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Certain Aspects of Some Arithmetic Functions in Number Theory [1]

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Abstract

The purpose of this paper is to present several inequalities about the arithmetic functions $\sigma^{(e)}$, $\tau^{(e)}$, $\sigma^{(e)*}$, $\tau^{(e)*}$ and other well-known arithmetic functions. Among these, we have the following:

$$\frac{\sqrt{\sigma_k^*(n)\cdot\sigma_l^*(n)}}{\sigma^*_{\frac{k-l}{2}}(n)} \leq \frac{n^{\frac{l-k}{4}}\cdot\sigma_k^*(n)+n^{\frac{k-l}{4}}\cdot\sigma_l^*(n)}{2\cdot\sigma_{\frac{k-l}{2}}(n)} \leq n^{\frac{l-k}{4}}\cdot\frac{n^{\frac{k+l}{2}}+1}{2},$$

for any $n,k,l\in\mathbb{N}^*$,

$$\frac{\sqrt{\sigma_k^{(e)*}(n)\cdot\tau^{(e)*}(n)}}{\sigma^{(e)*}_{\frac{k-l}{2}}(n)} \leq \frac{n^{\frac{l-k}{4}}\cdot\sigma_k^{(e)*}(n)+n^{\frac{k-l}{4}}\cdot\tau^{(e)*}(n)}{2\cdot\sigma^{(e)*}_{\frac{k-l}{2}}(n)} \leq$$

$\leq n^{\frac{l-k}{4}}\cdot\dfrac{n^{\frac{k+l}{2}}+1}{2}$, for any $n,k,l\in\mathbb{N}^*$, $\sigma_k^{(e)}(n)\cdot\sigma_l^{(e)}(n)\leq\tau^{(e)}(n)\cdot\sigma_{k+l}^{(e)}(n)$, for any $n,k,l\in\mathbb{N}^*$ and $\dfrac{\sigma_{k+1}^{(e)*}(n)}{\sigma_k^{(e)*}(n)}\geq\dfrac{\sigma^{(e)*}(n)}{\tau^{(e)*}(n)}\geq\tau(n)$, for any $n,k\in\mathbb{N}^*$, where $\tau(n)$ is the number of the natural divisors of n and $\sigma(n)$ is the sum of the divisors of n.

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1 Introduction

Let n be a positive integer, $n \geq 1$. We note with $\sigma_k(n)$ the sum of the kth powers of divisors of n, so, $\sigma_k(n) = \sum_{d|n} d^k$, whence we obtain the following equalities: $\sigma_1(n) = \sigma(n)$ and $\sigma_0(n) = \tau(n)$- the number of divisors of n (see [6]). If d is a unitary divisor of n, then we have $\left(d, \frac{n}{d}\right) = 1$. Let $\sigma_k^*(n)$ denote the sum of the kth powers of the unitary divisors of n. We note $d||n$.
Next we have to mention that the notion of "exponential divisor" was introduced M. V. Subbarao in [9].
Let $n > 1$ be an integer of canonical from $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r}$.
The integer $d = \prod_{i=1}^{r} p_i^{b_i}$ is called an *exponential divisor* (or e-*divisor*) of $n = \prod_{i=1}^{r} p_i^{a_i} > 1$, if $b_i | a_i$ for every $i = \overline{1, r}$. We note $d|_{(e)} n$. Let $\sigma^{(e)}(n)$ denote the sum of the exponential divisors of n and $\tau^{(e)}(n)$ denote the number of the exponential divisors of n. In [11] L. Tóth and N. Minculete introduced the notion of " exponential unitary divisors" or "e-unitary divisors". The integer $d = \prod_{i=1}^{r} p_i^{b_i}$ is called a *e-unitary divisor* of $n = \prod_{i=1}^{r} p_i^{a_i} > 1$ if b_i is a unitary divisor of a_i, so $\left(b_i, \frac{a_i}{b_i}\right) = 1$, for every $i = \overline{1, r}$. Let $\sigma^{(e)*}(n)$ denote the sum of e-unitary divisor of n, and $\tau^{(e)*}(n)$ denote the number of the e-unitary divisors of n. We note $d|_{(e)*}$. By convention, 1 is an e-unitary divisor of $n > 1$, the smallest e-unitary divisor of $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r} > 1$ is $p_1 p_2 ... p_r$, where $p_1 p_2 ... p_r = \gamma(n)$ is called the "core" of n.
Other aspects of these arithmetic function can be found in the papers [7] and [10].
In [6], J. Sándor shows that

