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A condition for convexity

Gheorghe Oros

Abstract

In this paper we obtain a sufficient condition for convexity, by using the modulus of $f'(z) + \alpha z f''(z) - 1$, where $f \in A_n$ and $\alpha > 2$.

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1 Introduction and preliminaries

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of analytic functions in the unit disc U. A function $f \in \mathcal{H}$ is said to be convex if it is univalent and $f(U)$ is a convex domain. It is well known that the function f is convex if and only if $f'(0) \neq 0$ and
Re $[zf''(z)/f'(z) + 1] > 0$, for $z \in U$

Let A_n denote the class of functions f of the form:

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \quad z \in U,$$

which are analytic in the unit disc U.

Let $A_1 = A$ and let

$$K = \left\{ f \in A : \ \mathrm{Re}\, \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\}$$

denote the class of normalized convex functions in U.

If f and g are analytic functions in U and g is univalent, then f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We shall use the following lemma to prove our result.

Lemma A. (See [1]). *Let G be a convex function in U, with $G(0) = a$, and let γ be a complex number, with* $\mathrm{Re}\,\gamma > 0$. *If $F(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ is analytic in U and $F \prec G$ then*

$$\frac{1}{z^\gamma} \int_0^z F(w) w^{\gamma - 1} dw \prec \frac{1}{n z^{\gamma/n}} \int_0^z G(w) w^{\left(\frac{\gamma}{n} - 1\right)} dw.$$

2 Main results

Theorem. *Let $f \in A_n$, $M \leq M_n(\alpha)$, $\alpha > 2$, where*

$$M_n(\alpha) = \frac{(\alpha - 2)(n\alpha + 1)}{\alpha(n + 1)}. \tag{1}$$

If

$$|f'(z) + \alpha z f''(z) - 1| < M, \tag{2}$$

then $f \in K$.

Proof. From (2) we deduce

$$f'(z) + \alpha z f''(z) \prec 1 + Mz, \tag{3}$$

with $M \leq M_n(\alpha)$ where $M_n(\alpha)$ is given by (1).

Letting $f'(z) = P(z)$, the subordination (3) becomes

$$P(z) + \alpha z P'(z) \prec 1 + Mz \equiv h(z).$$

By using Lemma A we deduce that $P(z)$ is subordinate to $Q(z)$, given by:

$$Q(z) = \frac{1}{n\alpha z^{\frac{1}{n\alpha}}} \int_0^1 (1 + Mz) z^{\left(\frac{1}{n\alpha} - 1\right)} dt = 1 + \frac{M}{n\alpha + 1}. \tag{4}$$

The fact that $P(z) \prec Q(z)$ is equivalent to:

$$|P(z) - 1| < \frac{M}{n\alpha + 1}. \tag{5}$$

If we let

$$R = \frac{M}{n\alpha + 1}, \tag{6}$$

then from (1) we deduce $R < 1$.

Let

$$1+\frac{zf''(z)}{f'(z)}=p(z).$$

We have

$$zf''(z)=[p(z)-1]f'(z)$$

and we deduce

$$zP'(z)=[p(z)-1]P(z).$$

By using this in (2) we obtain:

$$|P(z)+\alpha[p(z)-1]P(z)-1|<M \tag{7}$$

and we have to show that this relation implies

$$\text{Re } p(z)>0.$$

If we suppose that Re $p(z)\not>0$, then there is a $z\in U$ such that $p(z)=is$, $s\in\mathbb{R}$.

To show that relation (7) implies Re $p(z)>0$ it is sufficient to verify the inequality:

$$|P(z)+\alpha(is-1)P(z)-1|\geq M \tag{8}$$

for all real s and $z\in U$.

The inequality (8) is equivalent to

$$|P(z)|^2[(1-\alpha)^2+\alpha^2s^2]-2\text{Re } P(z)(1-\alpha+\alpha is)+1\geq M^2 \tag{9}$$

If we let

$$E = |P(z)|^2[(1-\alpha)^2 + \alpha^2 s^2] - 2\text{Re}\ P(z)(1-\alpha+\alpha i s) + 1$$

then we have

$$E \geq |P(z)|^2|1-\alpha+\alpha i s| - 2|P(z)| \cdot |1-\alpha+\alpha i s| + 1 =$$

$$= \left[|P(z)|\sqrt{(1-\alpha)^2+\alpha^2 s^2} - 1\right]^2 \geq [|P(z)|(\alpha-1)-1]^2.$$

By using this in (9) we have

$$[|P(z)|(\alpha-1)-1]^2 \geq M^2,$$

which holds if

$$|P(z)|(\alpha-1) \geq 1+M$$

or

$$|P(z)| \geq \frac{1+M}{\alpha-1}. \tag{10}$$

From (5) and (6) we deduce

$$|P(z)| > 1-R.$$

Hence the inequality (10) holds if

$$1-R \geq \frac{1+M}{\alpha-1}. \tag{11}$$

It is easy to check that the inequality (11) is satisfied if

$$M \leq \frac{(\alpha - 2)}{\alpha(n+1)}(n\alpha + 1) = M_n(\alpha).$$

Hence the inequality (8) holds and we deduce that $\mathrm{Re}\, p(z) > 0$, which shows that $f \in K$.

From the Theorem we have that, if $f \in A_n$ and satisfies the condition

$$\left| \frac{f(z)}{\alpha} + zf''(z) - \frac{1}{\alpha} \right| < \frac{(\alpha - 2)(n\alpha + 1)}{(n+1)\alpha^2}$$

with $\alpha > 2$, then $f \in K$.

If we make $\alpha \to \infty$ we obtain the following

Corollary. (See[2]). *If $f \in A_n$ and*

$$|f''(z)| \leq \frac{n}{n+1}$$

then $f \in K$.

References

[1] D.J. Hallenbeck and S. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc., 52(1975), 191-195.

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Department of Mathematics
University of Oradea
Str.Armatei Române, No.5
3700 Romania