Decomposition Theorems in Differential Geometry

H. Reckziegel and M. Schaaf*

Abstract

By definition a net on a manifold is a family of complementary foliations. Nets and net morphisms between netted manifolds allow to develop basic tools for the decomposability of netted manifolds. In this way new generalizations of de Rham's decomposition theorem are obtained. In a second part the decomposability of isometric immersions from a warped product into a pseudoriemannian manifold equipped with a warped product net is investigated. In particular Nölkers classification of warped product representations of the standard spaces of constant curvature and his generalization of Moore's decomposition theorem of isometric immersions $M_1 \times \cdots \times M_k \to \mathbb{R}^n$ are described.

Introduction

In mathematics one is often interested in decomposing an object into simpler components, for instance integers into prime numbers, group representations into irreducible ones et cetera. In Riemannian geometry the best known result in this direction is due to de Rham (see [Rh]):

Theorem (Decomposition theorem of de Rham). *Let M be a Riemannian manifold and suppose that there exists an orthogonal splitting*

$$TM = E_1 \oplus E_2 \oplus \cdots \oplus E_k$$

of the tangent bundle TM, where each E_i is a subbundle being parallel with respect to the Levi-Civita connection of M. Then every point $p \in M$ has a neighbourhood U, which is isometric to a product $\prod_{i=1}^k M_i$ of Riemannian manifolds M_i such that the submanifolds of $\prod_{i=1}^k M_i$, which are parallel to the factor M_i, correspond to integral manifolds of the distribution $E_i|_U$. In the case that M is simply connected and complete the assertion is true with $U = M$.

This theorem was generalized to pseudoriemannian and affinely connected manifolds without torsion (for instance in [Ka], [W1], [W2], [Ma]) and with torsion in [Me], to warped products in [H] and [N], to twisted products in [G], [Ko1] and [Ko2] and to more generally foliated manifolds in [BH1], [PR] and [BH2]. In all these results a manifold M with a *k-net* $\mathcal{E}$, that means with k complementary foliations L_i, and some additional geometric structure $\mathcal{S}$ is given, and it is proved that under suitable conditions the manifold is diffeomorphic to a product manifold $\prod_{i=1}^k M_i$ such that the

*Most of the new results were obtained by the second author while working on his doctoral thesis.

canonical net structure of $\prod_{i=1}^{k} M_i$ corresponds to the net $\mathcal{E}$ and that the structure $\mathcal{S}$ can be constructed from the geometry of the factors M_i in a natural way. In the first part of this article we treat the problem from this general point of view discussing the following questions:

- What are the conditions for the *local decomposability* of a *netted manifold*?
- Under which conditions can the *global decomposability* be derived from the local one?

Theorem 1 gives an answer to the first question. For the second question we can find a series of results, which are formulated and proved for 2-nets. In order to carry them over to k-nets with $k > 2$ we derive a strategy (based on the Lemma in section 3). It delivers no automatism; but by two examples we depict that it is fruitful:

- In Theorem 6 (a) we give a criterion for the decomposition of a manifold, which is equipped with a linear connection (torsion $\neq 0$ is allowed). As far as we know it represents the strongest generalization of de Rham's decomposition theorem in the category of linearly connected resp. of pseudoriemannian manifolds. For 2-nets it is due to Meumertzheim [Me].
- Theorem 6 (b) describes two different generalizations of a result, which for 2-nets originally is due to Blumenthal and Hebda [BH1] in the Riemannian and to Ponge and Reckziegel [PR] in the pseudoriemannian case. In this theorem the net is "partially" totally geodesic. In this article it serves as a preparation in order to prove that certain netted pseudoriemannian manifolds are warped products, see Theorem 7.

The diversity of geometric structures makes it impossible to derive a general method for their splitting in the sense which we described at the beginning. But asking for conditions under which a certain geometric structure splits, if they are alreay *defined on a product manifold*, turns out to be an efficient method, at least in some interesting situations. We demonstrate this by two examples: 1. if the canonical product net is parallel with respect to a linear connection (see Proposition 1), and 2. if it is a so called warped product net (see Definition 5 and Proposition 4).

In many situations the decomposition of a manifold is only the first step towards the solution of the problem under consideration, because the real interest lies in the *decomposability of a map.* An important result is the following

Theorem (Decomposition theorem of J. D. Moore). *See the Lemma in* [Mo]. *Let $f : M \to \mathbb{R}^n$ be an isometric immersion from a Riemannian product $M = \prod_{i=1}^{k} M_i$ of connected Riemannian manifolds into the Euclidean n-space $\mathbb{R}^n$, whose second fundamental form h satisfies*

(D) $$h|T_pL_i \times T_pL_j = 0 \quad \textit{for all} \quad p \in M \quad \textit{and} \quad i \neq j \;,$$

where $L_1, \ldots, L_k$ *are the canonical foliations of* M *induced by the product structure of* M. *Then there exist isometric immersions* $f_i : M_i \to \mathbb{R}^{m_i}$ *and an isometry* $\Phi : \prod_{i=1}^{k} \mathbb{R}^{m_i} \to \mathbb{R}^n$ *such that*

$$f = \Phi \circ (f_1 \times \ldots \times f_k) .$$

This theorem, for instance, was an essential step in Ferus' classification of symmetric submanifolds of the euclidean space $\mathbb{R}^n$; see [F]. When Backes and Reckziegel tried to classify symmetric submanifolds of the other standard spaces $N^n_\varkappa$ of constant curvature $\varkappa$, they had to develop other methods, because for $\varkappa \neq 0$ these spaces do not admit product representations. Instead, they were successful by introducing the concept of *extrinsic products* $\Phi : \prod_{i=1}^{k} N_i \to N^n_\varkappa$, see [BR]. With this tool Molzan was able in 1983 to prove the obvious generalization of Moore's theorem for the standard spaces $N^n_\varkappa$ with $\varkappa \neq 0$, see [Mz] or [Rz2]. But only in 1990 the real break-through was made by Nölker; he discovered that warped product representations of the spaces $N^n_\varkappa$ are a very efficient tool. They are more than a substitute for the ordinary product representations of $\mathbb{R}^n$, as is depicted by the following decomposition theorem for isometric immersions:

Theorem (Decomposition theorem of S. Nölker). *Let* $f : M \to N^n_\varkappa$ *be an isometric immersion from a connected warped product* $M = M_0 \times_\rho \prod_{i=1}^{k} M_i$ *into the standard space* $N^n_\varkappa$ *of constant curvature* $\varkappa$, *and suppose that the second fundamental form of* f *satisfies Condition* (D) *of Moore's decomposition theorem. Then there exist an isometry* $\Phi : N_0 \times_\sigma \prod_{i=1}^{k} N_i \to G$ *from a warped product* $N_0 \times_\sigma \prod_{i=1}^{k} N_i$ *onto an open, dense subset* $G \subset N^n_\varkappa$ *and isometric immersions* $f_i : M_i \to N_i$ *such that*

$$f = \Phi \circ (f_1 \times \ldots \times f_k) .$$

The second part of this article is devoted to this theorem. In section 4 we explain what a warped product is, and we generalize this notion introducing *warped product nets*; by definition they have the same infinitesimal structure as the product net of a warped product. We show that manifolds which are equipped with a warped product net "are" locally warped products and describe under which conditions this result is globally true; see Theorem 6 and the historical notes mentioned there. The proof of this decomposition theorem is based on the results of section 3, as we mentioned above. In section 5 we show that warped product nets are rigid and include Nölker's classification of all isometries $\Phi : N_0 \times_\sigma \prod_{i=1}^{k} N_i \to G \subset N^n_\varkappa$, which occur in his decomposition theorem.

Section 6 is concerned with the decomposition of isometric immersions. We found it convenient to introduce the notion of *net morphisms*: these are C^∞-maps between netted manifolds mapping the net of the source manifold into the net of the target manifold. If two netted manifolds are locally (resp. globally) decomposable, then every net morphism between them is also locally (resp. globally) decomposable, see Theorem 2. Therefore, it is of interest to look for conditions which guarantee that a given map is a net morphism. Already in a very general situation Condition (D) of Moore's und Nölker's theorem is the essential hypothesis, see Theorem 3. In Theorem 10 and 11 we formulate the results which we regard as the principal ones of this article; they describe those parts of Nölker's work which are valid for isometric immersions between arbitrary pseudoriemannian manifolds equipped with warped product nets. The proof of Theorem 10 would fill a separate paper and so we could not present it in this article. At the very end of this paper we describe which results depending on the special structure of $N^n_\varkappa$ are needed beside the last mentioned theorems to get the full decomposition theorem of Nölker.

1 Locally decomposable nets

As every n-dimensional manifold is locally diffeomorphic to $\mathbb{R}^n$, the notion of local decomposability makes only sense if we regard it with respect to an additional structure. Our experience with known decomposition theorems inspires us to introduce the notion of nets as some other geometers have already done before us; see for instance [Ko1].

In the following M always denotes an n-dimensional connected C^∞-manifold without boundary.

Definition 1. Let $TM = \bigoplus_{i=1}^k E_i$ be a splitting of the tangent bundle TM by a family $\mathcal{E} = (E_i)_{i=1,\dots,k}$ of subbundles.

(a) The family $\mathcal{E}$ is called a *net* or more informatively a *k-net* on M and the pair $(M, \mathcal{E})$ a *netted manifold*, if all subbundles E_i are integrable.