$$(1) \qquad \frac{\sqrt{\sigma_k(n) \cdot \sigma_l(n)}}{\sigma_{\frac{k-l}{2}}(n)} \leq n^{\frac{-(k-l)}{4}} \cdot \frac{n^{\frac{k+l}{2}} + 1}{2}, \text{ for all } n, k, l \in \mathbb{N}^*.$$

In [8], J. Sándor and L. Tóth proved the inequalities

$$(2) \qquad \frac{n^k + 1}{2} \geq \frac{\sigma_k^*(n)}{\tau^*(n)} \geq \sqrt{n^k},$$

and

$$\frac{\sigma^*_{k+m}}{\sigma^*_m(n)} \geq \sqrt{n^k}, \tag{3}$$

for all $n \geq 1$ and $k, m \geq 0$, real numbers.
In [3, 4], we found the inequalities

$$\frac{\sqrt{\sigma_k(n) \cdot \sigma_l(n)}}{\sigma_{\frac{k-l}{2}}(n)} \leq \frac{n^{\frac{l-k}{4}}\sigma_k(n) + n^{\frac{k-l}{4}}\sigma_l(n)}{2\sigma_{\frac{k-l}{2}}(n)} \leq n^{\frac{-(k-l)}{4}} \cdot \frac{n^{\frac{k+l}{2}} + 1}{2}, \tag{4}$$

for every $n, k, l \in \mathbb{N}$ with $n \geq 1$ and $\dfrac{k-l}{2} \in \mathbb{N}$,

$$\frac{\sqrt{\sigma_{k+2}(n) \cdot \sigma_k(n)}}{\sigma(n)} \leq \frac{\frac{1}{\sqrt{n}}\sigma_{k+2}(n) + \sqrt{n}\sigma_k(n)}{2\sigma(n)} \leq \frac{1}{\sqrt{n}} \cdot \frac{n^{k+1} + 1}{2}, \tag{5}$$

for every $n, k \in \mathbb{N}$ and $n \geq 1$,

$$\frac{\sqrt{\sigma^{(e)}_k(n)\tau^{(e)}(n)}}{\sigma_{\frac{k-l}{2}}(n)} \leq \frac{n^{\frac{l-k}{4}}\sigma^{(e)}_k(n) + n^{\frac{k-l}{4}}\tau^{(e)}(n)}{2\sigma^{(e)}_{\frac{k-l}{2}}(n)} \leq n^{\frac{-(k-l)}{4}} \cdot \frac{n^{\frac{k+l}{2}} + 1}{2}, \tag{6}$$

for every $n, k, l \in \mathbb{N}$ with $n \geq 1$ and $\dfrac{k-l}{2} \in \mathbb{N}$,

$$\frac{\sqrt{\sigma^{(e)}_{k+2}(n) \cdot \tau^{(e)}(n)}}{\sigma^{(e)}(n)} \leq \frac{\frac{1}{\sqrt{n}}\sigma^{(e)}_{k+2}(n) + \sqrt{n}\tau^{(e)}(n)}{2\sigma^{(e)}(n)} \leq \frac{1}{\sqrt{n}} \cdot \frac{n^{k+1} + 1}{2}, \tag{7}$$

for every $n, k \in \mathbb{N}$ $n \geq 1$,

$$\frac{\sqrt{\sigma^{(e)}_k(n) \cdot \tau^{(e)}(n)}}{\tau^{(e)}(n)} \leq \frac{\sigma^{(e)}_k(n) + \tau^{(e)}(n)}{2\tau^{(e)}(n)} \leq \frac{n^k + 1}{2}, \tag{8}$$

and

$$\frac{\sigma^{(e)}_k(n)}{\tau^{(e)}} \leq \left(\frac{n^k + 1}{2}\right)^2, \tag{9}$$

for every $n, k \in \mathbb{N}$ and $n \geq 1$.

