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Steepest descent approximations in Banach space[1]

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Abstract

Let E be a real Banach space and let $A : E \to E$ be a Lipschitzian generalized strongly accretive operator. Let $z \in E$ and x_0 be an arbitrary initial value in E for which the steepest descent approximation scheme is defined by

$$\begin{aligned} x_{n+1} &= x_n - \alpha_n(Ay_n - z), \\ y_n &= x_n - \beta_n(Ax_n - z), \quad n = 0, 1, 2 \ldots, \end{aligned}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

$$\begin{aligned} &(i) \quad 0 \le \alpha_n, \beta_n \le 1, \\ &(ii) \quad \sum_{n=0}^{\infty} \alpha_n = +\infty, \\ &(iii) \quad \lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n, \end{aligned}$$

converges strongly to the unique solution of the equation $Ax = z$.

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1 Introduction

Let E be a real Banach space and let E^* be its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{u \in E^* : \langle x, u\rangle = \|x\| \|u\|, \|u\| = \|x\|\},$$

where $\langle ., .\rangle$ denotes the generalized duality pairing.

A mapping A with domain $D(A)$ and range $R(A)$ in E is said to be strongly accretive if there exist a constant $k \in (0, 1)$ such that for all $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq k \|x - y\|^2,$$

and is called ϕ-strongly accretive if there is a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for any $x, y \in D(A)$ there exist $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \phi(\|x - y\|) \|x - y\|.$$

The mapping A is called generalized Φ-accretive if there exist a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for all $x, y \in D(A)$ there exist $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \Phi(\|x - y\|).$$

It is well known that the class of generalized Φ-accretive mappings includes the class of ϕ-strongly accretive operators as a special case (one set $\Phi(s) = s\phi(s)$ for all $s \in [0,\infty)$).

Let $N(A) := \{x \in D(A) : Ax = 0\} \neq \emptyset$.

The mapping A is called strongly quasi-accretive if there exist $k \in (0, 1)$ such that for all $x \in D(A), p \in N(A)$ there exist $j(x - p) \in J(x - p)$ such that

$$\langle Ax - Ap, j(x - p)\rangle \geq k \|x - p\|^2.$$

A is called ϕ-strongly quasi-accretive if there exist a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for all $x \in D(A)$, $p \in N(A)$ there exist $j(x-p) \in J(x-p)$ such that

$$\langle Ax - Ap, j(x-p)\rangle \geq \phi(\|x-p\|)\,\|x-p\|\,.$$

Finally, A is called generalized Φ-quasi-accretive if there exist a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for all $x \in D(A), p \in N(A)$ there exist $j(x-p) \in J(x-p)$ such that

$$\langle Ax - Ap, j(x-p)\rangle \geq \Phi(\|x-p\|). \tag{1}$$

A mapping $G : E \to E$ is called Lipschitz if there exists a constants $L > 0$ such that $\|Gx - Gy\| \leq L\,\|x-y\|$ for all $x, y \in D(G)$.

Closely related to the class of accretive-type mappings are those of pseudo-contractive types.

A mapping $T : E \to E$ is called strongly pseudo-contractive if and only if $I - T$ is strongly accretive, and is called strongly ϕ-pseudo-contractive if and only if $(I-T)$ is ϕ-strongly accretive. The mapping T is called generalized Φ-pseudo-contractive if and only if $(I-T)$ is generalized Φ-accretive.

In [5, page 9], Ciric et al. showed by taking an example that a generalized Φ-strongly quasi-accretive operator is not necessarily a ϕ-strongly quasi-accretive operator.

If $F(T) := \{x \in E : Tx = x\} \neq \emptyset$, the mapping T is called strongly hemi-contractive if and only if $(I-T)$ is strongly quasi-accretive; it is called ϕ-hemi-contractive if and only if $(I-T)$ is ϕ-strongly quasi-accretive; and T is called generalized Φ-hemi-contractive if and only if $(I-T)$ is generalized Φ-quasi-accretive.

The class of generalized Φ-hemi-contractive mappings is the most general (among those defined above) for which T has a unique fixed point. The relation between the zeros of accretive-type operators and the fixed points of pseudo-contractive-type mappings is well known [1,8,11].

The steepest descent approximation process for monotone operators was introduced independently by Vainberg [13] and Zarantonello [15]. Mann [9] introduced an iteration process which, under suitable conditions, converges to a zero in Hilbert space. The Mann iteration scheme was further developed by Ishikawa [6]. Recently, Ciric et al. [5], Zhou and Guo [16], Morales and Chidume [12], Chidume [3], Xu and Roach [14] and many others have studied the characteristic conditions for the convergence of the steepeast descent approximations.