The foliation of integral manifolds of E_i will be denoted by $L_i^{\mathcal{E}}$ and its leaf through a point $p \in M$ by $L_i^{\mathcal{E}}(p)$.

Although we do not assume in general any pseudoriemannian metric on M, we assign the subspace $E_i^\perp := \bigoplus_{j \neq i} E_j$ to E_i for every i so that we get the splitting $TM = E_i \oplus E_i^\perp$. Furthermore, for every non-empty subset $I \in \{1, \dots, k\}$ we put

$$E_I := \bigoplus_{i \in I} E_i \ . \tag{1}$$

For every vector field $X \in \mathfrak{X}(M)$ we will write $X = \sum_i X^i$ with $X^i \in \Gamma(E_i)$ in the following.

Besides, we will notice that for every vector bundle E the modul of global sections in E will be denoted by $\Gamma(E)$.

For the end of the definition let $\mathcal{E}$ be a net as above.

(b) The net $\mathcal{E}$ is said to be *∇-parallel* with respect to a linear connection ∇ of M iff all E_i are ∇-parallel subbundles.

(c) If M is a pseudoriemannian manifold, we will say that $\mathcal{E}$ is an *orthogonal net*, if TM is the orthogonal sum of the E_i, symbolized by $TM = \bigoplus\!\!\!\!\!\perp_{i=1}^k E_i$, that means, each E_i is metrically nondegenerate and orthogonal to every E_j with $j \neq i$.

Examples.

(a) On a product manifold $M = \prod_{i=1}^{k} M_i$ a canonical k-net $\mathcal{E}$ is given by the vector subbundles $E_i \subset TM$ which are parallel to the factors M_i in the natural sense. This net will be called the *product net* of M. If not mentioned otherwise a product manifold will always be regarded as a netted manifold in this way.

If $(M_1, \nabla^1), \ldots, (M_k, \nabla^k)$ are affine manifolds and (M, ∇) is its product, then its product net $\mathcal{E}$ is ∇-parallel. And if $M_1, \ldots, M_k$ are pseudoriemannian manifolds and M is its pseudoriemannian product, then $\mathcal{E}$ is an orthogonal net which is parallel with respect to the Levi-Civita connection of M.

(b) On $M := \mathbb{R}^n \setminus \{0\}$ everybody is familiar with the *spherical coordinate net* $\mathcal{E} = (E_1, E_2)$, i.e., with the orthogonal 2-net characterized by $E_1(p) := \mathbb{R}\, p$.

Locally this construction is possible on every Riemannian manifold N: Let $\exp_p : T_pN \supset V_\varepsilon(0) \to U_\varepsilon(p)$ describe a normal neighbourhood of a point $p \in N$; then the spherical coordinate net on $M := U_\varepsilon(p) \setminus \{p\}$ is the orthogonal 2-net characterized by $E_1(\exp_p(v)) := \mathbb{R} \cdot {\exp_p}_* \tilde{v}$ where $\tilde{v}$ is the vector in $T_v(T_pN)$ which under the canonical identification $T_v(T_pN) \cong T_pN$ correponds to v. In the case of the hyperbolic space $\mathbb{H}^n$ this net is defined on the whole pointed space $\mathbb{H}^n \setminus \{p\}$; in the case of the unit sphere S^n it is defined on $\mathrm{S}^n \setminus \{p, -p\}$.

In fact, the previous example is a special case of the following construction: Let M be a Riemannian manifold. Then every C^∞-function $\varphi : M \to \mathbb{R}$ without critical points induces an orthogonal 2-net $\mathcal{E} = (E_1, E_2)$ by $E_1 := \mathbb{R}\, \mathrm{grad}\varphi$. The leaves of the foliation $L_2^{\mathcal{E}}$ are the level hypersurfaces of φ.

(c) Let $f : \tilde{M} \to N_\varkappa$ be an isometric hypersurface immersion into a Riemannian manifold of constant curvature $\varkappa$ and suppose that there exists a globally defined unit normal field η of f. The shape operator of f with respect to η shall be denoted by A. Then there exists an open and dense subset of $\tilde{M}$ such that the following holds: If M is a connected component of this subset, then there exist an integer $k \geq 1$ and C^∞-functions $\lambda_1, \ldots, \lambda_k$, the so called *principal curvature functions* of $f|M$, such that for every point $p \in M$ the numbers $\lambda_i(p)$ are the pairwise distinct eigenvalues of A_p. The corresponding eigenspaces $E_1(p), \ldots, E_k(p)$ form an orthogonal k-net $(E_i)_{i=1,\ldots,k}$ on M, which we will call the *principal net* of $f|M$; see [Rz1] p.176. As the second fundamental form h of f is described by $h(X,Y) = \langle AX, Y \rangle \cdot \eta$, the immersion $f|M$ satisfies Condition (D) of the Theorem of Moore (see the introduction) automatically.

Remark 1. At first a general remark: If E is a subbundle of the tangent bundle of a manifold M, then a C^∞-map $f : N \to M$ is said to be *E-integral* iff $f_*TN \subset E$. In the case that E is *involutive* (that means *integrable*) and N is connected, for every E-integral map $f : N \to M$ the image $f(N)$ lies in an integral manifold L_0 of E and the map $f : N \to L_0$ is also C^∞-differentiable, see [V] p. 30 and 17.

Now, we suppose that M is a pseudoriemannian manifold with Levi-Civita connection ∇ and E is a *totally umbilic* subbundle of TM; then E is integrable. In many cases this criterion implies that a splitting of the tangent bundle of a pseudoriemannian manifold is a net. – The total umbilicity of E means that it is (metrically) nondegenerate and that there exists a section $H^E \in \Gamma(E^\perp)$ such that

$$(2) \qquad \forall\, X, Y \in \Gamma(E)\,,\; Z \in \Gamma(E^\perp) \;:\; \langle \nabla_X Y, Z\rangle \;=\; \langle X, Y\rangle \cdot \langle H^E, Z\rangle\;.$$

If L denotes the foliation induced by E, then $H^E|L$ is the mean curvature vector field of the submanifold L. Therefore, H^E is called the *mean curvature vector field of the subbundle* E.

If in the previous situation H^E is a parallel section of the subbundle $E^\perp$, that means if

$$\forall\, X \in \Gamma(E) \;:\; \nabla_X(H^E) \in \Gamma(E)\;,$$

then the bundle E is said to be *spherical*. Of course, we encouter this situation in particular if E is *autoparallel*, that means

$$\forall\, X\,, Y \in \Gamma(E) \;: \nabla_X Y \in \Gamma(E)\;,$$

in other words, if E is totally umbilic and $H^E = 0$.

It should also be mentioned that E is autoparallel resp. spherical resp. totally umbilic if and only if all integral manifolds of E (i.e. the leaves of L) are totally geodesic resp. spherical (i.e. they are extrinsic spheres) resp. totally umbilic submanifolds of M; see [HR] p. 271.

Now we come to the second crucial definition of this article:

Definition 2. Let M and N be C^∞-manifolds, which are equipped with nets $\mathcal{E} = (E_i)_{i=1,\dots,k}$ and $\mathcal{F} = (F_i)_{i=1,\dots,k}$, respectively, and $f : M \to N$ a C^∞-map.

(a) The map f is called a *net morphism* iff

$$(3) \qquad \forall\, p \in M\;,\; i = 1, \dots, k \;:\; f_* E_i(p) \subset F_i(f(p))\;.$$

Equivalently we can say: f is a net morphism iff for every $p \in M$ and $i = 1, \dots, k$ the restriction $f|L_i^{\mathcal{E}}(p)$ is a C^∞-map into the leaf $L_i^{\mathcal{F}}(f(p))$.

By this definition we obtain the *category of k-netted manifolds*. In accordance with the theory of categories a *net isomorphism* $f : (M, \mathcal{E}) \to (N, \mathcal{F})$ is a diffeomorphism $f : M \to N$ such that f and f^{-1} are net morphisms.

(b) If M and N are pseudoriemannian manifolds and f is simultaneously a net morphism and an isometric immersion, then we shall call f a *pseudoriemannian net morphism*.

Net morphisms will be treated in detail in the next section. But the following definition shows already, how useful this notion is.

Definition 3. A k-net $\mathcal{E}$ on M is said to be *locally decomposable* iff for every point $p \in M$ there exist a neighbourhood U of p in M and a net isomorphism f from $(U, \mathcal{E}|_U)$ onto a product manifold $\prod_{i=1}^k M_i$. The map $f^{-1} : \prod M_i \to U$ will also be called a *product representation* of $(U, \mathcal{E}|_U)$. – If in this situation we can choose $U = M$, then the net $\mathcal{E}$ is said to be *globally decomposable.* In this situations we will also speak of the local or global decomposability of the netted manifold $(M, \mathcal{E})$.

It is well known that every 2-net is locally decomposable (see [KN], p.182). But for $k > 2$ one can easily construct k-nets which are not locally decomposable.

Theorem 1. *For every splitting $TM = \bigoplus_{i=1}^k E_i$ the following five assertions are pairwise equivalent:*

(a) The family $\mathcal{E} = (E_i)_{i=1,\dots,k}$ is a locally decomposable net.

(b) For every subset $I \subset \{1, \dots, k\}$ the subbundle E_I is integrable; see (1).