2 Main results

An inequality which is due to J.B. Diaz and F.T. Matcalf is proved in [2], namely:

Lemma 1 *Let n be a positive integer, $n \geq 2$. For every $a_1, a_2, ..., a_n \in \mathbb{R}$ and for every $b_1, b_2, ..., b_n \in \mathbb{R}^*$ with $m \leq \dfrac{a_i}{b_i} \leq M$ and $m, M \in \mathbb{R}$, we have the following inequality:*

$$\sum_{i=1}^{n} a_i^2 + mM \sum_{i=1}^{n} b_i^2 \leq (m+M) \sum_{i=1}^{n} a_i b_i. \tag{10}$$

Theorem 1 *For every $n, k, l \in \mathbb{N}$ with $n \geq 1$ and $\dfrac{k-l}{2} \in \mathbb{N}$, the following relation*

$$\frac{\sqrt{\sigma_k^*(n) \cdot \sigma_l^*(n)}}{\sigma_{\frac{k-l}{2}}^*(n)} \leq \frac{n^{\frac{l-k}{4}} \cdot \sigma_k^*(n) + n^{\frac{k-l}{4}} \cdot \sigma_l^*(n)}{2 \cdot \sigma_{\frac{k-l}{2}}^*(n)} \leq n^{\frac{l-k}{4}} \cdot \frac{n^{\frac{k+l}{2}} + 1}{2} \tag{11}$$

is true.

Proof. For $n = 1$, we have equality in relation (11). For $n \geq 2$, in the Lemma above, making the substitutions $a_i = \sqrt{d_i^k}$ and $b_i = \dfrac{1}{\sqrt{d_i^l}}$, where d_i is the unitary divisors of n, for all $i = \overline{1, \tau^*(n)}$. Since $1 \leq \dfrac{a_i}{b_i} = \sqrt{d_i^{k+l}} \leq n^{\frac{k+l}{2}}$ and $a_i b_i = d_i^{\frac{k-l}{2}}$, we take $m = 1$ and $M = n^{\frac{k+l}{2}}$. Therefore, inequality (10) becomes

$$\sum_{i=1}^{\tau^*(n)} d_i^k + n^{\frac{k+l}{2}} \cdot \sum_{i=1}^{\tau^*(n)} \frac{1}{d_i^l} \leq \left(1 + n^{\frac{k+l}{2}}\right) \sum_{i=1}^{\tau^*(n)} d_i^{\frac{k-l}{2}},$$

which is equivalent to

$$\sigma_k^*(n) + n^{\frac{k+l}{2}} \cdot \frac{\sigma_l^*(n)}{n^l} \leq \left(1 + n^{\frac{k+l}{2}}\right) \cdot \sigma_{\frac{k-l}{2}}^*(n),$$

so that

$$\sigma_k^*(n) + n^{\frac{k-l}{2}} \cdot \sigma_l^*(n) \leq \left(1 + n^{\frac{k+l}{2}}\right) \cdot \sigma_{\frac{k-l}{2}}^*(n), \tag{12}$$

for every $n, k, l \in \mathbb{N}$ with $n \geq 2$.

The arithmetical mean is greater than the geometrical mean or they are equal, so for every $n, k, l \in \mathbb{N}$ with $n \geq 2$, we have

$$\sqrt{n^{\frac{k-l}{2}} \cdot \sigma_k^*(n) \cdot \sigma_l^*(n)} \leq \frac{\sigma_k^*(n) + n^{\frac{k-l}{2}} \cdot \sigma_l^*(n)}{2}. \tag{13}$$

