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Parametric solutions for some Diophantine equations

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Abstract

Under some hypotheses we show that the Diophantine equation (1) has infinitely many solutions described by a family depending on $k+2$ parameters. Some applications of the main result are given and some special equations are studied.

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1 Introduction

Consider the Diophantine equation

$$a_0x_0^{p_0} + a_1x_1^{p_1} + \ldots + a_kx_k^{p_k} = 0 \tag{1}$$

where $a_0, a_1, \ldots, a_k$ are integers, $a_0 > 0$, and $p_0, p_1, \ldots, p_k$ are positive integers. Concerning the equation (1) in the book [3] the following general result is presented:

Assume that p is relatively prime to the product $P_k = p_1 p_2 \dots p_k$. Then:

a) if $a_1 + a_2 + \ldots + a_k \neq 0$, the equation (1) has infinitely many solutions in integers;

b) if $a_1 + a_2 + \ldots + a_k < 0$, the equation (1) has infinitely many solutions in positive integers.

In both cases mentioned above, the solutions are described by a family depending on a parameter.

In the paper [4], the second author gave a much general result without restrictive conditions a) and b). Moreover, the solutions are described by a family depending on $k+2$ parameters. The main result in [4] is contained in the following:

Theorem. *Consider the equation (1) with $a_0 > 0$ and assume that p_0 is relatively prime to $m = lcm(p_1, p_2, \ldots, p_k)$. Then:*

a) the equation (1) has infinitely many solutions in integers;

b) if $a_i < 0$, for some $i \in \{1, 2, \ldots, k\}$, then the equation (1) has infinitely many solutions in positive integers.

In order to construct a family of solutions, let us denote

$$T_k = a_0^{p_0 - 1}(-a_1 n_1^{p_1} - a_2 n_2^{p_2} - \ldots - a_k n_k^{p_k}) \tag{2}$$

where $n_1, n_2, \ldots, n_k$ are arbitrary integers. Taking into account that p and m are relatively prime, it follows that for infinitely many pairs (q, r) of positive integers the relation

$$p_0 q = mr + 1 \tag{3}$$

holds. Then a family of solutions to equation (1) is given by

$$
\left\{
\begin{array}{l}
x_0 = n_0^{mp_0/p_0} \cdot a_0^{-1} \cdot T_k^q \\
\\
x_1 = n_0^{mp_0/p_1} \cdot n_1 \cdot T_k^{rm/p_1} \\
\\
x_2 = n_0^{mp_0/p_2} \cdot n_2 \cdot T_k^{rm/p_2} \\
\dots \\
x_k = n_0^{mp_0/p_k} \cdot n_k \cdot T_k^{rm/p_k}
\end{array}
\right. \tag{4}
$$

The solutions in (4) depend on the $k+2$ parameters $n_0, n_1, \dots, n_k, r$ (or q).

Remarks. 1) If $n_0, n_1, \dots, n_k$ are rational numbers, the formula (4) point out an infinite family of rational solutions to equation (1).

2) In the particular case $n_0 = a_0$, $n_1 = n_2 = \dots = n_k = 1$,if we replace m by $p_1 p_2 \dots p_k$,then the formula (4) is obtained in the book [3].

3) If $n_0, n_1, \dots, n_k$ are real numbers, then (4) gives us a polynomial parametrization of the algebraic hypersurface defined by (1) in the Euclidean space $\mathbb{R}^{k+1}$.

4) A simplified form of (4) is obtained when $n_0 = 1$:

$$x_0 = a_0^{-1} T_k^q,\ x_1 = n_1 T_k^{rm/p_1},\ x_2 = n_2 T_k^{rm/p_2}, \dots, x_k = n_k T_k^{rm/p_k} \tag{5}$$

i.e. an infinite family depending on $k+1$ parameters.

In the references [1], [2] and [3] there are many examples of Diophantine equations which are special cases of equation (1). Let us mention the following equations contained in [1]:

(a) $x^p + y^p = z^{p\pm 1}$, (b) $x^2 + y^3 = z^5$, (c) $x^p + y^p + z^p + u^p = v^{p\pm 1}$.

Also, we mention some equations contained in [3]:

(d) $x^2 + y^3 = z^4$, (e) $x^2 + y^3 + z^4 = t^2$, (f) $x^2 + y^4 = 2z^3$.

In what follows we will apply the result in the above mentioned Theorem for some of these equations as well as for some other generalized equations.

2 The equations $x^p + y^p = z^{np\pm1}$ and $x^p + y^p = z^{p^n\pm1}$

First of all we will change the notations in order to apply in a direct way the result in our Theorem.