(c) The subbundles E_i and their direct sums $E_i \oplus E_j$ $(i \neq j = 1, \dots, k)$ are integrable.

(d) The complementary subbundles $E_i^\perp$ $(i = 1, \dots, k)$ are integrable.

(e) The subbundles E_i are parallel with respect to a suitable torsionfree linear connection.

We could not find a formulation of these assertions explicitely, although it is not unknown. For instance, Koike uses the implication (c) $\Rightarrow$ (a) in [Ko1].

Proof. In any of the situations (a), (c), (d) and (e) we can easily prove (b). Therefore, it is sufficient to verify "(d) $\Rightarrow$ (a)" and "(c) $\Rightarrow$ (e)".

To "(d) $\Rightarrow$ (a)". We generalize the proof of the Lemma in [KN] p. 182. Let $n_i := \operatorname{rk} E_i$ for every i and $p \in M$ an arbitrary point. Because of (c) we can choose charts (U, x^i) of M with $p \in U$ and $x^i(p) = 0$, which are adapted to $E_i^\perp$, that means: $\frac{\partial}{\partial x^i_{n_i+1}}, \dots, \frac{\partial}{\partial x^i_n}$ form a local frame field of $E_i^\perp$. Then $f^i := (x^i_1, \dots, x^i_{n_i}) : U \to \mathbb{R}^{n_i}$ is a submersion with $\ker f^i_* = E_i^\perp|_U$; and if we choose U appropriately, $x = (f^1, \dots, f^k)|U : U \to \mathbb{R}^n$ is a chart with $x(U) = Q_\varepsilon := \{ q \in \mathbb{R}^n \mid \|q_1\|, \dots, \|q_n\| < \varepsilon \}$. Moreover, with respect to the diffeomorphism $x : U \to Q_\varepsilon$ the net $\mathcal{E}|_U$ corresponds to the product net on Q_ε induced by the splitting $\mathbb{R}^n = \bigoplus_{i=1}^k \mathbb{R}^{n_i}$, because we have $f^j_* v = 0$ for $i \neq j$, $q \in U$ and $v \in E_i(q)$.

To "(c) $\Rightarrow$ (e)". Starting with an arbitrary torsionfree linear connection ∇ on M we define a tensor field Δ of type $(1,2)$ on M by

$$\Delta(X, Y) := -\sum_{i \neq j}^{k} \left(\nabla_X Y^i + \nabla_{Y^j} X^i \right)^j$$

and a new linear connection $\tilde\nabla$ on M by $\tilde\nabla := \nabla + \Delta$. Then for every $i_0 \in \{1, \dots, k\}$, $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E_{i_0})$ we get

$$\tilde\nabla_X Y = \nabla_X Y - \sum_{j \neq i_0} \left((\nabla_X Y)^j + (\nabla_Y X^j)^{i_0} \right) = (\nabla_X Y)^{i_0} - \sum_{j \neq i_0} (\nabla_Y X^j)^{i_0} \in \Gamma(E_{i_0}) \ ;$$

hence the net $\mathcal{E}$ is $\tilde{\nabla}$-parallel and it remains to prove that $\tilde{\nabla}$ is torsionfree, which is equivalent to the symmetry of Δ. For that we compute for arbitrary $X, Y \in \mathfrak{X}(M)$

$$\begin{aligned}\Delta(X,Y)-\Delta(Y,X) &= \sum_{i\neq j}\Big(\sum_{l}\nabla_{Y^l}X^i+\nabla_{X^j}Y^i-\sum_{l}\nabla_{X^l}Y^i-\nabla_{Y^j}X^i\Big)^j \\ &= \sum_{j}\sum_{i,l\neq j}\Big(\nabla_{Y^l}X^i-\nabla_{X^l}Y^i\Big)^j = \sum_{j}\sum_{i,l\neq j}[Y^l,X^i]^j\ .\end{aligned}$$

This expression vanishes, since $[Y^l, X^i] \in \Gamma(E_l + E_i)$ because of (c), hence $[Y^l, X^i]^j = 0$ for all indices occuring in the sum.

2 Net morphisms

Let M and N be C^∞-manifolds, which are equipped with nets $\mathcal{E} = (E_i)_{i=1,\dots,k}$ and $\mathcal{F} = (F_i)_{i=1,\dots,k}$ respectively.

General notations. For every C^∞-map $f : M \to N$ we introduce the following definition resp. conventions:

(a) If M and N are equipped with linear connections ∇^M and ∇^N, then we define the *second fundamental form* h of f with respect to these connections by

$$h(X,Y) = {\nabla^N}_X f_* Y - f_* {\nabla^M}_X Y \quad \text{for all} \quad X,\, Y \in \mathfrak{X}(M)\ . \tag{4}$$

We notice that h is tensorial, and it is symmetric, if both connections are torsionfree. – In the case of isometric immersion between pseudoriemannian manifolds this definition coincides with the usual one.

(b) We shall say that f satisfies the *Decomposability Condition* (D) with respect to given linear connections and the net $\mathcal{E}$ iff the second fundamental form of f satisfies

$$h|E_i \times_M E_j \equiv 0 \quad \text{for all} \quad i \neq j \in \{1,\dots,k\}\ .$$

(c) We shall say that f satisfies the *Initial Condition* (I) with respect to the nets $\mathcal{E}$ and $\mathcal{F}$ and a point $\bar{p} \in M$ iff

$$f(L_i^{\mathcal{E}}(\bar{p})) \subset L_i^{\mathcal{F}}(f(\bar{p})) \quad \text{for all} \quad i \in \{1,\dots,k\}\ .$$

It should be pointed out that condition (D) is exactly that of Moore's decomposition theorem (see the introduction). – The following result shows that net morphisms between globally decomposable netted manifolds are also decomposable.

Theorem 2. *For two product manifolds $M := \prod_{i=1}^k M_i$ and $N := \prod_{i=1}^k N_i$ a C^∞-map $f : M \to N$ is decomposable, i.e., there exist C^∞-maps $f_i : M_i \to N_i$ for $i = 1,\dots,k$ such that $f = f_1 \times \dots \times f_k$, if and only if it is a net morphism with respect to their product nets.*

As a consequence of this result we see also that in the case of two locally decomposable k-netted manifolds $(M,\mathcal{E})$ and $(N,\mathcal{F})$ every net morphism $(M,\mathcal{E}) \to (N,\mathcal{F})$ is locally decomposable, too.

Proof. The "only if"-part of the assertion is trivial. For the "if"-part we denote the product nets of M and N by $\mathcal{E}$ resp. $\mathcal{F}$ and write $f = (\tilde{f}_1, \dots, \tilde{f}_k)$ with C^∞-maps $\tilde{f}_i : M \to N_i$. Then we have to show that for every i there exists a C^∞-map $f_i : M_i \to N_i$ such that

$$\forall\, p \in M \;:\; \tilde{f}_i(p) = f_i(p_i)\ . \tag{5}$$

To (5): For every fixed $j \neq i$ and every pair $(p,q) \in M \times M$ with $\big(\forall\, l \neq j \,:\, q_l = p_l\big)$ we have $q \in L_j^{\mathcal{E}}(p)$ and therefore $f(q) \in L_j^{\mathcal{F}}(f(p))$, in particular $\tilde{f}_i(q) = \tilde{f}_i(p)$. Therefore, we also obtain $\tilde{f}_i(q) = \tilde{f}_i(p)$ for every pair $(p,q) \in M \times M$ satisfying $q_i = p_i$, because we can replace, step by step, every q_j $(j \neq i)$ by p_j without changing the image under $\tilde{f}_i$. In this way we get that there exists a unique map f_i satisfying (5). This map is differentiable and $f = \prod_{i=1}^k f_i$.

Because of Theorem 2 the following characterization of net morphisms is of importance.

Theorem 3. *Let M and N be manifolds, which are equipped with torsionfree linear connections ∇^M and ∇^N and with nets $\mathcal{E} = (E_i)_{i=1,\dots,k}$ and $\mathcal{F} = (F_i)_{i=1,\dots,k}$, respectively. We suppose that $\mathcal{E}$ is ∇^M-parallel and $\mathcal{F}$ is ∇^N-parallel (see Theorem 1) and that the netted manifold $(M,\mathcal{E})$ is connected and globally decomposable and the netted manifold $(N,\mathcal{F})$ is locally decomposable. Then a C^∞-map $f : M \to N$ is a net morphism with respect to $\mathcal{E}$ and $\mathcal{F}$ if and only if it satisfies the conditions* (D) *and* (I) *with respect to the given connections and nets and some point $\bar{p} \in M$.*

One should compare this general theorem with Moore's Decomposition Theorem noted in the introduction. Since in our situation the net $\mathcal{F}$ is fixed in advance we have also to assume some initial condition, namely condition (I).

Proof. To "$\Rightarrow$": Let two indices $i \neq j \in \{1, \dots, k\}$ and two sections $X \in \Gamma(E_i)$, $Y \in \Gamma(E_j)$ be given. Then the assumption implies $f_*Y \in \Gamma_f(F_j)$, thus ${\nabla^N}_X f_*Y \in \Gamma_f(F_j)$; on the other hand we have ${\nabla^M}_X Y \in \Gamma(E_j)$, thus $f_*{\nabla^M}_X Y \in \Gamma_f(F_j)$. Combining these results we get $h(X,Y) \in \Gamma_f(F_j)$, and changing the role of X and Y also $h(Y,X) \in \Gamma_f(F_i)$. Because of the symmetry of h condition (D) is proved. Of course condition (I) is also satisfied.

