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Existence of Solutions and Star-shapedness in Generalized Minty Variational Inequalities in Banach Spaces[1]

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Abstract

The purpose of this paper is to introduce and study Generalized Minty Variational Inequalities in Banach spaces. We consider a problem of vector variational inequalities, referred as Generalized Minty VI(f'_-, K), in a real Banach space X, where K is a nonempty subset of X and f'_- is the lower Dini directional derivative of a real function f defined on an open set in X containing K. The results presented in this paper generalize the corresponding results of Giovanni P. Crespi, Ivan Ginchev and Matteo Rocca [Giovanni P. Crespi, Ivan Ginchev and Matteo Rocca, Existence of solutions and star-shapedness in Minty Variational Inequalities, Journal of Global Optimization (2005), 32, 485-494].

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1 Introduction

Variational Inequalities provide a very general and suitable mathematical model for a wide range of problems, in particular equilibrium problems ([1,5,8]). According to [6], Minty Variational Inequalities is the problem of finding a vector $x^* \in K$, such that:

$$\text{Minty } VI(F, K) \;\; < F(y), x^* - y > \leq 0, \quad \forall y \in K$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $K \subseteq \mathbb{R}^n$ is nonempty and $< ., . >$ denotes the inner product defined on $\mathbb{R}^n$. In particular the case where the function F has a primitive $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, defined (and differentiable) on an open set containing K (i.e. the problem Minty $VI(f', K)$) has been widely studied, mainly in relation with the minimization of the function f over the set K (see e.g. [5]). In [3] a vector extension of Minty $VI(f', K)$ is introduced and related to optimality.

Let X be a Banach space, X^* be the dual space of X, $< ., . >$ denote the duality pairing of X^* and X. Generalized Minty Variational Inequality (for short, Generalized Minty VI) is a problem of finding a vector $x^* \in K$, such that:

$$\text{Find } x_0 \in K, \text{ such that} < F(y), \; x^* - y > \leq 0, \; \forall y \in K$$

where $F : X \longrightarrow X^*$ is a mapping and $K \subseteq X$ is nonempty.
Throughout the paper f denotes a real function defined on an open set containing K. For such a function, the Dini directional derivative of f at the point $x \in K$ in the direction $u \in X$ is defined as an element of $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ by:

$$f'_-(x, u) = \liminf_{t \longrightarrow +\infty} \frac{f(x + tu) - f(x)}{t}.$$

Now we introduce the following problem:

$$\text{Generalized Minty } VI(f'_-, K) \; f'_-(y, x^* - y) \leq 0, \; \forall y \in K.$$

The problem is to find $x^* \in K$ for which the inequalities in Generalized Minty $VI(f'_-, K)$ are satisfied.

The main result of the paper is that, when K is star-shaped, $x^* \in \ker K$ and f is radially lower semicontinuous in K on the rays starting at x^*, the point x^* is a solution of Generalized Minty $VI(f'_-, K)$ if and only if f is increasing along such rays (for short, IAR). This condition means that the level sets of f are star-shaped and can be regarded as a convexity-type condition (recall for comparison that, by definition, a function is quasi-convex if and only if its level sets are convex). Therefore we see that IAR functions naturally arise when dealing with Generalized Minty Variational Inequalities.

Moreover, we show that the class of IAR functions has relevant properties with regard to optimization problems and as it happens for convex functions, relations with well-posedness can be established. The latter allows to argue that, when Generalized Minty $VI(f'_-, K)$ has a solution (or more generally Generalized Minty $VI(f'_-, K)$ is solvable), the primitive optimization problem has some well-posedness property.

The structure of this paper is as follows. In section 2, we give some preliminary results on IAR functions. In section 3, we study Generalized Minty Variational Inequalities and we prove that a radially lower semicontinuous function f belongs to the class $IAR(K, x^*)$ if and only if x^* solves Generalized Minty $VI(f'_-, K)$. In section 4, we investigate well-posedness properties of convex functions.

2 Preliminaries

In this section, we recall the notion of IAR function and we investigate some basic properties of this class of functions. Such properties can be viewed as extensions of analogous properties holding for convex functions.

