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Typically real harmonic functions

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Abstract

In this paper, the two parameter family of classes of tipically real harmonic functions are constructed by the Hadamard product in the open disk. Sharp coefficient relations and some properties are given for these functions.

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1 Introduction

Clunie and Sheil-Small [1] studied the class S_H of all harmonic complex valued, sense preserving univalent mappings f defined on the open unit disk U which are normalized by $f(0) = f_z(0) - 1 = 0$. Such functions admit the representation $f = h + \overline{g}$ where

$$(1) \qquad h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad and \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in U. f is locally one to one and sense preserving if and only if $\mid g'(z) \mid < \mid h'(z) \mid$ for every $z \in U$.

A harmonic mapping f is said to be typically real if $f(z)$ is real if, and only if, z is real. In the same article, the authors introduce the class T_H of all locally univalent sense preserving mappings $f = h + \overline{g}$ which are typically real and normalized by $f(0) = 0$, $\mid h'(0) \mid = 1$ and $f(r) > 0$ for $0 < r < 1$. The subclass T_H^0 of T_H consist of all mappings in T_H which have the additional property $g\prime(0) = 0$. If $f \in T_H$ then $(Imz)(Imf(z)) > 0$ for all $z \in U \backslash \mathbb{R}$.

In this paper, investigations of a new family of classes of typically real harmonic functions which is constructed by employing the Hadamard product.

2 The Class $T_H(\alpha, \beta)$

Let α and β be fixed real numbers different from $-1, -1/2$, $-1/3, ...$ and let

$$(2) \quad t_\alpha = \sum_{n=1}^{\infty} \frac{1}{1+(n-1)\alpha} z^n \ , \ t_\beta(z) = \sum_{n=1}^{\infty} \frac{1}{1+(n-1)\beta} z^n$$

for $z \in U$. Of course, $t_\alpha(z)$ and $t_\beta(z)$ are analytic functions in U.

Denote by $T_H(\alpha, \beta)$ the class of all functions of the form

$$F = f * (t_\alpha + \overline{t_\alpha}) * (t_\beta + \overline{t_\beta}) \tag{3}$$

where $f \in T_H$ and $f * (t_\alpha + \overline{t_\alpha})$ stands for the Hadamard product of the functions f and $t_\alpha + \overline{t_\alpha}$. Thus from (1), (2) and (3) we obtain that if $F \in T_H(\alpha, \beta)$, then

$$F(z) = z + \sum_{n=2}^{\infty} \frac{a_n}{[1+(n-1)\alpha][1+(n-1)\beta]} z^n + \tag{4}$$
$$+ \sum_{n=1}^{\infty} \frac{\overline{b}_n}{[1+(n-1)\alpha][1+(n-1)\beta]} \overline{z}^n =$$
$$= z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n},$$

$z \in U$ and, conversely, if F is of the form (4) where $a_2, ..., a_n, ...$ and $b_1, b_2, ..., b_n, ...$ are the corresponding coefficients of a function $f \in T_H$, then $F \in T_H(\alpha, \beta)$. Let $T_H^0(\alpha, \beta)$ denote the subclass of $T_H(\alpha, \beta)$ with $B_1 = 0$. It is possible to transform between these classes. Namely, if $F \in T_H(\alpha, \beta)$, then $F_0 = (F - \overline{B_1 F})/(1- \mid B_1 \mid^2) \in T_H^0(\alpha, \beta)$, and conversely, if $F_0 \in T_H^0(\alpha, \beta)$, then $F = F_0 + \overline{B_1 F} \in T_H(\alpha, \beta)$.

It is clear that,

$$T_H(0,0) = T_H,$$

$$T_H(\alpha, \beta) = T_H(\beta, \alpha),$$

$$T_H(\alpha,\beta) \subset T_H \quad for \quad \alpha,\beta \geq 0$$

and

$$T_H(\alpha,0) = \widetilde{T_H}(\alpha),$$

where $\widetilde{T_H}(\alpha) = f * (t_\alpha + \overline{t_\alpha}) \mid f \in T_H$ and this class is studied in (4).