To "$\Leftarrow$": We may suppose that M is a product manifold $\prod_{i=1}^k \tilde{M}_i$ and that $\mathcal{E}$ is its product net. In order to prove (3) let $p = (p_1, \dots, p_k) \in M$ be an arbitrary point, $i \in \{1, \dots, k\}$ and $v \in E_i(p)$. Then we put $\tilde{p} := (\bar{p}_1, \dots, \bar{p}_{i-1}, p_i, \bar{p}_{i+1}, \dots, \bar{p}_k) \in L_i^{\mathcal{E}}(\bar{p})$, choose a C^∞-curve $\alpha = (\alpha_1, \dots, \alpha_k) : [0,1] \to M$ with $\alpha(0) = p$, $\alpha(1) = \tilde{p}$, and $\alpha_i \equiv p_i$ (hence $\dot{\alpha} = \Gamma_\alpha(E_i^\perp)$) and consider the ∇^M-parallel vectorfield $X \in \mathfrak{X}_\alpha(M)$ along α with $X(0) = v$, which is a section in E_i along α. Because of (D) ${\nabla^N}_\partial f_*X = f_*{\nabla^M}_\partial X + h(\dot{\alpha}, X) = 0$, where ∂ denotes the canonical unit vector field on $\mathbb{R}$. Therefore, f_*X is a parallel vector field along $f \circ \alpha$. On the other hand from

$X(1) \in E_i(\tilde{p})$ and (I) we deduce $f_*X(1) \in F_i(f(\tilde{p}))$; hence f_*X is a section in the parallel subbundle F_i. In particular we get $f_*v \in F_i(f(p))$.

Using the previous theorem we can derive a certain associativity of the property that a map is a net morphism.

Theorem 4. *Let M and N be manifolds, which are equipped with locally decomposable nets $\mathcal{E} = (E_i)_{i=1,\dots,l}$ and $\mathcal{F} = (F_i)_{i=1,\dots,l}$, respectively. Furthermore, let a partition $\{1,\dots,l\} = \dot\bigcup_{i=1}^k I_i$ be given, define the "coarse" nets $\mathcal{E}' = (E_{I_i})_{i=1,\dots,k}$ and $\mathcal{F}' = (F_{I_i})_{i=1,\dots,k}$ (see (1)) and suppose that the netted manifold $(M,\mathcal{E}')$ is connected and globally decomposable. Let us denote the maximal integral manifold of E_{I_i} through a fixed point $\bar{p} \in M$ by M_i. If in this situation a net morphism $f : (M,\mathcal{E}') \to (N,\mathcal{F}')$ satisfies*

$$\forall\, i = 1,\dots,k \quad \forall\, j \in I_i \quad \forall\, p \in M_i \ : \ f_*E_j(p) \subset F_j(f(p)) \ , \tag{6}$$

then f is also a net morphism with respect to the "fine" nets $\mathcal{E}$ and $\mathcal{F}$.

Proof. We choose linear connections ∇^M and ∇^N on M resp. on N such that $\mathcal{E}$ is ∇^M-parallel and $\mathcal{F}$ is ∇^N-parallel. Then also the subbundles $E'_i := E_{I_i}$ and $F'_i := F_{I_i}$ are ∇^M- resp. ∇^N-parallel. As f is a net morphism with respect to $\mathcal{E}'$ and $\mathcal{F}'$, it satisfies Condition (D) with respect to the connections ∇^M and ∇^N and the net $\mathcal{E}'$ (because of Theorem 3). Furthermore, we may suppose that M is a product manifold $\prod_{i=1}^k \tilde{M}_i$ and that $\mathcal{E}'$ is its product net.

If now a point $p = (p_1,\dots,p_k) \in M$, an index $j \in I_i$ and a vector $v \in E_j(p)$ are given, then we follow the "$\Leftarrow$"-part of the proof of Theorem 3 step by step only replacing the subbundles E_i resp. F_i by E'_i resp. F'_i. In particulary, Condition (D) implies that f_*X is a ∇^N-parallel vector field along $f \circ \alpha$. But in our special situation the vector field X is even a section in the subbundle E_j. Therefore we have $X(1) \in E_j(\tilde{p})$, hence $f_*X(1) \in F_j(f(\tilde{p}))$ (because of (6)), which implies $f_*v = f_*X(0) \in F_j(f(p))$, since F_j is a ∇^N-parallel subbundle.

We will now derive a result which will become of importance in combination with warped product representations of standard spaces of constant curvature, which only on certain open subsets induce globally decomposable nets; see Theorem 9 and Theorem 11.

Theorem 5. *Let $M = \prod_{i=1}^k M_i$ be a connected product manifold, $\mathcal{E}$ its product net, $\bar{p} = (\bar{p}_1,\dots,\bar{p}_k) \in M$ and $\mathcal{F} = (F_i)_{i=1,\dots,k}$ a globally decomposable net defined on an open subset G of a C^∞-manifold N. Futhermore, let $f : M \to N$ be a C^∞-map with the following properties:*

(i) $\forall\, i = 1,\dots,k \ : \ f(L_i^{\mathcal{E}}(\bar{p})) \subset L_i^{\mathcal{F}}(f(\bar{p}))$,

(ii) for every family $(U_i)_{i=1,\dots,k}$ of connected open subsets $U_i \subset M_i$ with $\bar{p} \in U := \prod_{i=1}^k U_i$ and $f(U) \subset G$ the restriction $f|U : (U,\mathcal{E}|_U) \to (G,\mathcal{F})$ is a net morphism.

Then f is a net morphism from $(M,\mathcal{E})$ into $(G,\mathcal{F})$. Therefore, f can be represented by

$$f = \Phi \circ (f_1 \times \cdots \times f_k)$$

with C^∞-maps $f_i : M_i \to N_i$, if we have chosen a product representation $\Phi : \prod_{i=1}^k N_i \to (G,\mathcal{F})$ of the netted manifold $(G,\mathcal{F})$.

Remark 2. For instance, because of Theorem 3 f satisfies the hypothesis (ii), if besides the other assumptions on M and N there exist torsionfree linear connections ∇^M and ∇^N such that the nets $\mathcal{E}$ and $\mathcal{F}$ are parallel with respect to them and f satisfies Condition (D) with respect to them and $\mathcal{E}$.

Proof. We put $V := f^{-1}(G)$, choose a product representation $\Phi : \prod_{i=1}^k N_i \to (G,\mathcal{F})$ and write $\Phi^{-1} \circ f|V =: (\tilde{f}_1, \dots, \tilde{f}_k)$. Because of assumption (i) $L_i^{\mathcal{E}}(\bar{p})$ is a subset of V for every $i = 1, \dots, k$ and therefore we can define for these i the C^∞-maps

$$f_i : M_i \to N_i \ , \ p_i \to \tilde{f}_i(\bar{p}_1, \dots, \bar{p}_{i-1}, p_i, \bar{p}_{i+1}, \dots, \bar{p}_k)$$

and the closed subset

$$A \ := \ \{ p \in M \mid f(p) = \Phi \circ (f_1 \times \cdots \times f_k)(p) \}$$

of M, which obviously is contained in V. Furthermore, we define

$$\mathfrak{U}_i \ := \ \{ U \subset M_i \mid U \text{ is open, connected and relatively compact in } M_i \text{ and } \bar{p}_i \in U \}$$

and

$$\Sigma \ := \ \{ (U_1, \dots, U_k) \in \mathfrak{U}_1 \times \cdots \times \mathfrak{U}_k \mid \bar{U}_1 \times \cdots \times \bar{U}_k \subset V \} \ .$$

This last set is not empty, because $\bar{p} \in V$ and V is open. We show now

(7)
$$\forall (U_1, \dots, U_k) \in \Sigma \ , \ i = 1, \dots, k \ : \ \bar{U}_1 \times \cdots \times \bar{U}_{i-1} \times M_i \times \bar{U}_{i+1} \times \cdots \times \bar{U}_k \subset V \ .$$

For that we fix a family $(U_1, \dots, U_k) \in \Sigma$ and an $i \in \{1, \dots, k\}$ and define the subset

$$B \ := \ \{ p_i \in M_i \mid \bar{U}_1 \times \cdots \times \bar{U}_{i-1} \times \{p_i\} \times \bar{U}_{i+1} \times \cdots \times \bar{U}_k \subset V \}$$

of M_i, which is open, since the $\bar{U}_j$ are compact. This subset contains $\bar{U}_i$. The connected component C of B containing $\bar{p}_i$ is also open in M_i. Because of the assumption (ii) we can apply Theorem 2 to the restriction $\Phi^{-1} \circ f|(U_1 \times \cdots \times U_{i-1} \times C \times U_{i+1} \times \cdots \times U_k)$ and obtain $U_1 \times \cdots \times U_{i-1} \times C \times U_{i+1} \times \cdots \times U_k \subset A$, hence also $\bar{U}_1 \times \cdots \times \bar{U}_{i-1} \times \bar{C} \times \bar{U}_{i+1} \times \cdots \times \bar{U}_k \subset A$, i.e., $\bar{C}$ is contained in B and therefore $\bar{C} = C$. As M_i is connected, we get $C = M_i$ and assertion (7) is proved. From this result we obtain immediately