Consider the equation

$$x_0^{np\pm1} - x_1^p - x_2^p = 0, \tag{6}$$

where p and n are positive integers. In that case we have $a_0 = 1$, $a_1 = a_2 = -1$ and $p_0 = np \pm 1$ is relatively prime to p. There exist infinitely many positive integers q and r such that

$$(np \pm 1)q = pr + 1 \tag{7}$$

It is easy to show that

$$\begin{cases} r(t) = (np \pm 1)t \pm n \\ \\ q(t) = pt \pm 1 \end{cases} \tag{8}$$

where t is any positive integers and the signs $+$ and $-$ correspond. Using formula (5) we find the following family of solutions to equation (6):

$$\begin{cases} x_0 = (n_1^p + n_2^p)^{pt\pm1} \\ \\ x_1 = n_1(n_1^p + n_2^p)^{(np\pm1)t\pm n} \\ \\ x_2 = n_2(n_1^p + n_2^p)^{(np\pm1)t\pm n} \end{cases} \tag{9}$$

Let us note that if $n = 1$, then we obtain the equations (a). If $n_1 = 1$, $n_2 = k$, $t = 1$ and $n_1 = k$, $n_2 = 1$, $t = 1$, respectively, we find the solutions

$$x_0 = k^p + 1, \quad x_1 = k^p + 1, \quad x_2 = k(k^p + 1)$$

when we consider the sign +, and

$$x_0 = (k^p + 1)^{p-1}, \quad x_1 = (k^p + 1)^{p-2}, \quad x_2 = k(k^p + 1)^{p-2}$$

in case of the sign −. These solutions are given in the book [1].

Let us consider the equations

$$x_0^{p^n \pm 1} - x_1^p - x_2^p = 0, \tag{10}$$

where p and n are positive integers. In that case we have $p_1 = p_2 = p$, $p_0 = p^n \pm 1$ and p_0 is relatively prime to p. Hence

$$(p^n \pm 1)q = pr + 1 \tag{11}$$

for some positive integers r and q. All such pairs (r, q) are given by

$$\begin{cases} r(t) = (p^n \pm 1)t \pm p^{n-1} \\ \\ q(t) = pt \pm 1 \end{cases} \tag{12}$$

where t is any positive integer and signs + and − correspond. From formula (5) we find the following family of solutions to equation (10):

$$\begin{cases} x_0 = (n_1^p + n_2^p)^{pt \pm 1} \\ \\ x_1 = n_1(n_1^p + n_2^p)^{(p^n \pm 1)t \pm p^{n-1}} \\ \\ x_2 = n_2(n_1^p + n_2^p)^{(p^n \pm 1)t \pm p^{n-1}} \end{cases} \tag{13}$$

The signs + and − in (13) correspond to the signs + and − in (10). Let us note that if $n = 1$, then we obtain again the equations (a).

Remark. In the book [1] the following equation is given

$$x_0^{p-1} - x_1^p - x_2^p - x_3^p - x_4^p = 0 \tag{14}$$

It is clear that we have $a_0 = 1$, $a_1 = a_2 = a_3 = a_4 = -1$, $p_1 = p_2 = p_3 = p_4 = p$ and $p_0 = p - 1$. An infinite family of solutions to (14) depending on two parameters is obtained in [1] by multiplication principle applied to equation (a) where the sign − is considered. Now we can construct a larger family of solutions depending on five parameters. Indeed, from relation

$$(p-1)q = pr + 1 \tag{15}$$

we deduce

$$\begin{cases} r(t) = (p-1)t - 1 \\ \\ q(t) = pt - 1 \end{cases} \tag{16}$$

where t is any positive integer. Formula (5) gives us the following family of solutions:

$$\begin{aligned} &x_0 = S^{pt-1},\ x_1 = n_1 S^{(p-1)t-1},\ x_2 = n_2 S^{(p-1)t-1} \\ &x_3 = n_3 S^{(p-1)t-1},\ x_4 = n_4 S^{(p-1)t-1}, \end{aligned} \tag{17}$$

where $S = n_1^{p_1} + n_2^{p_2} + n_3^{p_3} + n_4^{p_4}$ and n_1, n_2, n_3, n_4, t are arbitrary positive integers.

3 The equation $x_0^p - x_1^{2p-1} - x_2^{2p+1} = 0$

Consider the equation

$$x_0^p - x_1^{2p-1} - x_2^{2p+1} = 0 \tag{18}$$

In that case we have $a_0 = 1$, $a_1 = a_2 = -1$, $p_1 = 2p - 1$, $p_2 = 2p + 1$, $p_0 = p$. It is clear that p_0 is relatively prime to $p_1 p_2 = 4p^2 - 1$. In the case $p = 2$ we obtain equation (b) also studied in the book [1].