Consequently, from the relations (12) and (13) and taking into account that the relation "$\leq$" is transitive, we deduce the inequality

$$\frac{\sqrt{\sigma_k^*(n) \cdot \sigma_l^*(n)}}{\sigma_{\frac{k-l}{2}}^*(n)} \leq \frac{n^{\frac{l-k}{4}} \cdot \sigma_k^*(n) + n^{\frac{k-l}{4}} \cdot \sigma_l^*(n)}{2 \cdot \sigma_{\frac{k-l}{2}}^*(n)} \leq n^{\frac{l-k}{4}} \cdot \frac{n^{\frac{k+l}{2}} + 1}{2}.$$

Remark 1 *For $k = l$ in inequality (11), we obtain the relation of J. Sándor and L. Tóth, namely*

$$\frac{n^k + 1}{2} \geq \frac{\sigma_k^*(n)}{\tau^*(n)}, \tag{14}$$

for every $n, k \in \mathbb{N}$ with $n \geq 1$.

Theorem 2 *For every $n, k, l \in \mathbb{N}$ with $n \geq 1$ and $\frac{k-l}{2} \in \mathbb{N}$, the following relation*

$$\frac{\sqrt{\sigma_k^{(e)*}(n) \cdot \tau^{(e)*}(n)}}{\sigma_{\frac{k-l}{2}}^{(e)*}(n)} \leq \frac{n^{\frac{l-k}{4}} \cdot \sigma_k^{(e)*}(n) + n^{\frac{k-l}{4}} \cdot \tau^{(e)*}(n)}{2 \cdot \sigma_{\frac{k-l}{2}}^{(e)*}(n)} \leq n^{\frac{l-k}{4}} \cdot \frac{n^{\frac{k+l}{2}} + 1}{2} \tag{15}$$

is true.

Proof. For $n = 1$, we have equality in relation (15). For $n \geq 2$, in the Lemma above, making the substitutions $a_i = \sqrt{d_i^k}$ and $b_i = \frac{1}{\sqrt{d_i^l}}$, where d_i is the e-unitary divisor of n, for all $i = \overline{1, \tau^{(e)*}(n)}$. Since $\frac{k-l}{2} \in \mathbb{N}$, we have $k \geq l$, so, we deduce $1 \leq \frac{a_i}{b_i} = \sqrt{d_i^{k+l}} \leq n^{\frac{k+l}{2}}$ and $a_i b_i = d_i^{\frac{k-l}{2}}$. Hence, we take $m = 1$ and $M = n^{\frac{k+l}{2}}$.
Therefore, inequality (10) becomes

$$\sum_{i=1}^{\tau^{(e)*}(n)} d_i^k + n^{\frac{k+l}{2}} \cdot \sum_{i=1}^{\tau^{(e)*}(n)} \frac{1}{d_i^l} \leq \left(1 + n^{\frac{k+l}{2}}\right) \sum_{i=1}^{\tau^{(e)*}(n)} d_i^{\frac{k-l}{2}},$$

which is equivalent to

$$\sigma^{(e)*}(n) + n^{\frac{k+l}{2}} \cdot \sum_{i=1}^{\tau^{(e)*}(n)} \frac{1}{d_i^l} \leq \left(1 + n^{\frac{k+l}{2}}\right) \sigma^{(e)*}_{\frac{k-l}{2}}.$$

But

$$\sum_{i=1}^{\tau^{(e)*}(n)} \frac{1}{d_i^l} \geq \sum_{i=1}^{\tau^{(e)*}(n)} \frac{1}{n^l} = \frac{\tau^{(e)*}(n)}{n^l}.$$

Therefore, we obtain the inequality

$$\sigma_k^{(e)*}(n) + n^{\frac{k+l}{2}} \cdot \frac{\tau^{(e)*}(n)}{n^l} \leq \left(1 + n^{\frac{k+l}{2}}\right) \cdot \sigma^{(e)*}_{\frac{k-l}{2}}(n)$$

which means that

$$\sigma_k^{(e)*}(n) + n^{\frac{k-l}{2}} \cdot \tau^{(e)*}(n) \leq \left(1 + n^{\frac{k+l}{2}}\right) \cdot \sigma^{(e)*}_{\frac{k-l}{2}}(n), \tag{16}$$

for every $n, k, l \in \mathbb{N}$ with $n \geq 2$.
The arithmetical mean is greater than the geometrical mean or they are equal, so for every $n, k, l \in \mathbb{N}$ with $n \geq 2$, we have

$$\sqrt{n^{\frac{k-l}{2}} \cdot \sigma_k^{(e)*}(n) \cdot \tau^{(e)*}(n)} \leq \frac{\sigma_k^{(e)*}(n) + n^{\frac{k-l}{2}} \cdot \tau^{(e)*}(n)}{2}. \tag{17}$$

