{4.3}$$

Since $\widetilde{T}^{-N}\widetilde{x}, \widetilde{T}^{-N}\widetilde{x}'$ are in the same atom of $\mathcal{C}_V$, they differ by some element $v \in V$ with $\|v\| \leq s$, hence $\widetilde{x}' = \widetilde{x} + T_{\mathbb{A}}^N v$. Since also $\widetilde{x}' = \widetilde{x} + w$, we have $w - T_{\mathbb{A}}^N v \in \Gamma$. Using the bounds we have on the components of $a = w - T_{\mathbb{A}}^N v$ we will use (2.3) to show that in fact $a = 0$. Indeed,

$$\prod_{j=1}^m |a|_j^{\delta_j} = \prod_{j=1}^e |w_j - \zeta_j^N v_j|_j^{\delta_j} \prod_{j=e+1}^f |w_j|_j^{\delta_j} \prod_{j=f+1}^m |w_j|_j^{\delta_j}.$$

For the first type of terms we get $|w_j - \zeta_j^N v_j|_j \leq |w_j|_j + |\zeta_j|_j^N |v_j|_j \leq 2s|\zeta_j|_j^N$. Since $s \leq 1$, we conclude $|w_j - \zeta_j^N v_j|_j^{\delta_j} \leq 4s^2 |\zeta_j|_j^{\delta_j N}$ for $1 \leq j \leq e$. For $e < j \leq f$ we use just $|w_j|_j^{\delta_j} \leq s$ as our estimate, and for $f < j \leq m$ we use (4.3) in the form of $|w_j|_j^{\delta_j} \leq s|\zeta_j|_j^{\delta_j M}$. Together this gives

$$\prod_{j=1}^m |a|_j^{\delta_j} \leq 4^e s^m \prod_{j=1}^e |\zeta_j|_j^{\delta_j N} \prod_{j=f+1}^m |\zeta_j|_j^{\delta_j M}.$$

Let $P = \prod_{j=1}^f |\zeta_j|_j^{\delta_j}$. Then $\prod_{j=1}^e |\zeta_j|_j^{\delta_j} = P^\kappa$, and $\prod_{j=f+1}^m |\zeta_j|_j^{\delta_j} = P^{-1}$ by (2.2). We rewrite the last estimate and get

$$\prod_{j=1}^m |a|_j^{\delta_j} \leq 4^e s^m P^{\kappa N - M} \leq 4^e s^m.$$

Since $4^e s^m < 1$, (2.3) implies $a = 0$ and so

$$\widetilde{x}' \in \widetilde{x} + B_s^V. \tag{4.4}$$

Assume now that $\widetilde{x} \in E_N$, i.e. that the atoms of $\widetilde{x}$ with respect to $\mathcal{A}$ and $\mathcal{C}_V$ differ. Since $\mathcal{A} \subset \mathcal{C}_V$ and $\mathcal{A} \vee \mathcal{P}^{[M,N)} = \mathcal{C}_V$, there exists $\widetilde{x}' \in [\widetilde{x}]_{\mathcal{A}}$ and some $i \in \{M, \dots, N-1\}$ such that $\widetilde{T}^{-i}\widetilde{x}$ and $\widetilde{T}^{-i}\widetilde{x}'$ belong to different elements of the partition $\mathcal{P}$. Let $\theta = \min_{j=1}^f |\zeta_j|_j$. By (4.4), $\widetilde{T}^{-i}\widetilde{x}' \in \widetilde{T}^{-i}\widetilde{x}' + B^V_{\theta^{-i}s}$, so $\widetilde{x} \in \partial^V_{\theta^{-i}s}\mathcal{P}$. Thus $E_N \subset \bigcup_{i=M}^{N-1} \partial^V_{\theta^{-i}s}\mathcal{P}$, and so applying (3.1) we see that $\mu(E_N) < C\theta^{-M}$ for some $C > 0$, as claimed. □