(8)
$$\forall (U_1, \dots, U_k) \in \Sigma \ , \ i = 1, \dots, k \ , \ \tilde{U}_i \in \mathfrak{U}_i \ : \ (U_1, \dots, U_{i-1}, \tilde{U}_i, U_{i+1}, \dots, U_k) \in \Sigma \ ,$$

and therefore

$$\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_k \subset \Sigma \, .$$

In fact, if $U_i \in \mathfrak{U}_i$ are given for $i = 1, \ldots, k$, we choose a $(U_1^0, \ldots, U_k^0) \in \Sigma$ and replace step by step U_1^0 by U_1, U_2^0 by U_2 and so on. Then assertion (8) shows that by this process we do not leave Σ. Hence $(U_1, \ldots, U_k) \in \Sigma$, in particular $U_1 \times \cdots \times U_k \subset V$. As the subsets of the last type cover the whole manifold M, we even have proved $V = M$, i.e., $f(M) \subset G$. All other claims of the theorem follow now from the assumption (ii) and Theorem 2. □

3 Global decomposability of netted manifolds

There exists a number of decomposition theorems of 2-netted manifolds; but often their generalization to k-netted manifolds is not discussed. The following lemma describes a tool how to get such generalizations.

Lemma. *We suppose that on a connected manifold M a locally decomposable net $\mathcal{E} = (E_j)_{j=1,\ldots,l}$, a point $\bar{p} \in M$ and a partition $\{1, \ldots, l\} = \dot{\cup}_{i=1}^k I_i$ with $I_i \neq \varnothing$ for every i are given. Furthermore, we suppose that the "coarse" net $\mathcal{E}' = (E_{I_i})_{i=1,\ldots,k}$ is globally decomposable; see* (1)*. Now, for every $i = 1, \ldots, k$ we denote the maximal integral manifold of E_{I_i} through $\bar{p}$ by M_i. If in this situation for every $i = 1, \ldots, k$ the "restricted" net $\mathcal{E}_i = (E_j|_{M_i})_{j \in I_i}$ is globally decomposable, too, then also the "fine" net $\mathcal{E}$ is globally decomposable.*

Proof. We may suppose that M is a product manifold $\prod_{i=1}^k \tilde{M}_i$ and that $\mathcal{E}'$ is its product net. As every restricted net $\mathcal{E}_i$ is supposed to be decomposable, there exists a diffeomorphism $f_i : M_i \to \prod_{j \in I_i} N_j$ which is a net isomorphism with respect to $\mathcal{E}_i$ and the product net of $\prod_{j \in I_i} N_j$. Introducing furthermore the canonical diffeomorphisms $g_i : \tilde{M}_i \to M_i$, $p_i \mapsto (\bar{p}_1, \ldots, \bar{p}_{i-1}, p_i, \bar{p}_{i+1}, \ldots, \bar{p}_k)$ we can define the diffeomorphism

$$f = (f_1 \circ g_1, \ldots, f_k \circ g_k) \; : \; M = \prod_{i=1}^k \tilde{M}_i \to \prod_{i=1}^k \Big(\prod_{j \in I_i} N_j \Big) \cong \prod_{j=1}^l N_j \, .$$

If now $\mathcal{F}$ is the product net on $\prod_{j=1}^l N_j$, then the data f, $\mathcal{E}$ and $\mathcal{F}$ satisfy the assumptions of Theorem 4. Therefore, this theorem shows that f is a net isomorphism $(M, \mathcal{E}) \to \big(\prod_{j=1}^l N_j, \mathcal{F} \big)$.

The following two theorems are obtained by applying the previous lemma.

Theorem 6. *Let M be a simply connected manifold, which is equipped with a* locally *decomposable net $\mathcal{E} = (E_i)_{i=1,\ldots,k}$.*

(a) **Decomposition of linearly connected netted manifolds.** *If the manifold M is geodesically complete with respect to a linear connection ∇, the net $\mathcal{E}$ is ∇-parallel and if the torsion and curvature tensors T and R of ∇ satisfy*[†]

$$T|E_i \times_M E_j = 0 \quad \text{and} \quad R|E_i \times_M E_j = 0 \quad \text{for every} \quad i \neq j \,, \tag{9}$$

then (M, ∇) "is" the product of linearly connected manifolds (M_i, ∇^i); more exactly: There exists an affine diffeomorphism $f : M \to \prod_{i=1}^k M_i$, which at the same time is a net isomorphism with respect to $\mathcal{E}$ and the product net of $\prod_{i=1}^k M_i$.

(b) **Decomposition of netted manifolds of Blumenthal/Hebda type.** *Let M be a pseudoriemannian manifold and $\mathcal{E}$ an orthogonal net satisfying* one *of the following conditions:*

  (i) *For every $i > 1$ the leaves of the foliation $L_i^{\mathcal{E}}$ are totally geodesic and geodesically complete.*

  (ii) *For every $i > 1$ the maximal integral manifolds of the subbundle $E_i^{\perp}$ are totally geodesic and geodesically complete.*

Then the netted manifold $(M, \mathcal{E})$ is decomposable.

Remark 3. Both results are known in the case $k = 2$. In fact, then (a) is due to Meumertzheim [Me] and (b) was first proved by Blumenthal/Hebda [BH2] in the Riemannian case and was afterward generalized to the pseudoriemannian case by Ponge/Reckziegel [PR]. We will also point out that in (b) we have not to demand the geodesic completeness of the whole manifold M and that for $k = 2$ the conditions (i) and (ii) are identical.

Sketch of the proof. In order to prove (a) and (b) for $k > 2$ we use the same strategy, namely we prove by induction assuming that the result is true for some $k \geq 2$: If a $(k+1)$-net $\mathcal{E} = (E_i)_{i=1,\ldots,k+1}$ (with the quoted properties) is given, we can decompose first the "coarse" 2-net $\mathcal{E}' = \left(\bigoplus_{i=1}^k E_i \,,\, E_{k+1}\right)$, that means, we may assume that M is a product $M_1 \times M_2$ and that $\mathcal{E}'$ is its product net. Then we choose a point $(\bar{p}_1, \bar{p}_2) \in M_1 \times M_2$ and consider the "restricted" net $\mathcal{E}_N = (E_i|N)_{i=1,\ldots,k}$ on the simply connected manifold $N := M_1 \times \{\bar{p}_2\}$ and check that on $(N, \mathcal{E}_N)$ we can apply the theorem (in the case k) and get in this way the global decomposability of $(N, \mathcal{E}_N)$. Thus the lemma proves the global decomposability of the netted manifold $(M, \mathcal{E})$. For (b) the proof is finished thereby (details are left to the reader). For (a) it remains to prove the following proposition, which we formulate for k instead of $k+1$.

Proposition 1. *If on a product manifold $M := \prod_{i=1}^k M_i$ a linear connection ∇ is given such that its product net $\mathcal{E} = (E_i)_{i=1,\ldots,k}$ is ∇-parallel and satisfies the formulas* (9)*, then there exist linear connections ∇^i on M_i such that (M, ∇) is the product of $(M_1, \nabla^1), \ldots, (M_k, \nabla^k)$.*

† $R(u,v) = 0$ means $\forall w : R(u,v)w = 0$.

The general concept of the proof for Proposition 1 can be found in [Me]; but our situation is more general, because we do not handle only a local situation. Nevertheless, we will not go in the details here.

4 Warped product nets

First we recall the definition of warped products.

Definition 4. Let $(M_0, \langle\cdot,\cdot\rangle_0), \ldots, (M_k, \langle\cdot,\cdot\rangle_k)$ be pseudoriemannian manifolds and $M := \prod_{i=0}^k M_i$. Using the canonical projections $\pi_i : M \to M_i$ we define for every C^∞-function $\rho := (\rho_1, \ldots, \rho_k) : M_0 \to \mathbb{R}_+^k$ a pseudoriemannian metric $\langle\cdot,\cdot\rangle$ on M by

$$\langle X, Y\rangle \;:=\; \langle \pi_{0*}X, \pi_{0*}Y\rangle_0 + \sum_{i=1}^k (\rho_i \circ \pi_0)^2 \cdot \langle \pi_{i*}X, \pi_{i*}Y\rangle_i \quad \text{for every} \quad X, Y \in \mathfrak{X}(M) \; .$$

The pseudoriemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is called the *warped product* of $M_0, \ldots, M_k$ with the warping function ρ, and it is denoted by $M_0 \times_\rho \prod_{i=1}^k M_i$.

If in the previous definition we choose a point $\bar{p}_0 \in M_0$, we can normalize the data $\langle\cdot,\cdot\rangle_i$ and ρ_i in such a way that we have

$$(10) \qquad \forall\, i = 1, \ldots, k \;:\; \rho_i(\bar{p}_0) = 1 \; ;$$

furthermore, if M is connected, these data are uniquely determined by $\langle\cdot,\cdot\rangle$ and the property (10).