Definition 2.1. *(i) Let K be a nonempty subset of X. The set* $\ker K$ *consisting of all $x \in K$ such that ($y \in K$, $t \in [0,1]$)$\Longrightarrow x + t(y-x) \in K$ is called the* kernel *of K.*
(ii) A nonempty set K is star-shaped if $\ker K \neq \emptyset$.

In the following we will use the abbreviation st-sh for the word star-shaped. It is known (see e.g. [7]) that the set $\ker K$ is convex for an arbitrary st-sh set K. We will assume, by definition, that the empty set is st-sh.

Definition 2.2. *A function f defined on X is called increasing along rays at a point x^* (for short, $f \in IAR(x^*)$) if the restriction of this function on the ray $R_{x^*,x} = \{x^* + \alpha x : \alpha \geq 0\}$ is increasing for each $x \in X$. (A function g of one variable is called increasing if $t_2 \geq t_1$ implies that $g(t_2) \geq g(t_1)$.*

Definition 2.3. *Let $K \subseteq X$ be a st-sh set and $x^* \in \ker K$. A function f defined on K is called increasing along rays at x^* (for short, $f \in IAR(K, x^*)$), if the restriction of this function on the intersection $R_{x^*,x} \cap K$ is increasing, for each $x \in K$.*

We consider the following problem:

$$P(f,K) \qquad \min f(x), \qquad x \in K \subseteq X.$$

A point $x^* \in K$ is a (global) solution of $P(f,K)$ when $f(x) - f(x^*) \geq 0,\ \forall x \in K$. The solution is strict if $f(x) - f(x^*) > 0,\ \forall x \in K \setminus \{0\}$. We will denote by $\operatorname{argmin}(f,K)$ the set of solutions of $P(f,K)$. Local solutions of $P(f,K)$ have a clear definition and we omit it.
The next results give some basic properties of functions which are increasing along rays.

Proposition 2.1. *$K \subseteq X$ be a st-sh set, $x^* \in \ker K$ and $f \in IAR(K, x^*)$. Then:*
(i) x^* is a solution of $P(f,K)$;

(ii) No point $x \in K,\ x \neq x^*$, can be a strict local solution of P(f,K);
(iii) $x^* \in \text{kerargmin}(f, K)$

Proof. Let (i) $x \in K$ and set $z(t) = x^* + t(x - x^*),\ t \in [0, 1]$. Since $x^* \in \text{ker}K$, then $z(t) \in \text{ker}K$, then $z(t) \in K,\ \forall t \in [0, 1]$ and since $f \in IAR(K, x^*)$, we have $f(z(t)) \geq f(x^*) = f(z(t)),\ \forall t \in [0, 1]$ and in particular $f(z(1)) = f(x) \geq f(x^*)$. Since $x \in K$ is arbitrary, then x^* is a global minimizer of f over K.
(ii) Let x and $z(t)$ be as above. Since $f \in IAR(K, x^*)$, it easily follows $f(z(t)) \leq f(x) = f(z(1)),\ \forall t \in [0, 1]$. If U is an arbitrary neighborhood of x, then for t 'near enough' to 1, we have $z(t) \in U$ and so x cannot be a strict local minimizer for f over K.
(iii) Let $x \in \text{argmin}(f, K),\ x \neq x^*$. Since $z(t) \in K$, we have $f(z(t)) \leq f(x),\ \forall t \in [0, 1]$ and readily follows that for every $t \in [0, 1],\ z(t) \in \text{argmin}(f, K)$.

3 Generalized Minty Variational Inequalities and IAR Functions

In this section, we prove that a radially lower semicontinuous function f belongs to the class $IAR(K, x^*)$ if and only if x^* solves Generalized Minty $VI(f'_-, K)$.

Definition 3.1. *Let $K \subseteq X$, $x^* \in \text{ker}K$ and let f be a function defined on an open set containing K. The function f is said to be radially lower semicontinuous in K along rays starting at x^*, if for each $x \in K$, the restriction of this function on the intersection $R_{x^*,x} \cap K$ is lower semicontinuous*

We will use the abbreviation $f \in RLSC(K, x^*)$ to denote that f satisfies the previous definition.

