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On Continuous Maps in Closure Spaces[1]

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Abstract

The aim of this paper is to study some properties of continuous maps in closure spaces.

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1 Introduction

A map $u : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied :

(N1) $u\emptyset = \emptyset$,

(N2) $A \subseteq uA$ for every $A \subseteq X$,

(N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

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A closure operator u on a set X is called *additive* (respectively,*idempotent*) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement is closed. The empty set and the whole space are both open and closed.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too.

2 Continuous Maps

Definition 2.1. Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

Proposition 2.2. *Let (X, u) and (Y, v) be closure spaces. If $f : (X, u) \to (Y, v)$ is continuous, then $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$.*

Proof. Let $B \subseteq Y$. Then $f^{-1}(B) \subseteq X$. Since f is continuous, we have $f(uf^{-1}(B)) \subseteq vf(f^{-1}(B)) \subseteq vB$. Therefore, $f^{-1}(f(uf^{-1}(B))) \subseteq f^{-1}(vB)$. Hence, $uf^{-1}(B) \subseteq f^{-1}(vB)$.

Clearly, if $f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

The following statement is evident:

Proposition 2.3. *Let (X, u) and (Y, v) be closure spaces. If $f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(G)$ is an open subset of (X, u) for every open subset G of (Y, v).*

Proposition 2.4. *Let (X, u), (Y, v) and (Z, w) be closure spaces. If $f : (X, u) \to (Y, v)$ and $g : (Y, v) \to (Z, w)$ are continuous, then $g \circ f : (X, u) \to ((Z, w)$ is continuous.*

Proof. Let $A \subseteq X$. Since $g \circ f(uA) = g(f(uA))$ and f is continuous, $g(f(uA)) \subseteq g(vf(A))$. As g is continuous, we get $g(vf(A)) \subseteq wg(f(A))$. Consequently, $g \circ f(uA) \subseteq wg \circ f(A)$. Hence, $g \circ f$ is continuous.

Proposition 2.5. *Let (X,u) and (Y,v) be closure spaces and let (A,u_A) be a closed subspace of (X,u). If $f:(X,u)\to(Y,v)$ is continuous, then $f\big|A:(A,u_A)\to(Y,v)$ is continuous.*

Proof. If $B\subseteq A$, then

$$\begin{aligned} f\big|A(u_AB) &= f\big|A(uB\cap A) \\ &= f\big|A(uB)=f(uB)\subseteq vf(B)=vf\big|A(B). \end{aligned}$$

Hence, $f\big|A$ is continuous.

Definition 2.6. Let (X,u) and (Y,v) be closure spaces. A map $f:(X,u)\to(Y,v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y,v) whenever F is a closed (resp. open) subset of (X,u).

Proposition 2.7. *A map $f:(X,u)\to(Y,v)$ is closed if and only if, for each subset B of Y and each open subset G of (X,u) containing $f^{-1}(B)$, there is an open subset U of (Y,v) such that $B\subseteq U$ and $f^{-1}(U)\subseteq G$.*

Proof. Suppose that f is closed. Let B be a subset of Y and G be an open subset of (X,u) such that $f^{-1}(B)\subseteq G$. Then $f(X-G)$ is a closed subset of (Y,v). Let $U=Y-f(X-G)$. Then U is an open subset of (Y,v) and $f^{-1}(U)=f^{-1}(Y-f(X-G))=X-f^{-1}(f(X-G))\subseteq X-(X-G)=G$. Therefore, U is an open subset of (Y,v) containing B such that $f^{-1}(U)\subseteq G$.

Conversely, suppose that F is a closed subset of (X,u). Then $f^{-1}(Y-f(F))\subseteq X-F$ and $X-F$ is an open subset of (X,u). By hypothesis, there is an open subset U of (Y,v) such that $Y-f(F)\subseteq U$ and $f^{-1}(U)\subseteq X-F$. Therefore, $F\subseteq X-f^{-1}(U)$. Consequently, $Y-U\subseteq f(F)\subseteq f(X-f^{-1}(U))\subseteq Y-U$, which implies that $f(F)=Y-U$. Thus, $f(F)$ is a closed subset of (Y,v). Hence, f is closed.