Proof of Theorem 4.1. We have $\mathcal{C}_V = \mathcal{P} \vee \widetilde{T}\mathcal{C}_V$, and so

$$\begin{aligned} \mathsf{h}_{\widetilde{\mu}}(\widetilde{T}, V) &= \frac{1}{N}\mathsf{H}_{\widetilde{\mu}}(\mathcal{C}_V | \widetilde{T}^N \mathcal{C}_V) = \frac{1}{N}\mathsf{H}_{\widetilde{\mu}}(\mathcal{P}^{[0,N)} | \mathcal{C}_V^N) \\ &= \frac{1}{N}\mathsf{H}_{\widetilde{\mu}}(\mathcal{P}^{[0,M)} | \mathcal{C}_V^N) + \frac{1}{N}\mathsf{H}_{\widetilde{\mu}}(\mathcal{P}^{[M,N)} | \mathcal{P}^{[0,M)} \vee \mathcal{C}_V^N). \end{aligned} \tag{4.5}$$

Clearly

$$\frac{1}{N}\mathsf{H}_{\widetilde{\mu}}(\mathcal{P}^{[0,M)}|\mathcal{C}_V^N) \leq \frac{M}{N}\frac{1}{M}\mathsf{H}_{\widetilde{\mu}}(\mathcal{P}^{[0,M)}|\mathcal{B}_Y), \tag{4.6}$$

and the last expression tends to $\kappa\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}|\mathcal{B}_Y)$ for $N \to \infty$. For the proof of the theorem we need to show that the second expression on the right hand side of (4.5) tends to zero.

Let $\mathcal{A}$ and E_N be as in Lemma 4.2. We wish to estimate

$$\mathsf{H}_{\widetilde{\mu}}(\mathcal{P}^{[M,N)}|\mathcal{A}) = \int \mathrm{I}_{\widetilde{\mu}}(\mathcal{P}^{[M,N)}|\mathcal{A})\,\mathrm{d}\widetilde{\mu} = -\int \log\widetilde{\mu}_{\widetilde{x}}^{\mathcal{A}}\big([\widetilde{x}]_{\mathcal{P}^{[M,N)}}\big)\,\mathrm{d}\widetilde{\mu}(\widetilde{x}).$$

The entropy of a partition is less than the logarithm of the cardinality of the partition. Applying this for $\widetilde{\mu}_{\widetilde{x}}^{\mathcal{A}}$ gives the estimate

$$h(\widetilde{x}) = -\int_{[\widetilde{x}]_{\mathcal{A}}} \log\widetilde{\mu}_{\widetilde{x}}^{\mathcal{A}}\big([\widetilde{x}]_{\mathcal{P}^{[M,N)}}\big)\,\mathrm{d}\widetilde{\mu}_{\widetilde{x}}^{\mathcal{A}} \leq \begin{cases}(N-M)\log|\mathcal{P}| & \text{if } \widetilde{x}\in E_N,\\ 0 & \text{otherwise.}\end{cases} \tag{4.7}$$

Integrating (4.7) and applying Lemma 4.2, we get

$$\tfrac{1}{N}\mathsf{H}_{\widetilde{\mu}}(\mathcal{P}^{[M,N)}|\mathcal{A}) = \tfrac{1}{N}\int h(\widetilde{x})\,\mathrm{d}\widetilde{\mu} \leq \tfrac{N-M}{N}\log|\mathcal{P}|\widetilde{\mu}(E_N) \to 0 \text{ as } N\to\infty. \tag{4.8}$$

Combining (4.6) with (4.8) we get (4.1). □

5. The entropy function, and coarse Lyapunov foliations

We now return to $\mathbb{Z}^d$-actions, and establish the following identity regarding the relation between the entropies of individual elements of the action. This identity is central to our approach.

Theorem 5.1. *Let α be an irreducible $\mathbb{Z}^d$-action on a solenoid X. Let μ be an α-invariant measure and let $\mathcal{A}\subset\mathcal{B}_X$ be an α-invariant σ-algebra. Then there exists a constant $s_{\mu,\mathcal{A}}$ with $\mathsf{h}_\mu(\alpha^{\mathbf{n}}|\mathcal{A}) = s_{\mu,\mathcal{A}}\mathsf{h}_\lambda(\alpha^{\mathbf{n}})$ for every $\mathbf{n}\in\mathbb{Z}^d$.*

To see how this relates to the last sections, let β be a continuous $\mathbb{Z}^d$-action on a compact space Y which is measurably isomorphic via ϕ to the factor of α induced by $\mathcal{A}$ (so that $\phi: X\to Y$ is $\mathcal{A}$-measurable and $\phi\circ\alpha^{\mathbf{n}} = \beta^{\mathbf{n}}\circ\phi$ a.e. for every $\mathbf{n}\in\mathbb{Z}^d$). Let $\widetilde{\alpha}$ be the product action on $\widetilde{X} = Y\times X$, equipped with the measure $\widetilde{\mu} = (\phi\times\mathrm{Id})_*\mu$, so that $\mathsf{h}_\mu(\alpha^{\mathbf{n}}|\mathcal{A}) = \mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}|\mathcal{B}_Y)$.

