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# Direct Results for Mixed Beta-Szász Type Operators

**Vijay Gupta** and **Alexandru Lupaş**

*Dedicated to Professor Dumitru Acu on his 60th anniversary*

**Abstract**

In this paper we study the mixed summation-integral type operators having Beta and Szász basis functions in summation and integration respectively, we obtain the rate of point wise convergence, a Voronovskaja type asymptotic formula and an error estimate in simultaneous approximation.

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## 1 Introduction

Recently Srivastava and Gupta [7] proposed a general family of summation-integral type operators which include some well known operators (see [4],

[6]) as special cases. Very recently Ispir and Yuksel [5] considered the Bezier variant of the operators studied in [7] and estimated the rate of convergence for bounded variation operators. Several other hybrid summation-integral type operators were proposed by V. Gupta and M. K. Gupta [3] and Z. Finta [1]. For $f \in C_\gamma[0,\infty) = \{f \in C[0,\infty) : |f(t)| \leq Me^{\gamma t}$, for some $M > 0, \gamma > 0\}$, we consider a mixed sequence of summation-integral type operators as

(1)

$$B_n(f,x) = \int_0^\infty W_n(x,t)f(t)dt = \sum_{v=1}^\infty b_{n,v}(x)\int_0^\infty f(t)s_{n,v-1}(t)dt + (1+x)^{-n-1}f(0)$$

where $W_n(x,t) = \sum_{v=1}^\infty b_{n,v}(x)s_{n,v-1}(t) + (1+x)^{-n-1}\delta(t)$, $\delta(t)$ being Dirac delta function

and

$$b_{n,v}(x) = \frac{1}{B(n,v+1)}\frac{x^v}{(1+x)^{n+v+1}}, s_{n,v}(t) = e^{-nt}\frac{(nt)^v}{v!},$$

are respectively Beta and Szász basis functions. It is easily verified that the operators (1) are linear positive operators these operators were recently proposed by the author in [3]. The behaviour of these operators are very similar to the operators studied by Gupta and Srivastava [2], but the approximation properties of the operators $B_n$ are different in comparison to the operators studied in [2]. The main difference is that the operators are discretely defined at the point zero. In the present paper we study some direct results for the operators $B_n$, we obtain a point wise rate of convergence, asymptotic formula of Voronovskaja type and an error estimate in simultaneous approximation.

## 2 Auxiliary Results

We need the following lemmas in the sequel.

**Lemma 1.** *For $m \in N^0 := (0, 1, 2, 3, ....)$, if the m-th order moment be defined as*

$$U_{n,m}(x) = \frac{1}{n} \sum_{v=0}^{\infty} b_{n,v}(x)(v(n+1)^{-1} - x)^m,$$

*then $U_{n,0}(x) = 1, U_{n,1}(x) = 0$ and $nU_{n,m+1}(x) = x[U_{n,m}^{(1)}(x) + mU_{n,m-1}(x)]$. Consequently*

$$U_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

**Lemma 2.** *Let the function $\mu_{n,m}(x), m \in N^0$, be defined as*

$$\mu_{n,m}(x) = \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} (t-x)^m s_{n,v-1}(t)dt + (1+x)^{-n-1}(-x)^m$$

*Then*

$$\mu_{n,0}(x) = 1, \mu_{n,1}(x) = x/n, \mu_{n,2}(x) = \frac{x(1+x)(n+2)+nx}{n^2}$$

*also we have the recurrence relation:*

$$n\mu_{n,m+1}(x) = x(1+x)[\mu_{n,m}^{(1)}(x) + m\mu_{n,m-1}(x)] + (m+x)\mu_{n,m}(x) + mx\mu_{n,m-1}(x).$$

*Consequently for each $x \in [0, \infty)$, we have from this recurrence relation that $\mu_{n,m}(x) = O(n^{-[(m+1)/2]})$.*