The following statement is obvious :

Proposition 2.8. *Let (X,u), (Y,v) and (Z,w) be closure spaces, let $f:(X,u)\to(Y,v)$ and $g:(Y,v)\to(Z,w)$ be maps. Then* *(i) If f and g are closed, then so is $g \circ f$.*

(ii) If $g \circ f$ is closed and f is continuous and surjection, then g is closed.

(iii) If $g \circ f$ is closed and g is continuous and injection, then f is closed.

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$, is the closure space $\big(\prod\limits_{\alpha \in I} X_\alpha, u\big)$ where $\prod\limits_{\alpha \in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections $\pi_\alpha : \prod\limits_{\alpha \in I}(X_\alpha, u) \to (X_\alpha, u)$, $\alpha \in I$, i.e., is defined by $uA = \prod\limits_{\alpha \in I} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod\limits_{\alpha \in I} X_\alpha$.

The following statement is evident :

Proposition 2.9. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then the projection map $\pi_\beta : \prod\limits_{\alpha \in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous.*

Proposition 2.10. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a closed subset of (X_β, u_β) if and only if $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$.*

Proof. Let $\beta \in I$ and let F be a closed subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(F) = F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$, hence $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ be a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$. Since π_β is closed, $\pi_\beta\Big(F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u_β).

Proposition 2.11. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is an open subset of (X_β, u_β) if and only if $G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is an open subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$.*

Proof. Let $\beta \in I$ and let G be an open subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(G)$ is an open subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(G) = G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$, therefore $G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is an open subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$.

Conversely, let $G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}}(X_\alpha, u_\alpha)$ be an open subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$. Then $\prod\limits_{\alpha \in I} X_\alpha - G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$. But $\prod\limits_{\alpha \in I} X_\alpha - G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha = (X_\beta - G) \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$, hence $(X_\beta - G) \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$. By Proposition 2.10, $X_\beta - G$ is a closed subset of (X_β, u_β). Consequently, G is an open subset of (X_β, u_β).

Proposition 2.12. *Let (X, u) be a closure space, $\{(Y_\alpha, v_\alpha) : \alpha \in I\}$ be a family of closure spaces and $f : (X, u) \to \prod\limits_{\alpha \in I}(Y_\alpha, v_\alpha)$ be a map. Then f is closed if and only if $\pi_\alpha \circ f$ is closed for each $\alpha \in I$.*

Proof. Let f be closed. Since π_α is closed for each $\alpha \in I$, also $\pi_\alpha \circ f$ is closed for each $\alpha \in I$.

Conversely, let $\pi_\alpha \circ f$ be closed for each $\alpha \in I$. Suppose that f is not closed. Then there exists a closed subset F of (X, u) such that $\prod\limits_{\alpha \in I} v_\alpha \pi_\alpha(f(F)) \nsubseteq f(F)$. Therefore, there exists $\beta \in I$ such that $v_\beta \pi_\beta(f(F)) \nsubseteq \pi_\beta f(F)$. But $\pi_\beta \circ f$ is closed, hence $\pi_\beta(f(F))$ is a closed subset of (Y_β, v_β). This is a contradiction.

Proposition 2.13. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ and $\{(Y_\alpha, v_\alpha) : \alpha \in I\}$ be families of closure spaces. For each $\alpha \in I$, let $f_\alpha : (X_\alpha, u_\alpha) \to (Y_\alpha, v_\alpha)$ be a surjection and let $f : \prod\limits_{\alpha \in I}(X_\alpha, u_\alpha) \to \prod\limits_{\alpha \in I}(Y_\alpha, v_\alpha)$ be defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. Then f is closed if and only if f_α is closed for each $\alpha \in I$.*

Proof. Let $\beta \in I$ and let F be a closed subset of (X_β, u_β). Then $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$. Since f is closed, $f\Big(F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big)$ is a closed subset of $\prod_{\alpha\in I}(Y_\alpha, v_\alpha)$. But $f\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = f_\beta(F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$, hence $f_\beta(F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$ is a closed subset of $\prod_{\alpha\in I}(Y_\alpha, v_\alpha)$. By Proposition 2.10, $f_\beta(F)$ is a closed subset of (Y_β, v_β). Hence, f_β is closed.

