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New sufficient conditions for univalence [1]

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Abstract

The object of the present paper is to obtain new sufficient conditions on $f''(z)$ which lead to some subclasses of univalent functions defined in the open unit disk.

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1 Introduction

Let $\mathcal{A}_n$ denote the class of functions of the form :

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots , \tag{1}$$

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which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Let $\mathcal{A}_1 = \mathcal{A}$, then a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$, the class of starlike functions of order α, $0 \leq \alpha < 1$, if and only if

$$\text{(2)} \qquad \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}).$$

Then $\mathcal{S}^* = \mathcal{S}^*(0)$ is the class of starlike functions in $\mathcal{U}$. Further, if $f(z) \in \mathcal{A}$ satisfies

$$\text{(3)} \qquad \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \qquad (z \in \mathcal{U})$$

for some $0 < \alpha \leq 1$, then $f(z)$ said to be strongly starlike function of order α in $\mathcal{U}$, and this class denoted by $\overline{\mathcal{S}}^*(\alpha)$. Note that $\overline{\mathcal{S}}^*(1) = \mathcal{S}^*$.

Furthermore, let $\mathcal{K}(\alpha)$, $0 \leq \alpha < 1$, which consists of functions $f(z) \in \mathcal{A}$ such that

$$\text{(4)} \qquad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U}),$$

and $\mathcal{K} = \mathcal{K}(0)$ is the class of convex functions in $\mathcal{U}$. Also, if $f(z) \in \mathcal{A}$ satisfies

$$\text{(5)} \qquad \left|\arg \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2}, \qquad (z \in \mathcal{U})$$

for some $0 < \alpha \leq 1$, then $f(z)$ said to be strongly convex function of order α in $\mathcal{U}$, and this class denoted by $\overline{\mathcal{K}}(\alpha)$. Note that $\overline{\mathcal{K}}(1) = \mathcal{K}$ and if $zf'(z) \in \overline{\mathcal{S}}^*(\alpha)$ then $f(z) \in \overline{\mathcal{K}}(\alpha)$.

All the above mentioned classes are subclasses of univalent fictions in $\mathcal{U}$.

There are many results for sufficient conditions of functions $f(z)$ which are analytic in $\mathcal{U}$ to be starlike, convex, strongly starlike and strongly convex functions have been given by several researchers .(see [9],[1],[2],[7],[6],[5]). In this paper we will study λ such that the conditions$|f''(z)| \leq \lambda$, $z \in \mathcal{U}$, implies that $f(z)$ belongs to one of the classes defined above.

In order to prove our main results, we shall need the following lemmas.

Lemma 1 *([9]) If $f(z) \in \mathcal{A}$ satisfies*

$$|f'(z) - 1| < 2a\sqrt{\frac{5 - 4\sqrt{1-a^2}}{16a^2+9}} \qquad (z \in \mathcal{U}) \tag{6}$$

where $a = \sin(\alpha\pi/2)$, $0 < \alpha \leq 1$, then $f(z) \in \overline{\mathcal{S}}^*(\alpha)$.

Lemma 2 *([4])If $f(z) \in \mathcal{A}_n$, satisfies*

$$|f'(z) - 1| < \frac{(1-\alpha)(n+1)}{\alpha + \sqrt{(n+1)^2+1}}, \qquad (z \in \mathcal{U}) \tag{7}$$

where $0 \leq \alpha < 1$, then $f(z) \in \mathcal{S}^*(\alpha)$.

Lemma 3 *([8])If $f(z) \in \mathcal{A}_n$, satisfies*

$$|f'(z) + \alpha z f''(z) - 1| < \frac{(\alpha-2)(n\alpha+1)}{\alpha(n+1)} \qquad (z \in \mathcal{U}) \tag{8}$$

where $\alpha > 2$, then $f(z) \in \mathcal{K}$.

2 Main Results

Employing the same method used by Nunokawa *et al.*[6], we prove the following

Theorem 1 *If $f(z) \in \mathcal{A}$ satisfies*

$$|f''(z)| \le 2a\sqrt{\frac{5-4\sqrt{1-a^2}}{16a^2+9}} \qquad (z \in \mathcal{U};\ 0 < \alpha \le 1) \tag{9}$$

where $a = \sin(\alpha\pi/2)$, then $f(z) \in \overline{\mathcal{S}}^*(\alpha)$.

Proof. Noting that

$$\begin{aligned} |f'(z)-1| &= \left|\int_0^z f''(\sigma)d\sigma\right| \le \int_0^{|z|} \left|f''(te^{i\theta})\right| dt \le 2a\sqrt{\frac{5-4\sqrt{1-a^2}}{16a^2+9}} \int_0^{|z|} dt \\ &= 2a\sqrt{\frac{5-4\sqrt{1-a^2}}{16a^2+9}}\,|z| < 2a\sqrt{\frac{5-4\sqrt{1-a^2}}{16a^2+9}} \end{aligned}$$

Hence, by Lemma 1, we conclude that $f(z) \in \overline{\mathcal{S}}^*(\alpha)$.

Corollary 1 *Let $f(z) \in \mathcal{A}, z \in \mathcal{U}$ then*

(i) $|f''(z)| \le 2\sqrt{5}/5 = 0.8944\ldots$ *implies* $f(z) \in \mathcal{S}^*$;

(ii) $|f''(z)| \le \sqrt{(5-2\sqrt{3})/13} = 0.3437\ldots$*implies* $f(z) \in \overline{\mathcal{S}}^*(1/3)$;.