In order to give some insight in the geometry of a warped product $M_0 \times_\rho \prod_{i=1}^k M_i$, we write down the formula for its Levi-Civita connection ∇ by comparing it with the Levi-Civita connection $\tilde{\nabla}$ of the ordinary pseudoriemannian product $\prod_{i=0}^k M_i$. For that we denote the product net of this product manifold by $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ and introduce the vector fields

$$(11) \qquad H_i \;:=\; -\mathrm{grad}(\ln \circ \rho_i \circ \pi_0) \; ,$$

where we get the same result by calculating the gradient with respect to the ordinary product metric or the warped product metric above. Then we get

$$(12) \qquad \nabla_X Y \;=\; \tilde{\nabla}_X Y + \sum_{i=1}^k \left(\langle X^i, Y^i\rangle\, H_i - \langle H_i, X\rangle\, Y^i - \langle H_i, Y\rangle\, X^i \right) .$$

In [N] one can find a proof for the Riemannian case, which we can copy in our situation. This formula enables us to prove by standard arguments

Proposition 2. *The product net* $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ *of the warped product* $M := M_0 \times_\rho \prod_{i=1}^k M_i$ *has the following proporties:*

(a) $\mathcal{E}$ *is an* orthogonal net *with respect to the warped product metric.*

(b) *For every* $i = 1, \dots, k$ *the subbundle* E_i *is spherical (in the sense of Remark 1). Furthermore, the mean curvature vector field* H^{E_i} *of* E_i *coincides with the vector field* H_i *defined by* (11), *and the subbundle* $E_i^{\perp}$ *is autoparallel.*

Motivated by this result we define

Definition 5. An orthogonal net $\mathcal{E} = (E_i)_{i=0,\dots,k}$ on a pseudoriemannian manifold M is called a *warped product net* or shortly WP-net iff for every $i = 1, \dots, k$ the subbundle E_i is spherical and its orthogonal complement $E_i^{\perp}$ is autoparallel.

Examples.

(a) The product net of a warped product is a WP-net because of Proposition 2.

(b) If M is a *hypersurface of a space of constant curvature* and if M satisfies some curvature conditions, then often the principal net (see Example (c) of section 1) is a WP-net. For instance, if M has *harmonic curvature* (that is equivalent to the fact that the Ricci tensor of M is a Codazzi tensor), then in a natural way we come to WP-nets, see [MR2], Theorem 2. Examples of manifolds with harmonic curvature are all Riemannian manifolds with parallel Ricci tensor, hence all Einstein manifolds, hence all manifolds with constant curvature. In [DN] Dillen and Nölker study *normally flat, semi-parallel submanifolds* of arbitrary codimension in spaces of constant curvature, and also there we encounter WP-nets, see [DN], p. 53.

Proposition 3 (First Part). *For every WP-net* $\mathcal{E} = (E_i)_{i=0,\dots,k}$ *on a pseudoriemannian manifold* M *the following is true:*

(a) *The subbundle* E_0 *is autoparallel.*

(b) $(M, \mathcal{E})$ *is locally decomposable.*

(c) *For every* $i > 0$ *the mean curvature vector field* H_i *of the subbundle* E_i *is a section in* E_0 *(see Remark 1).*

Proof. For (a): Notice that E_0 is the intersection of the autoparallel subbundles $E_i^{\perp}$ with $i = 1, \dots, k$.

For (b): Because of Theorem 1 it suffices to prove that $E_0^{\perp}$ is integrable, because $E_i^{\perp}$ is integrable for $i > 0$ by assumption (see Remark 1). In fact, it is sufficient to show for arbitrary $X \in \Gamma(E_i)$, $Y \in \Gamma(E_j)$ with $i, j > 0$ that $[X, Y] (= \nabla_X Y - \nabla_Y X) \in \Gamma(E_0^{\perp})$. In the case $i = j$ we know that $[X, Y] \in \Gamma(E_i)$, since E_i is integrable. If $i \neq j$, then for every $Z \in \Gamma(E_0)$ we have $\nabla_X Z \in \Gamma(E_j^{\perp})$ because $E_0 \oplus E_i \subset E_j^{\perp}$ and the latter bundle is autoparallel. Therefore,

$$\langle \nabla_X Y, Z \rangle = X \cdot \langle Y, Z \rangle - \langle Y, \nabla_X Z \rangle = 0 ,$$

i.e. $\nabla_X Y \in \Gamma(E_0^\perp)$. Thus we get also $[X,Y] \in \Gamma(E_0^\perp)$.

For (c): Let $i \neq j > 0$, $X,\, Y \in \Gamma(E_i)$ and $Z \in \Gamma(E_j)$ be given. Then X and Y are sections of the autoparallel subbundle $E_j^\perp$, and thus we also have $\nabla_X Y \in \Gamma(E_j^\perp)$, which implies $\langle \nabla_X Y, Z\rangle = 0$. Therefore, formula (2) shows $H^{E_i} \in \Gamma(E_0)$.

We will derive further properties on WP-nets by use of the following result, which in the case $k = 2$ is due to Ponge/Reckziegel, see [PR] Proposition 3 (c). Of course the main importance of this result comes from assertion (b) of Proposition 3.

Proposition 4. *If on a product manifold* $M = \prod_{i=0}^{k} M_i$ *the product net* $\mathcal{E} = (E_i)_{i=0,\dots,k}$ *is a WP-net with respect to a pseudoriemannian metric* $\langle\cdot,\cdot\rangle$, *then* $(M, \langle\cdot,\cdot\rangle)$ *is a warped product* $M_0 \times_\rho \prod_{i=1}^{k} M_i$ *with respect to suitable pseudoriemannian metrics* $\langle\cdot,\cdot\rangle_i$ *on* M_i *and a* C^∞*-function* $\rho = (\rho_1, \dots, \rho_k) : M_0 \to \mathbb{R}_+^k$.

Sketch of the proof. We fix an $i \in \{1,\dots,k\}$ and write $M = M^i \times M_i$ with $M^i := \prod_{j\neq i} M_i$. The product net $\mathcal{E}_i = (E_i^\perp, E_i)$ of this representation of M is a WP-net again. Therefore, we can apply the $k = 2$ case of the result and obtain that there exist a function $\rho_i : M^i \to \mathbb{R}_+$ and pseudoriemannian metrics $\langle\cdot,\cdot\rangle^i$ and $\langle\cdot,\cdot\rangle_i$ on M^i and M_i, respectively, such that $(M, \langle\cdot,\cdot\rangle) = M^i \times_{\rho_i} M_i$. Moreover, the mean curvature vector field H_i of E_i equals $-\mathrm{grad}(\ln \circ \rho_i \circ \pi^i)$, where $\pi^i : M^i \times M_i \to M^i$ denotes the canonical projection. Since we know from Proposition 3 (c) that H_i is a section in E_0 the function ρ_i only depends on the points of M_0, hence we get

$$\forall\, X,\, Y \in \mathfrak{X}(M) \;:\; \langle X, Y\rangle \;=\; \langle \pi^i_* X, \pi^i_* Y\rangle^i \;+\; (\rho_i \circ \pi_0)^2 \cdot \langle \pi_{i*} X, \pi_{i*} Y\rangle_i \,.$$

Now it is not difficult to deduce that M is the warped product $M_0 \times_\rho \prod_{i=1}^{k} M_i$. Details will be published elsewhere.

The history of the following generalization of de Rham's decomposition theorem starts in 1979 with the article [H] of Hiepko, who proved the Riemannian version for $k = 2$. His result was generalized to arbitrary k by Nölker in [N] (another generalization can be found in Gauchman's article [G]) and to pseudoriemannian manifolds in the case $k = 2$ by Ponge/Reckziegel in [PR].

Theorem 7 (Decomposition of WP-netted manifolds). *If* $\mathcal{E} = (E_i)_{i=0,\dots,k}$ *is a WP-net on a pseudoriemannian manifold* $(M, \langle\cdot,\cdot\rangle)$, *then for every point* $p \in M$ *there exist a neighbourhood* U *of* p *in* M *and a pseudoriemannian net isomorphism from* $(U, \mathcal{E}|_U)$ *onto a suitable warped product* $M_0 \times_\rho \prod_{i=1}^{k} M_i$. *If* M *is simply connected and the leaves of the foliation tangential to* $E_i^\perp$ *are geodesically complete for every* $i > 0$, *then this statement is true with* $U = M$.

Proof. From Proposition 3 (b) we know that for every $p \in M$ there exist a neighbourhood $U = U(p) \subset M$ and a diffeomorphism f_U from U onto a product manifold $\prod_{i=0}^{k} M_i$, which is a net isomorphism with respect to $\mathcal{E}|_U$ and the product net $\mathcal{F}$ of $\prod_{i=0}^{k} M_i$. This net $\mathcal{F}$ is a WP-net on $\left(\prod_{i=0}^{k} M_i\,,\, (f_U^{-1})^* \langle\cdot,\cdot\rangle\right)$. Therefore, we can apply Proposition 4, which shows that this pseudoriemannian manifold is a warped product.

Now we assume that the hypotheses for the global statement are fulfilled. The assumption that for every $i > 0$ the subbundle $E_i^\perp$ is autoparallel allows to apply part (b) of Theorem 6 (see Remark 1), and in this way we find that in the first part of the proof we can choose $U = M$.