Because p_0 is relatively prime to $4p^2 - 1$, we have

$$pq = (4p^2 - 1)r + 1 \tag{19}$$

and all pairs (r, q) of such positive integers are given by

$$\begin{cases} r(t) = pt + 1 \\ \\ q(t) = 4p^2 t + 4p - t \end{cases} \tag{20}$$

for any positive integer t. Applying formula (4) we obtain the following family of solutions to equation (18):

$$\begin{cases} x_0 = n_0^{4p^2-1}(n_1^{2p-1} + n_2^{2p+1})^{(4p^2-1)t+4p} \\ \\ x_1 = n_0^{p(2p+1)} n_1 (n_1^{2p-1} + n_2^{2p+1})^{(2p+1)(pt+1)} \\ \\ x_2 = n_0^{p(2p-1)} n_2 (n_1^{2p-1} + n_2^{2p+1})^{(2p-1)(pt+1)} \end{cases} \tag{21}$$

The family (21) depends on four parameters n_0, n_1, n_2, t.

In the case $p = 2$ we obtain a family of solutions to equation (b):

$$\begin{cases} x_0 = n_0^{15}(n_1^3 + n_2^5)^{15t+8} \\ \\ x_1 = n_0^{10} n_1 (n_1^3 + n_2^5)^{10t+5} \\ \\ x_2 = n_0^{6} n_2 (n_1^3 + n_2^5)^{6t+3} \end{cases} \tag{22}$$

where n_0, n_1, n_2, t are any positive integers.

4 The equation $b_m x_m^{2n+m} + b_{m+1} x_{m+1}^{2n+m+1} + \ldots + b_{m+p} x_{m+p}^{2n+m+p} = 0$

In the above equation n and p are positive integers and m is an integer. The coefficients b_i, $i = m, m+1, \ldots, m+p$, are integers. In what follows we will study three special cases of this equation. We use the notations in our Theorem.

4.1. Let us consider the equation

$$a_0 x_0^{2n+1} + a_1 x_1^{2n-1} + a_2 x_2^{2n} + a_3 x_3^{2n+2} + a_4 x_4^{2n+3} = 0 \tag{23}$$

where $a_0 > 0$, $a_1^2 + a_2^2 + a_3^2 + a_4^2 \neq 0$ and $n \geq 2$ is a positive integer.

We have $p_0 = 2n+1$, $p_1 = 2n-1$, $p_2 = 2n$, $p_3 = 2n+2$, $p_4 = 2n+3$ and p_0 is relatively prime to each of the integers p_1, p_2, p_3, p_4. Applying the result in our main Theorem we obtain:

Proposition 1. *a) The equation (23) has infinitely many solutions in integers.*

b) If $a_i < 0$ for some $i \in \{1, 2, 3, 4\}$, then the equation (23) has infinitely many solutions in positive integers.

Let us indicate how we can construct an infinite family of solutions. Because $p_0 = 2n+1$ is relatively prime to each p_1, p_2, p_3, p_4 it follows that

$$(2n+1)q = (2n-1)2n(2n+2)(2n+3)r + 1 \tag{24}$$

for some positive integers r and q. That is equivalent to

$$p_0 q = (p_0^2 - 4)(p_0^2 - 1)r + 1 \tag{25}$$

From (25) it follows that $4r + 1 = p_0 s$, where s is a positive integer. We can choose $s = 4t + p_0$ for any positive integer t and we find

$$(26) \qquad \begin{cases} r(t) = \dfrac{1}{4}(4p_0 t + p_0^2 - 1) \\ \\ q(t) = \dfrac{1}{4p_0}[(p_0^2 - 4)(p_0^2 - 1)(4p_0 t + p_0^2 - 1) + 4] \end{cases}$$

Using formula (4) or (5) we obtain an infinite family of integral solutions to equation (23).

As an example, let consider $n = 2$ i.e. the equation

$$(27) \qquad a_0 x_0^5 + a_1 x_1^3 + a_2 x_2^4 + a_3 x_3^6 + a_4 x_4^7 = 0$$

We take $T_4 = a_0^4(-a_1 n_1^3 - a_2 n_2^4 - a_3 n_3^6 - a_4 n_4^7)$, $r(t) = 5t + 6$, $q(t) = 504t + 605$, where n_1, n_2, n_3, n_4, t are arbitrary integers.