**Remark 1.** *From Lemma 2, we can easily obtain the following identity*

$$B_n(t^i, x) = \frac{(n+i)!}{n!n^i} x^i + i(i-1)\frac{(n+i-1)!}{n!n^i} x^{i-1} + O(n^{-2})$$

**Lemma 3.** *There exist the polynomials $Q_{i,j,r}(x)$ independent of $n$ and $v$ such that*

$$[x(1+x)]^r D^r[b_{n,v}(x)] = \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+1)^i [v-(n+1)x]^j Q_{i,j,r}(x) b_{n,v}(x)$$

*where $D = \frac{d}{dx}$.*

# 3 Simultaneous Approximation

**Theorem 1.** *Let $f \in C_\gamma[0,\infty), \gamma > 0$ and $f^{(r)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} B_n^{(r)}(f(t),x) = f^{(r)}(x), \tag{2}$$

**Proof.** *By Taylor's expansion of $f$, we have*

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r$$

*where $\varepsilon(t,x) \to 0$ as $t \to \infty$. Hence*

$$B_n^{(r)}(f(t),x) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt +$$

$$+ \int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r dt = E_1 + E_2, say.$$

*First to estimate $E_1$, using binomial expansion of $(t-x)^r$, and Lemma 2, we have*

$$E_1 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{v=0}^{i} \binom{i}{v} (-x)^{i-v} \int_0^\infty W_n^{(r)}(t,x) t^r dt =$$

$$= \frac{f^{(r)}(x)}{r!} \int_0^\infty W_n^{(r)}(t,x) t^r dt = f^{(r)}(x) + o(1), n \to \infty.$$

*Next using Lemma 3, we obtain*

$$|E_2| \le \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+1)^i \frac{|Q_{i,j,r}(x)|}{[x(1+x)]^r} \sum_{v=1}^{\infty} |v-(n+1)x|^j \, b_{n,v}(x)$$

$$\int_0^\infty s_{n,v-1}(t) \, |\varepsilon(t,x)| \, (t-x)^r dt +$$

$$+(-n-1)(-n-2)...(-n-r)(1+x)^{(-n-1-r)} \, |\varepsilon(0,x)| \, (-x)^r = E_3 + E_4.$$

*Since* $\varepsilon(t,x) \to 0$ *as* $t \to x$ *for a given* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|\varepsilon(t,x)| < \varepsilon$ *whenever* $0 < |t-x| < \delta$. *Further if* $s \ge max\{\gamma, r\}$, *where* $s$ *is any integer, then we can find a constant* $M_1$ *such that* $|\varepsilon(t,x)(t-x)^r| \le M_1 |t-x|^s$, *for* $|t-x| \ge \delta$. *Thus with* $M_2 = \sup\limits_{2i+j\le r} [x(1+x)]^{-r} |Q_{i,j,r}(x)|$, *we have*

$$E_3 \le M_2 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+1)^i \sum_{v=1}^{\infty} b_{n,v}(x) \, |v-(n+1)x|^j \cdot$$

$$\cdot \left\{ \varepsilon \int_{|t-x|<\delta} s_{n,v-1}(t) \, |t-x|^r + \int_{|t-x|\ge\delta} s_{n,v-1}(t) M_1 \, |t-x|^s \, dt \right\} = E_5 + E_6.$$

*Applying Schwarz inequality for integration and summation respectively and using Lemma 1 and Lemma 2, we obtain*

$$E_5 \le \varepsilon M_2 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+1)^i \sum_{v=1}^{\infty} b_{n,v}(x) \, |v-(n+1)x|^j \left\{ \int_0^\infty s_{n,v-1}(t) dt \right\}^{1/2} \cdot$$

$$\cdot \left\{ \int_0^\infty s_{n,v-1}(t)(t-x)^{2r} dt \right\}^{1/2} \le \varepsilon M_2 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+1)^i O(n^{j/2}) O(n^{-r/2}) = \varepsilon O(1).$$

