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Bernstein polynomial is a best approximation polynomial?

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Let $Y = C^{n+1}[a,b]$, $x_0, ..., x_n,$ a system of distinct points from $[a,b]$, and

$$I(f,g) := \int_a^b f^{(n+1)}(t)g^{(n+1)}(t)\,dt \quad ,$$

$$< f, g >_1 := \sum_{k=0}^{n} [x_0, x_1, ..., x_n\,; f][x_0, x_1, ..., x_n\,; g] + I(f,g) \ ,$$

$$< f, g >_2 := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)g^{(k)}(x_0)}{(k!)^2} + I(f,g) \ .$$

Further

$$(N_n f)(x) = \sum_{k=0}^{n} (x - x_0)(x - x_1)...(x - x_{k-1})[x_0, x_1, ..., x_k\,; f] \ ,$$

$$(T_n f)(x) \ := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \ .$$

It is known that $H_j = (Y, < .,. >_j\ )$, $j = 1, 2,$ are inner-product spaces. If $f \in Y$ and Π_n is the linear subspace of

Y which includes all real polynomials of the degree $\leq n$, then it is known (see for instance [1]) that for all $h,\ h \in \Pi_n$,

$$\|f - N_n f\|_1 \leq \|f - h\|_1 \quad ,$$

$$\|f - T_n f\|_2 \leq \|f - h\|_2 \quad ,$$

with

$$\|f\|_j = \sqrt{< f, f >_j} \ .$$

This means that in the above sense, these well known approximation operators furnish best approximation processes.

Problem. *Find a similar result for the Bernstein operators.*

This means , that we must find a scalar product $< .,. >_3$ *in a certain space* $Y_1, \Pi_n \subset Y_1 \subseteq C[0,1]$, *then a system of polynomials* $\{\varphi_0, \varphi_1, ..., \varphi_n\}$ *such that*

$$< \varphi_j, \varphi_k >_3 = \ \delta_{j,k}$$

and

$$(B_n f)(x) = \sum_{k=0}^{n} < f, \varphi_k >_3 \varphi_k(x) \ .$$

The same problem for Stancu operators [2].

References

[1] A. Lupaş, *Asupra unor polinoame de aproximare,* Gazeta Matematică seria A, anul **VIII** nr.2, (1987) 63-66.

[2] D.D.Stancu, *Approximation of functions by a new class of linear polynomial operators,* Rev.Roumaine Math.Pures Appl., **13** (1968) 1173-1194.