Theorem 3.1. *(Mean value theorem).Let $x^* \in \text{ker}K,\ f \in RLSC(K, x^*),\ y \in K$, and $t > 0$ such that $y + t(x^* - y) \in K$. Then there exists a number*

$\alpha \in (0, t]$, *such that:*

$$f(y + t(x^* - y)) - f(y) \leq t f'_-(y + \alpha(x^* - y), x^* - y).$$

Proof. Let $h(s) = f(y + t(x^* - y)) - \frac{s}{t}[f(y + t(x^* - y)) - f(y)]$. Then the mean value inequality is equivalent to the existence of a number $\alpha \in (0, t]$ such that:

$$h_-(\alpha) = \liminf_{r \longrightarrow +0} \frac{h(\alpha + r) - h(\alpha)}{r}.$$

Clearly we can write $y + s(x^* - y) = x^* + (1 - s)(y - x^*)$ and hence h is lower semicontinuous. According to Weierstrass Theorem, it attains its global minimum at some point $\hat{t} \in [0, 1]$. Indeed, we have $h(0) = h(t) = f(y)$ and therefore if the global minimum is achieved for $\hat{t} = 0$ it is also achieved for $\hat{t} = t$. Hence, for $\alpha = \hat{t}$ we have $h_-(\alpha) \geq 0$ and the Theorem is proved.

Theorem 3.2. *Let $K \subseteq X$ be a star-shaped set and $x^* \in \ker K$.*
(i) If x^* solves Generalized Minty $VI(f'_-, K)$ and $f \in RLSC(K, x^*)$, then $f \in IAR(K, x^*)$.
(ii) Conversely, if $f \in IAR(K, x^*)$, then x^* is a solution of $VI(f'_-, K)$.

Proof. (i) Let x^* be a solution of Generalized Minty $VI(f'_-, K)$, $y \in K$ and $y + t_2(x^* - y),\ y + t_1(x^* - y)$ be points in $R_{x^*,x} \cap K$, with $t_2 \geq t_1 \geq 0$. Applying Theorem 3.1 we have

$$f(y + t_2(x^* - y)) - f(y + t_1(x^* - y)) \leq (t_2 - t_1) f'_-(y + \alpha(x^* - y), x^* - y) \leq 0,$$

with $\alpha \in]t_1, t_2]$. It is easily seen that this proves $f \in IAR(K, x^*)$.
(ii) Assume that $f \in IAR(K, x^*)$ and let $y \in K$. For every $t \in [0, 1]$, we have

$$f(y + t(x^* - y)) = f(x^* + (1 - t)(y - x^*)).$$

This implies that

$$\frac{f(y + t(x^* - y)) - f(y)}{t} \leq 0.$$

Taking lim inf as $t \longrightarrow 0$, we obtain that x^* solves Generalized Minty $VI(f'_-, K)$.

In the previous Theorem the assumption $f \in RLSC(K, x^*)$ appears in only of the two opposite implications. A natural question arises, whether it cannot be dropped at all (see, [4, examples 2, 3, p.489]).

Corollary 3.1. *Let $x^* \in \ker K$ and let $f \in RLSC(K, x^*)$. If x^* solves Generalized Minty $VI(f'_-, K)$, then x^* solves Generalized Minty $P(f, K)$.*

Proof. It is immediate from Theorem 3.2 and Proposition 2.1.

Remark 3.1. *The previous Corollary extends a classical result which states that if K is a convex set, any solution of Generalized Minty $VI(f', K)$, then x^* solves Generalized Minty $P(f, K)$.*

4 Generalized Minty Variational Inequalities and Well-posedness

In this section, we show that any function $f \in IAR(K, x^*)$ enjoys some well-posedness properties, analogously to convex functions.