*Again using Schwarz inequality, Lemma 1 and Lemma 2, we get*

$$E_6 \le M_3 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+1)^i \sum_{v=1}^{\infty} b_{n,v}(x) \, |v-(n+1)x|^j \int_{|t-x|\ge\delta} s_{n,v-1}(t) \, |t-x|^s \, dt \le$$

$$\leq M_3 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+1)^i (\sum_{v=1}^{\infty} (v-(n+1)x)^{2j})^{1/2} (\sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} s_{n,v-1}(t)(t-x)^{2s} dt)^{1/2} =$$

$$= \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+1)^i O(n^{j/2}) O(n^{-s/2}) = O(n^{(r-s)/2}) = o(1).$$

*Thus due to arbitrariness of* $\varepsilon > 0$ *it follows that* $E_3 = o(1)$ *Also* $E_4 \to 0$ *as* $n \to \infty$ *and hence* $E_2 = o(1)$. *Collecting the estimates of* $E_1$ *and* $E_2$ , *we get the required result.*

**Theorem 2.** *Let* $f \in C_\gamma[0,\infty), \gamma > 0$ *and* $f^{(r+2)}$ *exists at a point* $x \in (0,\infty)$, *then*

$$\lim_{n\to\infty} n[B_n^{(r)}(f,x) - f^{(r)}(x)] = \frac{r(r+1)}{2} f^{(r)}(x) +$$

$$+[x(1+r)+r] f^{(r+1)}(x) + \frac{(x^2+x)}{2} f^{(r+2)}(x).$$

**Proof.** *Using Taylor's expansion of* $f$, *we have*

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} (t-x)^i + \varepsilon(t,x)(t-x)^{r+2}$$

*where* $\varepsilon(t,x) \to 0$ *as* $t \to x$. *Applying Lemma 2, we have*

$$n[B_n^{(r)}(f,x) - f^{(r)}(x)]$$

$$= [\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \int_0^{\infty} W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x)] +$$

$$+n \int_0^{\infty} W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^{r+2} dt = J_1 + J_2.$$

$$J_1 = n \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \begin{pmatrix} i \\ j \end{pmatrix} (-x)^{i-j} \int_0^{\infty} W_n^{(r)}(t,x) t^j dt - n f^{(r)}(x) =$$

$$= \frac{f^{(r)}(x)}{r} n \left[ B_n^{(r)}(t^r, x) - (r!) \right] +$$

$$+ \frac{f^{(r+1)}(x)}{(r+1)!} n \left[ (r+1)(-x) B_n^{(r)}(t^r, x) + B_n^{(r)}(t^{r+1}, x) \right] + \frac{f^{(r+2)}(x)}{(r+2)!} n \cdot$$

$$\cdot \left[ \frac{(r+1)(r+2)}{2} x^2 B_n^{(r)}(t^r, x) + (r+2)(-x) B_n^{(r)}(t^{r+1}, x) + B_n^{(r+2)}(t^{r+2}, x) \right]$$

*Using Remark 1 for each $x \in (0, \infty)$, we have*

$$J_1 = n f^{(r)}(x) \left[ \frac{(n+r)!}{n! n^r} - 1 \right]$$

$$+ n \frac{f^{(r+1)}(x)}{(r+1)!} \left[ (r+1)(-x)(-r!) \left\{ \frac{(n+r)!}{n! n^{r+1}} \right\} + \right.$$

$$\left. + \left\{ \frac{(n+r+1)!}{n! n^{r+1}} (r+1)! x + r(r+1) \frac{(n+r)!}{n! n^{r+1}} (r!) \right\} \right] +$$

$$+ n \frac{f^{(r+2)}(x)}{(r+2)!} \left[ \frac{(r+2)(r+1)x^2}{2} (r!) \frac{(n+r)!}{n^r n!} \right.$$

$$+ (r+2)(-x) \left\{ \frac{(n+r+1)!}{n^{r+1} n!} (r+1)! x + r(r+1) \frac{(n+r)!}{n! n^{r+1}} (r!) \right\}$$

$$\left. + \left\{ \frac{(n+r+2)!}{n! n^{r+2}} \frac{(r+2)!}{2} x^2 + (r+1)(r+2) \frac{(n+r+1)!}{n! n^{r+2}} (r+1)! x \right\} + O(n^{-2}) \right]$$