Consequently, from the relations (16) and (17), we deduce the inequality

$$\frac{\sqrt{\sigma_k^{(e)*}(n) \cdot \tau^{(e)*}(n)}}{\sigma^{(e)*}_{\frac{k-l}{2}}(n)} \leq \frac{n^{\frac{l-k}{4}} \cdot \sigma_k^{(e)*}(n) + n^{\frac{k-l}{4}} \cdot \tau^{(e)*}(n)}{2 \cdot \sigma^{(e)*}_{\frac{k-l}{2}}(n)} \leq n^{\frac{l-k}{4}} \cdot \frac{n^{\frac{k+l}{2}} + 1}{2}.$$

Remark 2 *For $k = l$, we obtain the relation*

$$\frac{\sigma_k^{(e)*}(n)}{\tau^{(e)*}(n)} \leq \left(\frac{n^k + 1}{2}\right)^2, \tag{18}$$

for every $n, k \in \mathbb{N}$ with $n \geq 1$.

Remark 3 *For $k = l = 1$, we obtain the relation*

$$\sqrt{\frac{\sigma^{(e)*}(n)}{\tau^{(e)*}(n)}} \leq \frac{\sigma^{(e)*}(n) + \tau^{(e)*}(n)}{2 \cdot \tau^{(e)*}(n)} \leq \frac{n+1}{2} \tag{19}$$

for every $n, k \in \mathbb{N}$ with $n \geq 1$.

Remark 4 *From inequality* (19), *we deduce another simple inequality, namely*

$$\frac{\sigma^{(e)*}(n)}{\tau^{(e)*}(n)} \leq n \tag{20}$$

for every $n \geq 1$.

Theorem 3 *For every* $n, k, l \in \mathbb{N}$ *with* $n \geq 1$, *there are the following relations:*

$$\sigma_k^{(e)}(n) \cdot \sigma_l^{(e)}(n) \leq \tau^{(e)}(n) \cdot \sigma_{k+l}^{(e)}(n), \tag{21}$$

$$\frac{\sigma_k^{(e)}(n)}{\sigma_l^{(e)}(n)} \geq \left(\frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)}\right)^{k-l} \geq \tau^{k-l}(n), \tag{22}$$

$$\frac{\sigma_{k+1}^{(e)}(n)}{\sigma_k^{(e)}(n)} \geq \tau(n) \tag{23}$$

and

$$\frac{\sigma_{k+1}^{(e)}(n)}{\sigma_k^{(e)}(n)} \geq \frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)} \geq \tau(n) \tag{24}$$

Proof. For $n = 1$, we obtain equality in the relation above.
Let $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r} > 1$. We apply Chebyshev's Inequality for oriented system and, we deduce the inequality

$$\sigma_k^{(e)}(n) \cdot \sigma_l^{(e)}(n) = \sum_{d|_{(e)}n} d^k \cdot \sum_{d|_{(e)}n} d^l \leq \tau^{(e)}(n) \sum_{d|_{(e)}n} d^{k+l} = \tau^{(e)}(n) \sigma_{k+l}^{(e)},$$

so

$$\sigma_k^{(e)}(n) \cdot \sigma_l^{(e)}(n) \leq \tau^{(e)}(n) \cdot \sigma_{k+l}^{(e)}(n).$$