Conversely, let f_β be closed for each $\beta \in I$. Suppose that f is not closed. Then there exists a closed subset F of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ such that $\prod_{\beta\in I} v_\beta\pi_\beta(f(F)) \nsubseteq f(F)$. Therefore, there exists $\beta \in I$ such that $v_\beta f_\beta(\pi_\beta(F)) \nsubseteq f_\beta(\pi_\beta(F))$. But $\pi_\beta(F)$ is a closed subset of (X_β, u_β) and f_β is closed, $f_\beta(\pi_\beta(F))$ is a closed subset of (Y_β, v_β). This is a contradiction.

Proposition 2.14. *Let (X, u) be a closure space, $\{(Y_\alpha, v_\alpha) : \alpha \in I\}$ be a family of closure spaces and $f : (X, u) \to \prod_{\alpha\in I}(Y_\alpha, v_\alpha)$ be a map. Then f is continuous if and only if $\pi_\alpha \circ f$ is continuous for each $\alpha \in I$.*

Proof. Let f be continuous. Since π_α is continuous for each $\alpha \in I$, $\pi_\alpha \circ f$ is continuous for each $\alpha \in I$.

Conversely, let $\pi_\alpha \circ f$ be continuous for each $\alpha \in I$. Suppose that f is not continuous. Then there exists a subset A of X such that $f(uA) \nsubseteq \prod_{\alpha\in I} v_\alpha\pi_\alpha(f(A))$. Therefore, there exists $\beta \in I$ such that $\pi_\beta(f(uA)) \nsubseteq v_\beta\pi_\beta(f(A))$. This is contradicts the continuity of $\pi_\beta \circ f$. Consequently, f is continuous.

Proposition 2.15. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ and $\{(Y_\alpha, v_\alpha) : \alpha \in I\}$ be families of closure spaces. For each $\alpha \in I$, let $f_\alpha : (X_\alpha, u_\alpha) \to (Y_\alpha, v_\alpha)$ be a map and let $f : \prod_{\alpha\in I}(X_\alpha, u_\alpha) \to \prod_{\alpha\in I}(Y_\alpha, v_\alpha)$ be defined by $f((x_\alpha)_{\alpha\in I}) = (f_\alpha(x_\alpha))_{\alpha\in I}$. Then f is continuous if and only if f_α is continuous for each $\alpha \in I$.*

Proof. Let f be continuous, let $\beta \in I$ and let $A \subseteq X_\beta$. Then

$$\begin{aligned}
f_\beta(u_\beta A) &= \pi_\beta\Big(f_\beta(u_\beta A) \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} f_\alpha(u_\alpha X_\alpha)\Big) \\
&= \pi_\beta\Big(f\Big(u_\beta A \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} u_\alpha X_\alpha\Big)\Big) \\
&= \pi_\beta\Big(f\Big(u\Big(A \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha\Big)\Big)\Big) \\
&\subseteq \pi_\beta\Big(vf\Big(A \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha\Big)\Big) \\
&= \pi_\beta\Big(v\Big(f_\beta(A) \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} f_\alpha(X_\alpha)\Big)\Big) \\
&= \pi_\beta\Big(v_\beta f_\beta(A) \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} v_\alpha f_\alpha(X_\alpha)\Big) \\
&= v_\beta f_\beta(A).
\end{aligned}$$

Hence, f_β is continuous.

Conversely, let f_α be continuous for each $\alpha \in I$ and let $A \subseteq \prod_{\alpha\in I} X_\alpha$. Then

$$\begin{aligned}
f(uA) &= \prod_{\alpha\in I} f_\alpha(\prod_{\alpha\in I} u_\alpha \pi_\alpha(A)) \\
&= \prod_{\alpha\in I} f_\alpha(u_\alpha \pi_\alpha(A)) \\
&\subseteq \prod_{\alpha\in I} v_\alpha f_\alpha(\pi_\alpha(A)) \\
&= \prod_{\alpha\in I} v_\alpha \pi_\alpha(f(A)) \\
&= vf(A).
\end{aligned}$$

Therefore f is continuous.

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