(iii) $|f''(z)| \le \sqrt{(10-4\sqrt{2})/17} = 0.5054\ldots$ *implies* $f(z) \in \overline{\mathcal{S}}^*(1/2)$; *and*

(iii) $|f''(z)| \le \sqrt{21}/7 = 0.6546\ldots$ *implies* $f(z) \in \overline{\mathcal{S}}^*(2/3)$.

Remark 1 *The result from Corollary 1 (i) was obtained by Nunokawa et al.[6].*

Now, we derive

Theorem 2 *If $f(z) \in \mathcal{A}$ satisfies*

$$|f''(z)| \leq a\sqrt{\frac{5-4\sqrt{1-a^2}}{16a^2+9}} \qquad (z \in \mathcal{U};\ 0 < \alpha \leq 1) \tag{10}$$

where $a = \sin(\alpha\pi/2)$, then $f(z) \in \overline{\mathcal{K}}(\alpha)$.

Proof. It follows that

$$\begin{aligned}
|(zf'(z))' - 1| &= |f'(z) + zf''(z) - 1| \leq |f'(z) - 1| + |zf''(z)| \\
&\leq \left|\int_0^z f''(t)dt\right| + |zf''(z)| \leq \int_0^{|z|} |f''(t)dt| + a\sqrt{\frac{5-4\sqrt{1-a^2}}{16a^2+9}}\,|z| \\
&\leq 2a\sqrt{\frac{5-4\sqrt{1-a^2}}{16a^2+9}}\,|z| < 2a\sqrt{\frac{5-4\sqrt{1-a^2}}{16a^2+9}}
\end{aligned}$$

Therefore, using Lemma 1, we see that $zf'(z) \in \overline{\mathcal{S}}^*(\alpha)$, or $f(z) \in \overline{\mathcal{K}}(\alpha)$.

Corollary 2 *Let $f(z) \in \mathcal{A}, z \in \mathcal{U}$ then*

(i) $|f''(z)| \leq \sqrt{5}/5 = 0.4472\ldots$ *implies* $f(z) \in \mathcal{K}$;

(ii) $|f''(z)| \leq \frac{1}{2}\sqrt{(5-2\sqrt{3})/13} = 0.1718\ldots$ *implies* $f(z) \in \overline{\mathcal{K}}(1/3)$;.

(iii) $|f''(z)| \leq \frac{1}{2}\sqrt{(10-4\sqrt{2})/17} = 0.2527\ldots$ *implies* $f(z) \in \overline{\mathcal{K}}(1/2)$;

and

(iii) $|f''(z)| \leq \frac{1}{2}\sqrt{21}/7 = 0.3273\ldots$ *implies* $f(z) \in \overline{\mathcal{K}}(2/3)$.

Remark 2 *The result from Corollary 2 (i) was obtained by Nunokawa et al.[6].*

Applying the same method as in the proof of Theorem 1 and using Lemmas 2 instead of Lemma 1, we obtain the following theorem

Theorem 3 *If* $f(z) \in \mathcal{A}_n$ *satisfies*

$$|f''(z)| \leq \frac{(1-\alpha)(n+1)}{\alpha + \sqrt{(n+1)^2+1}} \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1) \tag{11}$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

Letting $\alpha = 0$ in Theorem 3, we obtain

Corollary 3 *If* $f(z) \in \mathcal{A}_n$ *satisfies*

$$|f''(z)| \leq \frac{n+1}{\sqrt{(n+1)^2+1}} \qquad (z \in \mathcal{U}) \tag{12}$$

then $f(z) \in \mathcal{S}^*$.

Remark 3 *Letting* $n = 1$ *in Corollary 3 , we obtain the result (i) from Corollary 2.5 which was obtained by Nunokawa et al.[6].*

Finally, we prove

Theorem 4 *If* $f(z) \in \mathcal{A}_n$, *satisfies*

$$|f''(z)| < \frac{(\alpha-2)(n\alpha+1)}{\alpha(\alpha+1)(n+1)} \qquad (z \in \mathcal{U}) \tag{13}$$

where $\alpha > 2$, *then* $f(z) \in \mathcal{K}$.

Proof. It follows that

$$\begin{aligned} &|f'(z) + \alpha z f''(z) - 1| \leq |f'(z) - 1| + \alpha\, |zf''(z)| \\ &\leq \left| \int_0^z f''(t)dt \right| + \alpha\, |zf''(z)| \leq \int_0^{|z|} |f''(t)dt| + \frac{(\alpha-2)(n\alpha+1)}{(\alpha+1)(n+1)}\, |z| \\ &\leq \frac{(\alpha-2)(n\alpha+1)}{\alpha(n+1)}\, |z| < \frac{(\alpha-2)(n\alpha+1)}{\alpha(n+1)}, \end{aligned}$$

using Lemma 3, we have $f(z) \in \mathcal{K}$.

Letting $\alpha \to \infty$ in Theorem 4, we have the following result obtained by Mocanu[3].

Corollary 4 *If* $f(z) \in \mathcal{A}_n$, *satisfies*

$$|f''(z)| < \frac{n}{n+1} \qquad (z \in \mathcal{U}) \tag{14}$$

then $f(z) \in \mathcal{K}$.

Letting $n = 1$ in Corollary 4, we have the following result obtained by Obradovič[7].

Corollary 5 *If* $f(z) \in \mathcal{A}$, *satisfies*

$$|f''(z)| < \frac{1}{2} \qquad (z \in \mathcal{U}) \tag{15}$$

then $f(z) \in \mathcal{K}$.

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