Morales and Chidume proved the following theorem:

Theorem 1.. *Let X be a uniformly smooth Banach space and let $T : X \to X$ be a ϕ- strongly accretive operator, which is bounded and demicontinous. Let $z \in X$ and let x_0 be an arbitrary initial value in X for which $\liminf_{t\to\infty} \phi(t) > \|Tx_0\|$. Then the steepest descent approximation scheme*

$$x_{n+1} = x_n - (Tx_n - z),\ n = 0, 1, 2 \ldots,$$

converges strongly to the unique solution of the equation $Tx = z$ provided that the sequence $\{\alpha_n\}$ of positive real numbers satisfies the following:

$$\begin{aligned} &(i) \quad \{\alpha_n\} \quad \text{is bounded above by some fixed constant,} \\ &(ii) \quad \sum_{n=0}^{\infty} \alpha_n = +\infty, \\ &(iii) \quad \sum_{n=0}^{\infty} \alpha_n b(\alpha_n) < +\infty, \end{aligned}$$

where $b : [0, \infty) \to [0, \infty)$ is a nondecreasing continuous function.

In [5], Ciric et al. proved the following theorem:

Theorem 2.. *Let X be a uniformly smooth Banach space and let $T : X \to X$ be a bounded and demicontinous generalized strongly accretive operator. Let $z \in X$ and let x_0 be an arbitrary initial value in X for which $\|Tx_0\| <$*

$\sup\{\Phi(t)/t : t > 0\}$. *Then a steepest descent approximation scheme defined by*

$$\begin{aligned} x_{n+1} &= x_n - \alpha_n(Ty_n - z), \ n = 0, 1, 2\ldots, \\ y_n &= x_n - \beta_n(Tx_n - z), \ n = 0, 1, 2, \ldots, \end{aligned}$$

where the sequence $\{\alpha_n\}$ *of positive real numbers satisfies the following conditions:*

(i) $\alpha_n \le \lambda$, *where* λ *is some fixed constant,*

(ii) $\sum_{n=0}^{\infty} \alpha_n = +\infty$,

(iii) $\alpha_n \to 0$ *as* $n \to \infty$, *converges strongly to the unique solution of the equation* $Tx = z$.

The purpose of this paper is to continue a study of sufficient conditions for the convergence of the steepest descent approximation process to the zero of a generalized strongly accretive operator. We also extend and improve the results which include the steepest descend method considered by Ciric et al. [5], Morales and Chidume [12], Chidume [3] and Xu and Roach [14] for a bounded ϕ-strongly quasi-accretive operator and also the generalized steepest descend method considered by Zhou and Guo [16] for a bounded ϕ-strongly quasi-accretive operator.

2 Main results

The following lemmas are now well known.

Lemma 1. *[2] Let* $J : E \to 2^E$ *be the normalized duality mapping. Then for any* $x, y \in E$, *we have*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \text{for all} \quad j(x + y) \in J(x + y).$$

Suppose there exist a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ *with* $\Phi(0) = 0$.

Lemma 2. *[10] Let $\Phi : [0,\infty) \to [0,\infty)$ be a strictly increasing function with $\Phi(0) = 0$ and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be nonnegative real sequences such that*

$$\lim_{n\to\infty} b_n = 0, \quad c_n = o(b_n), \quad \sum_{n=0}^{\infty} b_n = \infty.$$

Suppose that for all $n \geq 0$,

$$a_{n+1}^2 \leq a_n^2 - \Phi(a_{n+1})b_n + c_n,$$

then $\lim\limits_{n\to\infty} a_n = 0$.

Theorem 3.. *Let E be a real Banach space and let $A : E \to E$ be a Lipschitzian generalized strongly accretive operator. Let $z \in E$ and x_0 be an arbitrary initial value in E for which the steepest descent approximation scheme is defined by*

$$\begin{aligned} x_{n+1} &= x_n - \alpha_n(Ay_n - z), \\ y_n &= x_n - \beta_n(Ax_n - z), \quad n = 0,1,2\ldots, \end{aligned} \tag{2}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

$$\begin{aligned} &(i) \quad 0 \leq \alpha_n, \beta_n \leq 1, \\ &(ii) \quad \sum_{n=0}^{\infty} \alpha_n = +\infty, \\ &(iii) \quad \lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n, \end{aligned}$$

converges strongly to the unique solution of the equation $Ax = z$.