*In order to complete the proof of the theorem it is sufficient to show that $J_2 \to 0$ as $n \to \infty$ , which can easily be proved along the lines of the proof of Theorem 1 and by using Lemma 1, Lemma 2 and Lemma 3.*

**Remark 2.** *In particular if $r = 0$, we obtain the following conclusion of the above asymptotic formula in ordinary approximation which was obtained in [4, Th. 2], for bounded functions:*

$$\lim_{n \to \infty} n[B_n(f, x) - f(x)] = x f^{(1)}(x) + \frac{(x^2 + x)}{2} f^{(2)}(x).$$

**Theorem 3.** *Let $f \in C_\gamma[0,\infty)$ and $r \le m \le (r+2)$. If $f^{(m)}$ exists and is continuous on $(a-\eta, b+\eta)$, then for $n$ sufficiently large*

$$\|B_n^{(r)}(f,x) - f^{(r)}\| \le M_4 n^{-1} \sum_{i=r}^{m} \|f^{(i)}\| + M_5 \omega(f^{(r+1)}, n^{-1/2}) + O(n^{-2}),$$

*where the constants $M_4$ and $M_5$ are independent of $f$ and $n$, $\omega(f,\delta)$ is the modulus of continuity of $f$ on $(a-\eta, b+\eta)$ and $\|.\|$ denotes the sup-norm on the interval $[a,b]$.*

**Proof.** *By Taylor's expansion of $f$, we have*

$$f(t) = \sum_{i=0}^{m} (t-x)^i \frac{f^{(i)}(x)}{i!} + (t-x)^m \zeta(t) \frac{f^{(m)}(\xi) - f^{(m)}(x)}{m!} + h(t,x)(1-\zeta(t)),$$

*where $\zeta$ lies between $t$ and $x$ and $\zeta(t)$ is the characteristic function on the interval $(a-\eta, b+\eta)$.*

*For $t \in (a-\eta, b+\eta), x \in [a,b]$, we have*

$$f(t) = \sum_{i=0}^{m} (t-x)^i \frac{f^{(i)}(x)}{i!} + (t-x)^i \frac{f^{(m)}(\xi) - f^{(m)}(x)}{m!}.$$

For $t \in [0,\infty) \setminus (a-\eta, b+\eta)$ , we define

$$h(t,x) = f(t) - \sum_{i=0}^{m} (t-x)^i \frac{f^{(i)}(x)}{i!}$$

Thus

$$B_n^{(r)}(f,x) - f^{(r)}(x) = \left\{ \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x) \right\} +$$

$$+ \left\{ \int_0^\infty W_n^{(r)}(t,x) \frac{f^{(m)}(\xi) - f^{(m)}(x)}{m!} (t-x)^m \zeta(t) dt \right\} +$$

$$+\left\{\int_0^\infty W_n^{(r)}(t,x)h(t,x)(1-\zeta(t))dt\right\} = K_1 + K_2 + K_3$$

Using Remark 1, we obtain

$$K_1 = \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \int_0^\infty W_n^{(r)}(t,x) t^j dt - f^{(r)}(x) =$$

$$= \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j}.$$

$$\cdot \frac{\partial^r}{\partial x^r} \cdot \left[ \frac{(n+j)!}{n^j\ n!} x^j + j(j-1) \frac{(n+j-1)!}{n^j\ n!} x^{j-1} + O(n^{-2}) \right] - f^{(r)}(x)$$

Hence

$$\|K_1\| \leq M_4 n^{-1} \sum_{i=r}^{m} \left\| f^{(i)} \right\| + O(n^{-2}),$$

uniformly in $x \in [a,b]$. Next

$$|K_2| \leq \int_0^\infty W_n^{(r)}(t,x) \frac{\left| f^{(m)}(\xi) - f^{(m)}(x) \right|}{m!} |t-x|^m \zeta(t) dt \leq$$

$$\leq \frac{\omega(f^{(m)}\delta)}{m!} \int_0^\infty \left| W_n^{(r)}(t,x) \right| \left(1 + \frac{|t-x|}{\delta}\right) |t-x|^m dt.$$