For every eigenspace $\mathbb{K}_j$ the corresponding *Lyapunov vector* is the linear functional defined by $\mathbf{v}_j(\mathbf{n}) = \log|\zeta_j(\mathbf{n})|_j$. For any non-zero linear function $\mathbf{w}$ the subspace $V_{\mathbf{w}} = \sum_{\mathbf{v}_j\in\mathbb{R}^+\mathbf{w}}\mathbb{K}_j$ is a *coarse Lyapunov subspace*. The coarse Lyapunov subspaces are the biggest sums of eigenspaces which are as a whole contracted, expanded or isometric for every element $\alpha^{\mathbf{n}}$ of the action. This is reflected by the entropy contribution of a coarse Lyapunov subspace.

Lemma 5.2. *Let $V = V_{\mathbf{w}}$ be a nontrivial coarse Lyapunov subspace. Then there exists some $s_{\widetilde{\mu}}(V)\geq 0$ with $\mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V) = s_{\widetilde{\mu}}(V)\mathsf{h}_\lambda(\alpha^{\mathbf{n}}, V)$ for all $\mathbf{n}\in\mathbb{Z}^d$.*

Indeed, if $\mathbf{w}(\mathbf{n})\leq 0$, i.e. $\alpha^{\mathbf{n}}$ does not expand V, Lemma 5.2 is satisfied trivially. It is also clear that for every $\mathbf{n}$ and N,

$$\mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{N\mathbf{n}}, V) = N\mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V). \tag{5.1}$$

The interpretation of $\mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V)$ in terms of the volume growth of suitably scaled boxes given by Proposition 3.4 implies that if $\mathbf{w}(\mathbf{n})\geq\mathbf{w}(\mathbf{m})>0$, then $\mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V)\geq$

$\mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{m}}, V)$; in conjunction with (5.1) we get that $\mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V) = c\max(0, \mathbf{w}(\mathbf{n}))$ for some constant c depending on $\widetilde{\mu}$, V and α; since $\mathsf{h}_{\lambda}(\widetilde{\alpha}^{\mathbf{n}}, V)$ is given by a similar formula, we get the desired identity.

Proposition 5.3. *Let W^+ be the sum of all expanding eigenspaces for $\alpha^{\mathbf{n}}$ and let $W^+ = V_1 + \cdots + V_e$ be its decomposition into coarse Lyapunov subspaces. Then* $\mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}|\mathcal{B}_Y) = \mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^+) = \mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V_1) + \cdots + \mathsf{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V_e)$.

We give no details here, but mention the main reason: $\widetilde{\mu}_{\widetilde{x},W^+}$ is the product measure of the conditionals $\widetilde{\mu}_{\widetilde{x},V_j}$ for $j = 1, \ldots, e$, which together with Proposition 3.4 implies the above proposition. The product structure of the conditionals appeared first in a different context in [3]; see also [21, Sect. 6]. Proposition 5.3 is related to a more general result regarding commuting diffeomorphisms by Hu [10, Thm. B].

Lemma 5.2 and Proposition 5.3 show that Theorem 5.1 is equivalent to the next lemma.