Proof. Following the tehnique of Chidume and Chidume [4], without loss of generality we may assume that $z = 0$. Define by p the unique zero of A.

By $\lim\limits_{n\to\infty} \alpha_n = 0 = \lim\limits_{n\to\infty} \beta_n$, imply there exist $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\alpha_n \leq \delta$ and $\beta_n \leq \delta'$;

$$\begin{aligned} 0 < \delta &= \min\left\{\frac{1}{3L}, \frac{\Phi(2\Phi^{-1}(a_0))}{36L^2\left[\Phi^{-1}(a_0)\right]^2}\right\}, \\ 0 < \delta' &= \min\left\{\frac{1}{2L}, \frac{\Phi(2\Phi^{-1}(a_0))}{24L^2\left[\Phi^{-1}(a_0)\right]^2}\right\}. \end{aligned}$$

Define $a_0 := \|Ax_{n_0}\| \, \|x_{n_0} - p\|$. Then from (1), we obtain that $\|x_{n_0} - p\| \leq \Phi^{-1}(a_0)$.

By induction, we shall prove that $\|x_n - p\| \leq 2\Phi^{-1}(a_0)$ for all $n \geq n_0$. Clearly, the inequality holds for $n = n_0$. Suppose it holds for some $n \geq n_0$, i.e., $\|x_n - p\| \leq 2\Phi^{-1}(a_0)$. We prove that $\|x_{n+1} - p\| \leq 2\Phi^{-1}(a_0)$. Suppose that this is not true. Then $\|x_{n+1} - p\| > 2\Phi^{-1}(a_0)$, so that $\Phi(\|x_{n+1} - p\|) > \Phi(2\Phi^{-1}(a_0))$. Using the recursion formula (2), we have the following estimates

$$\begin{aligned}
\|Ax_n\| &= \|Ax_n - Ap\| \leq L \|x_n - p\| \leq 2L\Phi^{-1}(a_0), \\
\|y_n - p\| &= \|x_n - p - \beta_n Ax_n\| \leq \|x_n - p\| + \beta_n \|Ax_n\| \\
&\leq 2\Phi^{-1}(a_0) + 2L\Phi^{-1}(a_0)\beta_n \leq 3\Phi^{-1}(a_0), \\
\|x_{n+1} - p\| &= \|x_n - p - \alpha_n Ay_n\| \leq \|x_n - p\| + \alpha_n \|Ay_n\| \\
&\leq \|x_n - p\| + L\alpha_n \|y_n - p\| \\
&\leq 2\Phi^{-1}(a_0) + 3L\Phi^{-1}(a_0)\alpha_n \leq 3\Phi^{-1}(a_0).
\end{aligned}$$

With these estimates and again using the recursion formula (2), we obtain by Lemma 1 that

$$\begin{aligned}
(3) \qquad \|x_{n+1} - p\|^2 &= \|x_n - p - \alpha_n Ay_n\|^2 \\
&\leq \|x_n - p\|^2 - 2\alpha_n \langle Ay_n, j(x_{n+1} - p)\rangle \\
&= \|x_n - p\|^2 - 2\alpha_n \langle Ax_{n+1}, j(x_{n+1} - p)\rangle \\
&+ 2\alpha_n \langle Ax_{n+1} - Ay_n, j(x_{n+1} - p)\rangle \\
&\leq \|x_n - p\|^2 - 2\alpha_n \Phi(\|x_{n+1} - p\|) \\
&+ 2\alpha_n \|Ax_{n+1} - Ay_n\| \, \|x_{n+1} - p\| \\
&\leq \|x_n - p\|^2 - 2\alpha_n \Phi(\|x_{n+1} - p\|) \\
&+ 2\alpha_n L \|x_{n+1} - y_n\| \, \|x_{n+1} - p\|,
\end{aligned}$$

where

$$\begin{aligned}
\|x_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \|x_n - y_n\| = \alpha_n \|Ay_n\| + \beta_n \|Ax_n\| \\
&\leq L\alpha_n \|y_n - p\| + L\beta_n \|x_n - p\| \leq L\Phi^{-1}(a_0)(3\alpha_n + 2\beta_n),
\end{aligned}$$