Definition 4.1. *(i) A sequence $x^k \in K$ is a minimizing sequence for $P(f, K)$, when*

$$\lim_{k \longrightarrow \infty} f(x^k) = \inf_{x \in K} f(x).$$

(ii) A sequence $x^k \in K$ is a generalized minimizing sequence for $P(f, K)$, when

$$\lim_{k \longrightarrow \infty} f(x^k) = \inf_{x \in K} f(x), \qquad \lim_{k \longrightarrow \infty} \operatorname{dist}(x^k, K) = 0$$

(here $\operatorname{dist}(x, K)$ denotes the distance from the point x to the set K.)

Definition 4.2. *(i) Problem $P(f, K)$ is Tykhonov well-posed when it admits a unique solution x^* and every minimizing sequence for $P(f, K)$ converges to x^*.*
(ii) Problem $P(f, K)$ is Levitin-Polyak well-posed when it admits a unique solution x^* and every generalized minimizing sequence for $P(f, K)$ converges to x^*.

Let us denote with $\operatorname{argmin}(f, K)$ the set of solutions of $P(f, K)$ and consider the sets

$$L^f(\epsilon) := \{x \in K : f(x) \leq \inf_{y \in K} f(y) + \epsilon\}$$

and

$$L_s^f(\epsilon) := \{x \in K : \operatorname{dist}(x, K) \leq \epsilon,\ f(x) \leq \inf_{y \in K} f(y) + \epsilon\}.$$

Definition 4.3. *Problem $P(f, K)$ is said Tykhonov well-posed in the generalized sense when* $\operatorname{argmin}(f, K) \neq \emptyset$ *and every minimizing sequence for $P(f, K)$ has some subsequence that converges to an element of* $\operatorname{argmin}(f, K)$.

Of course, $P(f, K)$ is Tykhonov well-posed if and only if $\operatorname{argmin}(f, K)$ is a singleton and $P(f, K)$ is well-posed in the generalized sense.

Definition 4.4. *Problem $P(f, K)$ is stable when* $\operatorname{argmin}(f, K) \neq \emptyset$ *and for every sequence x^k minimizing for $P(f, K)$ we have*

$$\lim_{k \longrightarrow \infty} \operatorname{dist}(x^k, \operatorname{argmin}(f, K)) = 0.$$

The following results extends to IAR functions some classical well-posedness properties of convex functions.

Theorem 4.1. *Let K be a closed subset of X, $x^* \in \ker K$ and let $f \in IAR(K, x^*)$ be a lower semicontinuous function. If* $\operatorname{argmin}(f, K)$ *is bounded, then $P(f, K)$ is stable.*

Proof. Let $x^k \in K$ be a minimizing sequence for $P(f, K)$, but, by contradiction, assume that $\lim_{k \longrightarrow \infty} \operatorname{dist}(x^k, \operatorname{argmin}(f, K)) \neq 0$. Then, for infinitely many k we have

$$x^k \notin \operatorname{argmin}(f, K) + \delta B,$$

for some positive δ (here B denotes the open ball in X). Without loss of generality, we can assume that this holds for every k. If x^k is a bounded

sequence, one can think that x^k converges to a point $\bar{x} \notin \text{argmin}(f, K)$, but this is absurd.
We shall therefore assume x^k is unbounded. If this holds, for every k there exists $t_k \in (0, 1)$ such that $y^k = t_k x^* + (1 - t_k)x^k \in \text{bd}[\text{argmin}(f, K) + \delta B]$ (here bd A denotes the boundary of the set A). Since $\text{argmin}(f, K)$ is bounded and K is closed, one can think that $y^k \longrightarrow \bar{y} \in K$ with $\bar{y} \notin \text{argmin}(f, K)$. Hence $\forall \epsilon > 0$ and for k 'large enough', since $f \in IAR(K, x^*)$, we get

$$f(x^*) \leq f(y^k) \leq f(x^k) \leq \inf_{x \in K} f(x) + \epsilon = f(x^*) + \epsilon$$

and the lower semicontinuity of f gives the contradiction $f(x^*) = f(\bar{y})$.

Corollary 4.1. *Let K be a closed convex subset of X, $x^* \in$*ker*K, $f \in IAR(K, x^*)$ be lower semicontinuous and* $\text{argmin}(f, K)$ *be compact. Then $P(f, K)$ is Tykhonov well-posed in the generalized sense.*

Proof. It is easily follows observing that when $\text{argmin}(f, K)$ is compact, then stability is equivalent to Tykhonov generalized well-posedness.