Lemma 5.4. *If α is irreducible, there exists some $s_{\mu,\mathcal{A}}$ such that $s_{\widetilde{\mu}}(V) = s_{\mu,\mathcal{A}}$ for all coarse Lyapunov subspaces V.*

Proof. Since by definition every coarse Lyapunov subspace is expanded by some $\alpha^{\mathbf{n}}$, there exists $T = \alpha^{\mathbf{n}}$ such that no coarse Lyapunov subspace is isometric for T. Suppose $V_1, \ldots, V_e$ are the expanded Lyapunov subspaces and $V_{e+1}, \ldots, V_f$ the contracted ones. We apply Theorem 4.1 for T and some V_j with $j \leq e$. Note that the denominator and the numerator of the fraction in this theorem are exactly $\mathsf{h}_{\lambda}(T)$ and $\mathsf{h}_{\lambda}(T, V_j)$. Therefore $\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}, V_j) \leq s\mathsf{h}_{\lambda}(T, V_j)$ with $s = \mathsf{h}_{\mu}(T|\mathcal{A})/\mathsf{h}_{\lambda}(T)$. However, by Proposition 5.3 the sum over these inequalities for $j \leq e$ gives the inequality $\mathsf{h}_{\mu}(T|\mathcal{A}) \leq s\mathsf{h}_{\lambda}(T) = \mathsf{h}_{\mu}(T|\mathcal{A})$. This shows, that all the inequalities have to be equalities, i.e. $\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}, V_j) = s\mathsf{h}_{\lambda}(T, V_j)$. Since $\mathsf{h}_{\lambda}(T, V_j) > 0$, we conclude that $s_{\widetilde{\mu}}(V_j) = s$ for $j = 1, \ldots, e$.

Using T^{-1} in the above argument does not change s, and proves $s_{\widetilde{\mu}}(V_j) = s$ for $j = e+1, \ldots, f$, thus concluding the proof of Theorem 5.1. □

6. Outline of the proof of Theorem 1.1

Once Theorem 5.1 is proved, Theorem 1.1 can be proved in a way similar to Rudolph's proof of Theorem 1.2. We give a variant of this method, which avoids the need to explicitly employ the suspension construction of Katok and Spatzier.

Let α be a totally irreducible (hence arithmetic) $\mathbb{Z}^d$-action, and let $\alpha_{\mathbb{A}}$ be the corresponding $\mathbb{Z}^d$-action on the covering space $\mathbb{A}$.

Lemma 6.1. *α is* **not** *virtually cyclic if and only if there exist at least two linear independent Lyapunov vectors.*

Sketch of the proof. If α has no two linearly independent Lyapunov vectors, then $H = \bigcap_j \ker \mathbf{v}_j$ is a hyperplane. Suppose $\mathbf{n} \in \mathbb{Z}^d$ is close to H, i.e. satisfies $\mathbf{v}_j(\mathbf{n}) \in (-\epsilon, \epsilon)$ for $j = 1, \ldots, m$. Recall that the numbers $\zeta_{j,\mathbf{n}} \in \mathbb{K}_j$ in Theorem 2.1 are the images of a single algebraic number ζ. For small enough ϵ it follows that ζ must be an algebraic unit whose real and complex embeddings have absolute value close to one. It follows from Dirichlet's unit theorem that ζ must be a unit root and $\mathbf{n} \in H$. Thus α is virtually cyclic. □

Let V and W be the coarse Lyapunov subspaces corresponding to two linearly independent Lyapunov vectors. Then $V+W$ is contracted by some $\alpha^{\mathbf{n}}$, $\mathbf{n}\in\mathbb{Z}^d$. As in the discussion following Proposition 5.3, the conditional measure $\mu_{x,V+W}=\mu_{x,V}\times\mu_{x,W}$ is a product measure a.s., which implies the following.

Lemma 6.2. *There exists a null set $N\subset X$ such that $\mu_{x,V}=\mu_{x',V}$ if $x,x'\notin N$ and $x'\in x+W$.*

Let $\mathcal{A}'$ be the smallest σ-algebra with respect to which $x\mapsto\mu_{x,V}$ is measurable. Then invariance of μ under α implies that $\mu_{\alpha^{\mathbf{n}}x,V}\propto(\alpha_{\mathbb{A}}^{\mathbf{n}})_*\mu_{x,V}$ a.s.; indeed, $\mu_{\alpha^{\mathbf{n}}x,V}=\frac{(\alpha_{\mathbb{A}}^{\mathbf{n}})_*\mu_{x,V}}{(\alpha_{\mathbb{A}}^{\mathbf{n}})_*\mu_{x,V}(B_1^V)}$, and so $\mathcal{A}'$ is α-invariant. Using Lemma 6.2 one can find a countably generated α-invariant σ-algebra $\mathcal{A}$ which is equal modulo μ to $\mathcal{A}'$ so that every $A\in\mathcal{A}$ is a union of full F_W-leaves.