From [1], we shall use the inequality

$$\frac{a_1^k + a_2^k + \dots + a_n^k}{a_1^l + a_2^l + \dots + a_n^l} \geq \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^{k-l},$$

for every $a_1, a_2, \dots, a_n > 0$ and for all $k, l \in \mathbb{N}$ with $k \geq l$, and by replacing $a_1, a_2, \dots,$ with the exponential divisors of n, we obtain the following inequality:

$$\frac{\sum\limits_{d|_{(e)}n} d^k}{\sum\limits_{d|_{(e)}n} d^l} \geq \left(\frac{\sum\limits_{d|_{(e)}n} d}{\tau^{(e)}(n)}\right)^{k-l}$$

which is equivalent to

$$\frac{\sigma_k^{(e)}(n)}{\sigma_l^{(e)}(n)} \geq \left(\frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)}\right)^{k-l}.$$

We know from [5] that $\dfrac{\sigma^{(e)}(n)}{\tau^{(e)}(n)} \geq \tau(n)$ and from the inequality $\dfrac{\sigma_k^{(e)}(n)}{\sigma_l^{(e)}(n)} \geq \left(\dfrac{\sigma^{(e)}(n)}{\tau^{(e)}(n)}\right)^{k-l}$, we deduce an interesting inequality, namely

$$\frac{\sigma_k^{(e)}(n)}{\sigma_l^{(e)}(n)} \geq \tau^{k-l}(n).$$

We observe that making the substitution $k \to k+1$ and $l \to k$ in inequality

$$\frac{\sigma_k^{(e)}(n)}{\sigma_l^{(e)}(n)} \geq \left(\frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)}\right)^{k-l},$$

we have

$$\frac{\sigma_{k+1}^{(e)}(n)}{\sigma_k^{(e)}(n)} \geq \tau(n).$$

If we assign values of k from 1 to $k-1$, we have the following relations:

$$\begin{aligned} \sigma_k^{(e)}(n) &\geq \tau(n)\sigma_{k-1}^{(e)}(n), \\ \sigma_{k-1}^{(e)}(n) &\geq \tau(n)\sigma_{k-2}^{(e)}(n), \\ &\cdots \\ \sigma_2^{(e)}(n) &\geq \tau(n)\sigma_1^{(e)}(n), \end{aligned}$$

and taking the product of these relations, we deduce the inequality

$$\sigma_k^{(e)}(n) \geq \tau^{k-1}(n)\sigma^{(e)}(n) \geq \tau^k(n)\tau^{(e)}(n).$$

Therefore, we obtain

$$\sigma_k^{(e)}(n) \geq \tau^k(n)\tau^{(e)}(n).$$

In relation $\dfrac{\sigma_k^{(e)}(n)}{\sigma_l^{(e)}(n)} \geq \left(\dfrac{\sigma^{(e)}(n)}{\tau^{(e)}(n)}\right)^{k-l}$, making the substitutions $k \to k+1$ and $l \to k$, we obtain the inequality

$$\frac{\sigma_{k+1}^{(e)}(n)}{\sigma_k^{(e)}(n)} \geq \frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)} \geq \tau(n).$$

Theorem 4 *For every $n, k, l \in \mathbb{N}$ with $n \geq 1$, there are the following relations:*

$$\sigma_k^{(e)*}(n) \cdot \sigma_l^{(e)*}(n) \leq \tau^{(e)}(n) \cdot \sigma_{k+l}^{(e)*}(n), \tag{25}$$

$$\frac{\sigma_k^{(e)*}(n)}{\sigma_l^{(e)*}(n)} \geq \left(\frac{\sigma^{(e)*}(n)}{\tau^{(e)*}(n)}\right)^{k-l} \geq \tau^{k-l}(n), \tag{26}$$

$$\frac{\sigma_{k+1}^{(e)*}(n)}{\sigma_k^{(e)*}(n)} \geq \tau(n) \tag{27}$$

and

$$\frac{\sigma_{k+1}^{(e)*}(n)}{\sigma_k^{(e)*}(n)} \geq \frac{\sigma^{(e)*}(n)}{\tau^{(e)*}(n)} \geq \tau(n). \tag{28}$$

Proof. We make the same proof as in Theorem 3, by repacing the exponential divisors with the e-unitary divisors.

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