4.2. Consider the equation

$$(28) \quad a_0 x_0^{2n+1} + a_1 x_1^{2n-3} + a_2 x_2^{2n-1} a_3 x_3^{2n} + a_4 x_4^{2n+2} + a_5 x_5^{2n+3} + a_6 x_6^{2n+5} = 0$$

where n is a positive integer ≥ 3, the coefficients a_i are integers, $a_0 > 0$ and $a_1^2 + a_2^2 + \ldots + a_6^2 \neq 0$. We have $p_0 = 2n + 1$ and it is relatively prime to any $p_1 = 2n - 3$, $p_2 = 2n - 1$, $p_3 = 2n$, $p_4 = 2n + 2$, $p_5 = 2n + 3$, $p_6 = 2n + 5$. From our main Theorem it follows:

Proposition 2. *a) The equation (28) has infinitely many solutions in integers.*

b) If $a_i < 0$ for some $i \in \{1, 2, 3, 4, 5, 6\}$, then the equation (28) has infinitely many solutions in positive integers.

We can construct an infinite family of solutions in the following way. The integers p_0 and $p_1 p_2 p_3 p_4 p_5 p_6$ are relatively prime, hence

$$(29) \qquad (2n_1)q = (2n - 3)(2n - 1)(2n)(2n + 2)(2n + 3)(2n + 5)r + 1$$

for some positive integers r and q. That is equivalent to

$$p_0 q = (p_0^2 - 16)(p_0^2 - 4)(p_0^2 - 1)r + 1 \tag{30}$$

It follows $64r - 1 = p_0 s$, where s is a positive integer. In order to find convenient pairs (r, q) of positive integers satisfying (30) let us use the following obvious property: *For any positive integers $n, k \geq 1$, the integer $(2n+1)^{2^k} - 1$ is divisible by 2^{k+2}.* In that case we can consider

$$\begin{cases} r(t) = \dfrac{1}{64}(64t - 1)p_0^{16} + 1 \\ \\ q(t) = \dfrac{1}{p_0}[(p_0^2 - 16)(p_0^2 - 4)(p_0^2 - 1)r(t) + 1], \end{cases} \tag{31}$$

where t is any positive integer.

In the particular case $n = 3$, we have

$$T_6 = a_0^6(-a_1 n_1^3 - a_2 n_2^5 - a_3 n_3^6 - a_4 n_4^8 - a_5 n_5^9 - a_6 n_6^{11})$$

and

$$r(t) = \frac{1}{64}[(64t - 1)7^{16} + 1], \quad q(t) = \frac{1}{7}(33 \cdot 45 \cdot 48 r(t) + 1)$$

A family of integral solutions to equation (28) can be obtained by using formula (4) or (5).

4.3. Let us consider the equation

$$a_0 x_0^{2n+3} + a_1 x_1^{2n-1} + a_2 x_2^{2n} + a_3 x_3^{2n+1} + a_4 x_4^{2n+2} + \tag{32}$$

$$+ a_5 x_5^{2n+4} + a_6 x_6^{2n+5} + a_7 x_7^{2n+6} + a_8 x_8^{2n+7} = 0,$$

where $n \geq 2$ is a positive integer, the coefficients a_i are integers, $a_0 > 0$ and $a_1^2 + a_2^2 + \ldots + a_8^2 \neq 0$. Assume that n is not divisible by 3. Then $p_0 = 2n+3$ is relatively prime to all p_i, $i = 1, 2, \ldots, 8$. We have many integral solutions.

b) If $a_i < 0$ for some $i \in \{1, 2, \ldots, 8\}$, then equation (32) has infinitely many solutions in positive integers.

We will indicate the effective construction of an infinite family of integral solutions. Taking into account that $p_0 = 2n + 3$ is relatively prime to the product $p_1 p_2 \ldots p_8$, we have

$$(2n+3)q = \tag{33}$$

$$= (2n-1)(2n)(2n+1)(2n+2)(2n+4)(2n+5)(2n+6)(2n+7)r + 1,$$

for some positive integers r and q. The relation (33) is equivalent to

$$p_0 q = (p_0^2 - 16)(p_0^2 - 9)(p_0^2 - 4)(p_0^2 - 1)r + 1 \tag{34}$$

Therefore, the relation $9 \cdot 64r + 1 = p_0 s$, for a positive integer s.

Taking into account that $(2n+3)^{16} - 1 = (2(n+1)+1)^{16} - 1$ is divisible by 64 (see the general property in 4.2) and $[(2n+3)^2 - 1]^2$ is divisible by 9, it follows that we can choose $r(t)$ and $q(t)$ as

$$\begin{cases} r(t) = \dfrac{1}{9 \cdot 64}(p_0 t + 1)(p_0^{16} - 1)(p_0^2 - 1)^2 \\ \\ q(t) = \dfrac{1}{p_0}[(p_0^2 - 16)(p_0^2 - 9)(p_0^2 - 4)(p_0^2 - 1)r(t) + 1] \end{cases} \tag{35}$$

where t is any positive integer. Using (3) we can derive an infinite family of integral solutions to equation (32) directly from formula (4) or (5).

References

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