As in the preceding section, let β be a continuous realization of this factor, and recall that the action α on (X,μ) is isomorphic to $\widetilde{\alpha}$ on $(\widetilde{X},\widetilde{\mu})$ via the map $\Phi=\phi\times\mathrm{Id}$.

Proposition 6.3. $\mu_{x,W}=\widetilde{\mu}_{\Phi(x),W}$ *for a.e.* $x\in X$.

Proof. Suppose $\mathcal{C}\subset\mathcal{B}_X$ is a σ-algebra subordinate to W. Since for every x, $[x]_{\mathcal{C}}\subset x+W$, $[x]_{\mathcal{C}}\subset[x]_{\mathcal{A}}$ and so $\mathcal{A}\subset\mathcal{C}$. Since $\phi(x)$ is constant on atoms of $\mathcal{C}$, it follows that there is a σ-algebra $\widetilde{\mathcal{C}}$ subordinate to W in $\widetilde{X}$ such that $\mathcal{C}=\Phi^{-1}\widetilde{\mathcal{C}}$. Since Φ is a measurable isomorphism, the conditionals satisfy $\Phi_*\mu_x^{\mathcal{C}}=\mu_{\Phi(x)}^{\widetilde{\mathcal{C}}}$. It now follows from Proposition 3.2 that indeed $\mu_{x,W}=\widetilde{\mu}_{\Phi(x),W}$ a.s. □

Proposition 6.4. *Suppose the entropy* $\mathsf{h}_\mu(\alpha^{\mathbf{n}})$ *is* **positive** *for some* $\mathbf{n}\in\mathbb{Z}^d$. *Then for μ a.e. x, there is a nonzero $v\in V$ so that $\mu_{x,V}\propto(+_v)_*\mu_{x,V}$.*

Proof. Let N_0 be the set from Proposition 3.2(1) applied to X. By definition of $\mathcal{A}'$ and since $\mathcal{A}=\mathcal{A}'$ (mod μ) there is a set N' of measure zero (which we may as well assume contains N_0) such that if $x,x'\notin N'$ and $[x]_{\mathcal{A}}=[x']_{\mathcal{A}}$ (equivalently, $\phi(x)=\phi(x')$), then $\mu_{x,V}=\mu_{x',V}$. Let $\widetilde{N}=\Phi(N')$.

Let $s_\mu(V)$ and $s_\mu(W)$ be as in Lemma 5.4. Then $s_\mu(V)=s_\mu(W)>0$ and similarly $s_{\widetilde{\mu}}(V)=s_{\widetilde{\mu}}(W)$. Since $s_\mu(W)$ and $s_{\widetilde{\mu}}(W)$ are determined by the conditional measures $\mu_{x,W}$ and $\widetilde{\mu}_{\widetilde{x},W}$, respectively, by Proposition 6.3 $s_\mu(W)=s_{\widetilde{\mu}}(W)$. We conclude that $s_\mu(V)=s_{\widetilde{\mu}}(V)>0$, and in particular $\widetilde{\mu}_{\widetilde{x},V}(\{0\})=0$ a.s.

Let $\widetilde{\mathcal{C}}$ be any V-subordinate σ-algebra. For a.e. x (in particular for $x\notin N'$), $\widetilde{\mu}_{\Phi(x)}^{\mathcal{C}}(\widetilde{N})=0$, so a.s. there is a nonzero $v\in V$ such that $x,x+v\notin N'$ but $\Phi(x+v)\in[\Phi(x)]_{\mathcal{C}}$. In particular, $\phi(x)=\phi(x')$, which implies that $\mu_{x,V}=\mu_{x+v,V}$. Since $x,x+v\notin N_0$, Proposition 3.2 gives that $\mu_{x,V}\propto(+v)_*\mu_{x+v,V}$ and so $\mu_{x,V}\propto(+v)_*\mu_{x,V}$. □

From Proposition 6.4 one can conclude with standard techniques (see for instance [14, Lemma 3.4] or [16, Lemma 5.6]) that $\mu_{x,V}$ is actually translation invariant in the strict sense under some nonzero element of V a.s. Note that so far we have only used that α is irreducible. Using the total irreducibility of α it is not hard to conclude at this stage that μ is Haar measure on X.

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