and consequently from (3), we get

$$\begin{aligned}
(4) \quad \|x_{n+1}-p\|^2 &\leq \|x_n-p\|^2 - 2\alpha_n\Phi(\|x_{n+1}-p\|) \\
&+ 2L^2\Phi^{-1}(a_0)(3\alpha_n^2+2\alpha_n\beta_n)\|x_{n+1}-p\| \\
&\leq \|x_n-p\|^2 - 2\alpha_n\Phi(2\Phi^{-1}(a_0)) \\
&+ 6L^2\left[\Phi^{-1}(a_0)\right]^2\left(3\alpha_n^2+2\alpha_n\beta_n\right) \\
&\leq \|x_n-p\|^2 - 2\alpha_n\Phi(2\Phi^{-1}(a_0)) + \alpha_n\Phi(2\Phi^{-1}(a_0)) \\
&= \|x_n-p\|^2 - \alpha_n\Phi(2\Phi^{-1}(a_0)).
\end{aligned}$$

Thus

$$\alpha_n\Phi(2\Phi^{-1}(a_0)) \leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2,$$

implies

$$\Phi(2\Phi^{-1}(a_0))\sum_{n=n_0}^{j}\alpha_n \leq \sum_{n=n_0}^{j}(\|x_n-p\|^2-\|x_{n+1}-p\|^2) = \|x_{n_0}-p\|^2,$$

so that as $j\to\infty$ we have

$$\Phi(2\Phi^{-1}(a_0))\sum_{n=n_0}^{\infty}\alpha_n \leq \|x_{n_0}-p\|^2 < \infty,$$

which implies that $\sum_{n=0}^{\infty}\alpha_n < \infty$, a contradiction. Hence, $\|x_{n+1}-p\| \leq 2\Phi^{-1}(a_0)$; thus $\{x_n\}$ is bounded. Consider

$$\begin{aligned}
\|y_n-x_n\| &= \|x_n-\beta_nAx_n-x_n\| = \beta_n\|Ax_n\| \leq L\beta_n\|x_n-p\| \\
&\leq 2L\Phi^{-1}(a_0)\beta_n \to 0 \text{ as } n\to\infty,
\end{aligned}$$

implies the sequence $\{y_n-x_n\}$ is bounded. Since $\|y_n-p\| \leq \|y_n-x_n\| + \|x_n-p\|$, further implies the sequence $\{y_n\}$ is bounded.

Now from (4), we get

$$\begin{aligned}
(5) \quad \|x_{n+1}-p\|^2 &\leq \|x_n-p\|^2 - 2\alpha_n\Phi(\|x_{n+1}-p\|) \\
&+ 4L^2\left[\Phi^{-1}(a_0)\right]^2(3\alpha_n^2+2\alpha_n\beta_n).
\end{aligned}$$

Denote

$$a_n = \|x_n - p\|,$$
$$b_n = 2\alpha_n,$$
$$c_n = 4L^2\left[\Phi^{-1}(a_0)\right]^2(3\alpha_n^2 + 2\alpha_n\beta_n).$$

Condition $\lim_{n\to\infty}\alpha_n = 0$ ensures the existence of a rank $n_0 \in \mathbb{N}$ such that $b_n = 2\alpha_n \leq 1$, for all $n \geq n_0$. Now with the help of $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0 = \lim_{n\to\infty}\beta_n$ and Lemma 2, we obtain from (5) that

$$\lim_{n\to\infty}\|x_n - p\| = 0,$$

completing the proof.

Theorem 4.. *Let E be a real Banach space and let $A : E \to E$ be a Lipschitzian generalized strongly quasi-accretive operator such that $N(A) \neq \emptyset$. Let $z \in E$ and x_0 be an arbitrary initial value in E for which the steepest descent approximation scheme is defined by*

$$\begin{aligned} x_{n+1} &= x_n - \alpha_n(Ay_n - z), \\ y_n &= x_n - \beta_n(Ax_n - z), \quad n = 0, 1, 2\ldots, \end{aligned}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

$$\begin{aligned} &(i) \quad 0 \leq \alpha_n, \beta_n \leq 1, \\ &(ii) \quad \sum_{n=0}^{\infty}\alpha_n = +\infty, \\ &(iii) \quad \lim_{n\to\infty}\alpha_n = 0 = \lim_{n\to\infty}\beta_n, \end{aligned}$$

converges strongly to the unique solution of the equation $Ax = z$.

Remark 1. *One can easily see that if we take $\alpha_n = \frac{1}{n^\sigma}; 0 < \sigma < \frac{1}{2}$, then $\sum\alpha_n = \infty$, but also $\sum\alpha^2 \not< \infty$. Hence the results of Chidume and Chidume in [4] are not true in general and consequently the results presented in this manuscript are independent of interest.*

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