Next, we shall show that for $q = 0, 1, 2, .....$

$$\sum_{v=1}^{\infty} b_{n,v}(x) \left| v - (n+1)x \right|^j \int_0^\infty s_{n,v-1}(t) \left| t-x \right|^q dt = O(n^{(j-q)/2})$$

Now by using Lemma 1 and Lemma 2, we have

$$\sum_{v=1}^{\infty} b_{n,v}(x) \left| v - (n+1)x \right|^j \int_0^\infty s_{n,v-1}(t) \left| t-x \right|^q dt \leq$$

$$\leq \left(\sum_{v=1}^{\infty} b_{n,v}(x)(v-(n+1)x)^{2j}\right)^{1/2} \left(\sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} s_{n,v-1}(t)(t-x)^{2q} dt\right)^{1/2} =$$

$$= O(n^{j/2})O(n^{-q/2}) = O(n^{(j-q)/2}),$$

uniformly in x. Thus by Lemma 3, we obtain

$$\sum_{v=1}^{\infty} |b_{n,v}(x)| \int_0^{\infty} s_{n,v-1}(t)\,|t-x|^q\,dt \leq$$

$$\leq M_6 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+1)^i \left[\sum_{v=1}^{\infty} b_{n,v}(x)\,|v-(n+1)x|^j \int_0^{\infty} s_{n,v-1}(t)\,|t-x|^q\,dt\right] =$$

$$= O(n^{(r-q)/2}),$$

uniformly in $x$, where $M_6 = \sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \sup_{x\in[a,b]} |Q_{i,j,r}(x)|\,[x(1+x)]^{-r}$. Choosing $\delta = n^{-1/2}$ , we get for any $s > 0$

$$\|K_2\| \leq \frac{\omega(f^{(m)}, n^{-1/2})}{m!}[O(n^{(r-m)/2}) + n^{1/2}O(n^{(r-m-1)/2}) + O(n^{-s})] \leq$$

$$\leq M_5\omega(f^{(m)}, n^{-1/2})n^{-(m-r)/2}.$$

Since $t \in [0,\infty) \setminus (a-\eta, b+\eta)$, we can choose a $\delta > 0$ in such a way that $|t-x| \geq \delta$ for all $x \in [a,b]$. Applying Lemma 3, we obtain

$$\|K_3\| \leq \sum_{v=1}^{\infty} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+1)^i \frac{|Q_{i,j,r}(x)|}{[x(1+x)]^r}\,|v-(n+1)x|^j\,b_{n,v}(x)\cdot$$

$$\cdot \int_{|t-x|\geq\delta} s_{n,v-1}(t)\,|h(t,x)|\,dt+$$

$$+(-n-1)(-n-2)...(-n-r)(1+x)^{(-n-1-r)}\,|h(0,x)|$$

If $\beta$ is any integer greater than equal to $\{\gamma, m\}$, then we can find a constant $M_7$ such that $|h(t,x)| \leq M_7 |t-x|^\beta$ for $|t-x| \geq \delta$. Now applying Lemma 1 and Lemma 2, it is easily verified that $J_3 = O(n^{-q})$ for any $q > 0$ uniformly on $[a,b]$. Combining the estimates $K_1, K_2$ and $K_3$, we get the required result.

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School of Applied Sciences,
Netaji Subhas Institute of Technology,
Azad Hind Fauj Marg,
Sector 3 Dwarka New Delhi-110045, India
E-mail: *vijay@nsit.ac.in*

University of Sibiu,
Faculty of Sciences,
Department of Mathematics,
Str. I. Ratiu nr. 7, 550012 - Sibiu, Romania
E-mail: *alexandru.lupas@ulbsibiu.ro*