Corollary 4.2. *Let K be a closed convex subset of X, $x^* \in$*ker*K, $f \in IAR(K, x^*)$ be lower semicontinuous and* $\text{argmin}(f, K)$ *be singleton. Then $P(f, K)$ is Tykhonov well-posed.*

The assumption that $\text{argmin}(f, K)$ is bounded is essential to prove Theorem 4.1, as it is shown in [4, example 4, p. 492].

Lemma 4.1. *Let $x^* \in$*ker*K. Then* $\text{dist}(., K) \in IAR(x^*)$.

Proof. Without loss of generality we assume $x^* = 0$. Consider a point $x \in X$ and two positive scalars t_1, t_2, with $t_2 \geq t_1$ and set

$$\text{dist}(t_2 x, K) = \inf_{y \in K} \| t_2 x - y \| = l.$$

Consider a sequence $y^k \in K$, such that $\| t_k x - y^k \| \leq l + \frac{1}{k}$. Since $\frac{t_1}{t_2} y^k \in K$, we obtain

$$\text{dist}(t_1 x, K) \leq \frac{t_1}{t_2} \| t_2 x - y^k \| \leq \frac{t_1}{t_2}(l + \frac{1}{k}) \leq l + \frac{1}{k}$$

and for $k \longrightarrow \infty$ we get $\text{dist}(t_1 x, K) \leq l$ which completes the proof.

Theorem 4.2. *Assume that K is a closed set, $x^* \in kerK$, f is a lower semicontinuous function and there exists $\tau > 0$ such that $f \in IAR(K_\tau, x^*)$, where $K_\tau = K + \tau B$. If $P(f, K)$ is Tykhonov well-posed, then* diam $L_s^f(\epsilon) \longrightarrow 0$*, as $\epsilon \longrightarrow \infty$.*

Proof. Assume on the contrary that $\lim_{k \longrightarrow \infty}$ diam $L_s^f(\epsilon) \neq 0$. Hence there exists a positive number δ such that $\forall \epsilon > 0$ one can find a point $x(\epsilon)$ with $\text{dist}(x, K) \leq \epsilon$ and $f(x(\epsilon)) \leq f(x^*) + \epsilon$, but $x(\epsilon) \notin x^* + \delta B$. Let $\epsilon = \frac{1}{k}$, $x^k := x(\epsilon)$ and assume first that x^k is bounded. Hence we can assume that x^k converges to some $\bar{x}$. Since $\text{dist}(x^k, K) \leq \dfrac{1}{k}$ and K is closed, then $\bar{x} \in K$. Furthermore we have $f(x^k) \leq f(x^*) + \frac{1}{k}$ and recalling that f is lower semicontinuous and that x^* minimizes f over K, we get $f(\bar{x}) = f(x^*)$, which contradicts the assumption of Tykhonov well-posedness.
Let assume, therefore, x^k is unbounded. Hence for k 'large enough', $x^k \in K_\tau$ and we can find $\lambda > 0$ such that$x^k \notin x^* + \delta B$. Let now $y^k = t_k x^k + (1 - t_k)x^* \in \text{bd}(x^* + \delta B)$, for $t \in (0, 1)$. Since $x^* \in \text{ker}K$, then $\text{dist}(., K) \in IAR(x^*)$ and from

$$\text{dist}(y^k, K) \leq \text{dist}(x^k, K) \leq \frac{1}{k},$$

we get $\lim_{k \longrightarrow \infty} \text{dist}(y^k, K) = 0$. Since $f \in IAR(K_\tau, x^*)$, for k 'large enough' we have

$$f(x^*) \leq f(y^k) \leq f(x^k) \leq f(x^*) + \frac{1}{k}$$

and hence y^k is a generalized minimizing sequence. Now the well-posedness is contradicted since we can assume $y^k \longrightarrow \bar{y} \in K$, $\bar{y} \neq x^*$ and the lower semicontinuity of f implies $f(\bar{x}) = f(x^*)$, which completes the proof.

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