As WP-netted manifolds locally "are" warped products, as we have just seen, we get the following general results:

Proposition 3 (Second Part). *In the situation of the First Part of Proposition 3 the further assertion is true:*

(d) For $i \neq j > 0$ the mean curvature vector fields H_i and H_j satisfy

(13)
$$\langle H_i, H_j\rangle\langle X, X\rangle\,\langle Y, Y\rangle = -\langle R(X,Y)Y, X\rangle \quad \textit{for all} \quad X \in \Gamma(E_i)\,,\ Y \in \Gamma(E_j)\ ,$$

where R denotes the curvature tensor of M.

Proof. The assertion follows from the formula of the curvature tensor of a warped product, see [N], Lemma 2. □

5 Rigidity of warped product nets

The rigidity of totally geodesic submanifolds and of spherical submanifolds in arbitrary pseudoriemannian manifolds, which was proved in [MR1], yields the following result on the rigidity of warped product nets.

Theorem 8. *Let $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ and $\mathcal{F} = (F_i)_{i=0,\ldots,k}$ be two WP-nets on a connected pseudoriemannian manifold M and let $H_i^{\mathcal{E}}$ resp. $H_i^{\mathcal{F}}$ denote the mean curvature vector field of E_i resp. F_i for every i. If there exists at least one point $\bar p \in M$ such that*
$$\forall\, i = 0,\ldots,k\ :\ E_i(\bar p) = F_i(\bar p) \quad \textit{and} \quad H_i^{\mathcal{E}}(\bar p) = H_i^{\mathcal{F}}(\bar p)\ ,$$
then $\mathcal{E}$ and $\mathcal{F}$ coincide.

It should be pointed out that this result can also be derived immediately from Theorem 10.

Proof. We first prove the theorem in the case $k = 1$: For that we define the closed subset
$$A\ :=\ \{\, p \in M \mid E_1(p) = F_1(p)\ \text{ and }\ H_1^{\mathcal{E}}(p) = H_1^{\mathcal{F}}(p)\,\}\ .$$
Of course, we have also

$$\forall\, p \in A\ :\ E_0(p) = E_1(p)^\perp = F_1(p)^\perp = F_0(p)\ . \tag{14}$$

As M is supposed to be connected, we therefore have only to prove that A is open, too. For that let $p \in A$ be given. Now we will apply [MR1], Theorem 2 and its

Application and Remark. For that we notice that the leaves of any foliation are C$^\infty$-pathwise closed in the sense of this paper and that the leaves $L_1^{\mathcal{E}}(p)$ and $L_1^{\mathcal{F}}(p)$ are spherical submanifolds with the same tangent space and mean curvature normal at the point p (the latter is just expressed by $p \in A$). Hence, according to [MR1] we get $L_1^{\mathcal{E}}(p) = L_1^{\mathcal{F}}(p)$ and consequently $L_1^{\mathcal{E}}(p) \subset A$. Because of (14) we know also $E_0(q) = F_0(q)$ for every $q \in L_1^{\mathcal{E}}(p)$. Therefore we can apply again [MR1] in order to prove that the totally geodesic submanifolds $L_0^{\mathcal{E}}(q)$ and $L_0^{\mathcal{F}}(q)$ coincide. In this way we obtain

$$(15) \qquad E_0(p') = F_0(p') \quad \text{for every point} \quad p' \in G := \bigcup L_0^{\mathcal{E}}(q)\ ,\ q \in L_1^{\mathcal{E}}(p)\ .$$

Because of Proposition 3 (b) this set G is an open subset of M. Of course, (15) implies also $E_1(p') = F_1(p')$ for every $p' \in G$. Hence, the foliations $L_1^{\mathcal{E}}$ and $L_1^{\mathcal{F}}$ coincide on G implying $G \subset A$. Thereby we have completed the proof in the case $k = 2$.

In the case of an arbitrary k we apply the result of the foregoing proof to the WP-2-nets $\mathcal{E}_i = (E_i^\perp, E_i)$ and $\mathcal{F}_i = (F_i^\perp, F_i)$ for every $i > 0$ and obtain $E_i = F_i$ for every $i > 0$, which also implies $E_0 = \big(\bigoplus_{i=1}^k E_i\big)^\perp = \big(\bigoplus_{i=1}^k F_i\big)^\perp = F_0$.

The previous theorem has shown that a WP-net $\mathcal{E} = (E_i)_{i=0,\dots,k}$ is uniquely determined by the subspaces $E_1(p), \dots, E_k(p)$ and the correspondig mean curvature vectors $H_1(p), \dots, H_k(p)$ at a single point p. Therefore, it is a natural question to ask for the maximally expanded WP-net if such initial data are given at one point. On a “generic” n-dimensional pseudoriemannian manifold we can not expect to find any WP-net. For instance, on the complex space forms there do not exist any (locally defined) WP-nets in spite of their high symmetry. In contrast, the standard spaces of constant curvature are very rich in locally defined WP-nets. They were classified by Nölker in [N], Theorem 7:

Theorem 9. *Let $N := N_\varkappa^n$ be an n-dimensional standard space of constant curvature $\varkappa$ (by definition it is simply connected and complete). If then a point $p \in N$, an orthogonal splitting $T_pN = \bigoplus_{i=0}^k V_i$ and vectors $z_1, \dots, z_k \in V_0$ with $\langle z_i, z_j\rangle = -\varkappa$ for $i \neq j$ are given, then there exists an open dense neighbourhood G of p in N and on it a WP-net $\mathcal{F} = (F_i)_{i=0,\dots,k}$ with the initial data $F_i(p) = V_i$ and $H_i^{F_i}(p) = z_i$ for every $i > 0$; this WP-net cannot be expanded any further and the WP-netted manifold $(G, \mathcal{F})$ is globally decomposable, that means: There exists a warped product $N_0 \times_\sigma \prod_{i=1}^k N_i$ and a pseudoriemannian net isomorphism $\Phi : N_0 \times_\sigma \prod_{i=1}^k N_i \to (G, \mathcal{F})$.*

Every other WP-net, which is defined on an open subset of N, is part of a WP-net of the above type.

In [N] one can find an explicit construction of these “WP-representations” and a description of their symmetry properties. The best known examples are the spherical coordinate nets which we have mentioned in Example (b) of section 1.

For $N = \mathrm{N}_{\varkappa\neq 0}^n$ these “WP-representations” are excellent substitutes for the usual product representations of $\mathbb{R}^n$, see the introduction.

6 Decomposition of isometric immersions

The purpose of this section is to derive a decomposition of isometric immersions, which are defined on warped products. First we list the main properties of pseudoriemannian net morphisms, see Definition 2. We recall that every WP-netted manifold "is" locally a warped product so that the following proposition can locally be applied to pseudoriemannian net morphisms between arbitrary WP-netted manifolds.

Proposition 4. *Let* $M := M_0 \times_\rho \prod_{i=1}^k M_i$ *and* $N := N_0 \times_\sigma \prod_{i=1}^k N_i$ *be two warped products with their product nets* $\mathcal{E} = (E_i)_{i=0,\dots,k}$ *and* $\mathcal{F} = (F_i)_{i=0,\dots,k}$, *respectively, and* $f : (M, \mathcal{E}) \to (N, \mathcal{F})$ *a pseudoriemannian net morphism, which because of Theorem 2 has a representation* $f = f_0 \times \cdots \times f_k$. *Then the following holds true:*

(a) $f_0 : M_0 \to N_0$ *is an isometric immersion.*

(b) For every $i = 1, \dots, k$ *the map* $f_i : M_i \to N_i$ *is a* homothetical *immersion, that means: there exists a constant* $c_i \in \mathbb{R}_+$ *such that* $\langle f_{i*}X, f_{i*}Y\rangle_i = c_i^2 \cdot \langle X, Y\rangle_i$ *for all* $X, Y \in \mathfrak{X}(M_i)$. *By normalizing the metrics on the single factors* M_i *and* N_i *and the warping functions* ρ_i *and* σ_i *with respect to one point* $p_0 \in M_0$ *and its image* $f_0(p_0) \in N_0$ *we can get* $c_1 = \cdots = c_k = 1$; *in this way all immersions* f_i *become isometric.*

(c) $\rho_i \equiv c_i \cdot \sigma_i \circ f_0$ *with the constant* c_i *from assertion (b).*

Proof. The assertions follow immediately from the special kind of warped product metrics and the fact that f is isometric.

Proposition 5. *Let* M *and* N *be pseudoriemannian manifolds, which are endowed with WP-nets* $\mathcal{E} = (E_i)_{i=0,\dots,k}$ *resp.* $\mathcal{F} = (F_i)_{i=0,\dots,k}$. *Then every pseudoriemannian net morphism* $f : (M, \mathcal{E}) \to (N, \mathcal{F})$ *satisfies Condition* (D) *with respect to the Levi-Civita connections of* M *and* N *and the given nets (see section 2).*

Proof. We use that locally the netted manifolds M and N "are" warped products. Then the assertion can be derived from the representation of the second fundamental form of f given in [N], Lemma 12. □

In some sense the following theorem is the converse of the foregoing proposition; it generalizes an essential part of Nölker's decomposition theorem (noted in the introduction) to general WP-netted manifolds N.

Theorem 10. *Let* M *and* N *be a pseudoriemannian manifolds, which are equipped with WP-nets* $\mathcal{E} = (E_i)_{i=0,\dots,k}$ *and* $\mathcal{F} = (F_i)_{i=0,\dots,k}$, *respectively. We suppose that the netted manifold* $(M, \mathcal{E})$ *is connected and globally decomposable. Furthermore, let* $f : M \to N$ *be an isometric immersion satisfying the conditions* (D) *and* (I) *with respect to the Levi-Civita connections of* M *and* N, *the given nets and some point* $\bar{p} \in M$ *(see section 2). Then* f *is a pseudoriemannian net morphism, and therefore it is locally decomposable (as in Proposition 4).*

Proof.[The strategy of the proof] The hard part of the proof consists in the verification of the case $k = 1$. We will describe it at another place. If now k is greater than 1 and the theorem is known to be true for $k-1$ we continue in the following way: We define the WP-2-nets $\mathcal{E}' = \left(E_k^{\perp}, E_k\right)$ and $\mathcal{F}' = \left(F_k^{\perp}, F_k\right)$, the integral manifold $\tilde{M}$ of $E_k^{\perp}$ through the point $\bar{p}$ and the k-nets $\tilde{\mathcal{E}} := (E_i|_{\tilde{M}})_{i=0,\ldots,k-1}$ on $\tilde{M}$ and $\tilde{\mathcal{F}} := (F_0 \oplus F_k, F_1, \ldots, F_{k-1})$ on N. The restricted map $\tilde{f} := f|\tilde{M} : (\tilde{M}, \tilde{\mathcal{E}}) \to (N, \tilde{\mathcal{F}})$ satisfies the hypothesis of Theorem 10 and is therefore a pseudoriemannian net morphism. We can also construct the two nets $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ as "coarse" nets (in the sense of Theorem 4) assigned to the "fine" nets $\mathcal{E}'' := (E_0|_{\tilde{M}}, \ldots, E_{k-1}|_{\tilde{M}}, 0)$ (where 0 represents the trivial subbundle of $\tilde{M}$) and $\mathcal{F}$ by means of the partition $\{0, \ldots, k\} = \{0, k\} \cup \{1\} \cup \cdots \cup \{k-1\}$. Therefore, Theorem 4 proves that $\tilde{f}$ is also a net morphism $(\tilde{M}, \mathcal{E}'') \to (N, \mathcal{F})$, which implies that $\tilde{f}$ is $F_k^{\perp}$-integral, in other words: $f(L_0^{\mathcal{E}'}(\bar{p})) = \tilde{f}(\tilde{M}) \subset L_0^{\mathcal{F}'}(f(\bar{p}))$. The inclusion $f(L_1^{\mathcal{E}'}(\bar{p})) \subset L_1^{\mathcal{F}'}(f(\bar{p}))$ is identical with the hypothesis $f(L_k^{\mathcal{E}}(\bar{p}) \subset L_k^{\mathcal{F}}(f(\bar{p}))$. Thus, the induction hypothesis proves that f is a net morphism $(M, \mathcal{E}') \to (N, \mathcal{F}')$, and we can apply again Theorem 4 now using the partition $\{0, \ldots, k\} = \{0, \ldots, k-1\} \cup \{k\}$. In this way we find that f even is a net morphism with respect to $\mathcal{E}$ and $\mathcal{F}$.

Of course, there is to do some more work in order to check the assumptions of the theorems which we have applied. But that is rather easy. We will only notice that $\tilde{M}$ is totally geodesic and that therefore the second fundamental form of $\tilde{f}$ is obtained by restricting the second fundamental form of f to the tangent bundle $T\tilde{M}$.

Combining the Theorems 2 and 10 we obtain immediately

Corollary. *Let $f : M \to N$ be an isometric immersion between two warped products $M = M_0 \times_\rho \prod_{i=1}^{k} M_i$ and $N = N_0 \times_\sigma \prod_{i=1}^{k} N_i$. If M is connected, f satisfies the conditions* (D) *and* (I) *with respect to the Levi-Civita connections, the product nets of M and N and some point $\bar{p} \in M$ (see section 2) and if the warping functions are normalized by*

$$\forall\, i = 1, \ldots, k \ :\ \rho_i(\bar{p}) = \sigma_i(f(\bar{p})) = 1\ ,$$

then there exist isometric immersions $f_i : M_i \to N_i$ for $i = 0, \ldots, k$ such that $f = f_0 \times \cdots \times f_k$.

In Theorem 9 we have seen that on standard spaces of constant curvature there exist WP-nets, whose natural domain of definition is only an open subset of the space. The following result enables us to handle such situations in general.

Theorem 11. *Let $\mathcal{F} = (F_i)_{i=0,\ldots,k}$ be a WP-net defined on an open subset G of a pseudoriemannian manifold N and suppose that the netted manifold $(G, \mathcal{F})$ is globally decomposable. Furthermore, let $M := M_0 \times_\rho \prod_{i=1}^{k} M_i$ be a connected warped product, $\mathcal{E}$ its product net and $f : M \to N$ an isometric immersion satisfying the conditions* (D) *and* (I) *with respect to the Levi-Civita connections of M and N, the nets $\mathcal{E}$ and $\mathcal{F}$ and some point $\bar{p} \in M$ (see section 2). Then the image $f(M)$ lies completely* *in G. Moreover, if we have chosen a pseudoriemannian net isomorphism Φ from a warped product $\tilde{N} := N_0 \times_\sigma \prod_{i=1}^k N_i$ onto G, put $\bar{q} := \Phi^{-1}(f(\bar{p}))$ and if the warping functions are normalized by*

$$\forall\, i = 1, \ldots, k \ :\ \rho_i(\bar{p}) = \sigma_i(\bar{q}) = 1\ ,$$

then there exist isometric immersions $f_i : M_i \to N_i$ for $i = 0, \ldots, k$ such that

$$f = \Phi \circ (f_0 \times \cdots \times f_k)\ .$$

Proof. The essential new claim in this theorem is the inclusion $f(M) \subset G$; it is an immediate consequence of Theorem 5, which can be applied because of Thorem 10. All other assertions follow from the foregoing corollary applied to the map $\Phi^{-1} \circ f$.

At the end of the article we will return to Nölker's theorem noted in the introduction. In order to indicate the main steps for its proof we have to introduce the notion of the *spherical hull* of a subset A of a standard space $N^n_\varkappa$ of constant curvature $\varkappa$ (see Theorem 9): It is the (complete) spherical submanifold $S \subset N^n_\varkappa$ of smallest dimension containing the subset A. The existence and uniqueness of spherical hulls follow from the knowledge about spherical submanifolds of $N^n_\varkappa$. It should be mentioned that in general pseudoriemannian spaces one encounters difficulties with this notion. - The following result by Nölker (see [N], section 6) is crucial in his proof:

Theorem 12. *Let M be a connected pseudoriemannian manifold, which is equipped with a globally decomposable WP-net $\mathcal{E}$, i.e., M "is" a warped product, and let f denote an isometric immersion from M into a standard space $N^n_\varkappa$ of constant curvature $\varkappa$ satisfying Condition* (D) *with respect to the Levi-Civita connections of M and N and the net $\mathcal{E}$. If we choose a point $\bar{p} \in M$, put $\bar{q} := f(\bar{p})$, denote for $i > 0$ the spherical hull of the image $f(L_i^{\mathcal{E}}(\bar{p}))$ by S_i and put $V_i := T_{\bar{q}} S_i$, then V_i is perpendicular to V_j for all $i \neq j$, the mean curvature normal z_i of S_i at $\bar{q}$ lies in $V_0 := \left(\bigoplus_{i=1}^k V_i\right)^\perp$ for every $i > 0$ and $\langle z_i, z_j \rangle = -\varkappa$ for all $i \neq j > 0$. (Compare this result with Theorem 9.)*

A very personal comment: We understand Nölker's proof, and yet we find that the result is a wonder.

How Nölker's Thoerem of the introduction can be proved: Let an isometric immersion
$f \ :\ M_0 \times_\rho \prod_{i=1}^k M_i \ \to\ N^n_\varkappa$ satisfying Condition (D) be given. Then choose a point $\bar{p} \in M$.

1st step: Construct the "initial data" $(V_i)_{i=0,\ldots,k}$ and $(z_i)_{i=1,\ldots,k}$ by means of Theorem 12; in particular, $f(L_i^{\mathcal{E}}(\bar{p})) \subset S_i$ for every $i > 0$.

2nd step: Construct the warped product net $\mathcal{F} = (F_i)_{i=0,\ldots,k}$ on the largest possible open part G of $N^n_\varkappa$ belonging to the "initial data" $(V_i)_{i=0,\ldots,k}$ and $(z_i)_{i=1,\ldots,k}$ by means of Theorem 9; in particular, $S_i = L_i^{\mathcal{F}}(\bar{q})$ for every $i > 0$; prove also $f(L_0^{\mathcal{E}}(\bar{p})) \subset L_0^{\mathcal{F}}(f(\bar{p}))$. Hence, Condition (I) is satisfied by f with respect to the product net of $M_0 \times_\rho \prod_{i=1}^k M_i$ and $\mathcal{F}$.

3rd step: Apply Theorem 11.

End of proof.

This schedule shows which parts of the proof are specific for the standard spaces of constant curvature and which parts are of general character. So one can see what one has to do in order to carry over Nölker's main theorem to other situations.

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University of Cologne,
Department of Mathematics,
Weyertal 86 - 90
D-50931 Köln , Germany
E-mail: reckziegel@mi.uni-koeln.de
mschaaf@mi.uni-koeln.de