Theorem 1 *If $F = H + \overline{G} \in T_H(\alpha,\beta)$, then there exists $f \in T_H$ such that*

$$\alpha\beta[z^2H''(z) + \overline{z^2G''(z)}] + [\alpha + \beta(1-\alpha)] \tag{5}$$

$$[zH'(z) + \overline{zG'^{(z)}}] + (1-\alpha)(1-\beta)F(z) = f(z)$$

and, conversely, for any function $f \in T_H$, the solution of equation (5) of form (4) belongs to the class $T_H(\alpha,\beta)$.

Proof. Let $F \in T_H(\alpha,\beta)$. Since

$$\alpha z t'_\alpha(z) + (1-\alpha)t_\alpha(z) = t_0 z$$

and

$$\beta z t'_\beta(z) + (1-\beta)t_\beta(z) = t_0(z),$$

we get

$$f(z) = h(z) + \overline{g(z)} = f(z) * [\alpha(zt'_\alpha(z) + \overline{zt'_\alpha(z)}) + (1-\alpha)(t_\alpha(z) + \overline{t_\alpha(z)})]$$

$$*[\beta(zt'_\beta(z) + \overline{zt'_\beta(z)}) + (1-\beta)(t_\beta(z) + \overline{t_\beta(z)})]$$

for $f \in T_H$.

Hence we have

$$h(z) = \alpha\beta h(z) * zt'_\alpha(z) * zt'_\beta(z) + \alpha(1-\beta)h(z) * zt'_\alpha(z) * t_\beta(z)$$

$$+\beta(1-\alpha)h(z) * t_\alpha(z) * zt'_\beta(z) + (1-\alpha)(1-\beta)h(z) * t_\alpha(z) * t_\beta(z)$$

and

$$g(z) = \alpha\beta g(z) * zt'_\alpha(z) * zt'_\beta(z) + \alpha(1-\beta)g(z) * zt'_\beta(z) * t_\beta(z)$$

$$+\beta(1-\alpha)g(z) * t_\alpha(z) * zt'_\beta(z) + (1-\alpha)(1-\beta)g(z) * t_\alpha(z) * t_\beta(z).$$

Thus,

$$h(z) = \alpha\beta z^2 H''(z) + [\alpha + \beta(1-\alpha)]zH'(z) + \tag{6}$$

$$+(1-\alpha)(1-\beta)H(z)$$

and

$$g(z) = \alpha\beta z^2 G''(z) + [\alpha + \beta(1-\alpha)]zG'(z) + \tag{7}$$

$$+(1-\alpha)(1-\beta)G(z).$$

Consequently, we obtain (5).

Conversely, let $f \in T_H$. Then from (6) and (7), H and G are a pair of solution for the equation given in (5). Furthermore, H and G are analytic in U , since h and g are analytic in U. Hence we have

$$H(z)=\sum_{n=0}^{\infty}A_nz^n \quad and \quad G(z)=\sum_{n=0}^{\infty}B_nz^n.$$

From (6) and (7),

$$z+\sum_{n=2}^{\infty}a_nz^n=\sum_{n=0}^{\infty}[1+(n-1)\alpha][1+(n-1)\beta]A_nz^n$$

and

$$\sum_{n=1}^{\infty}b_nz^n=\sum_{n=0}^{\infty}[1+(n-1)\alpha][1+(n-1)\beta]B_nz^n.$$

Thus,

$$H(z)=h(z)*t_\alpha(z)*t_\beta(z),\; G(z)=g(z)*t_\alpha(z)*t_\beta(z)$$

and we get

$$F=H+\overline{G}=(h+\overline{g})*(t_\alpha+\overline{t}_\alpha)*(t_\beta+\overline{t}_\beta)\in T_H(\alpha,\beta).$$

By Theorem (1) and [5, Theorem 3.4] we get in the following result.

Corollar 1 $F=H+\overline{G}\in T_H(\alpha,\beta)$ *if and only if*

$$Re\{(1-z^2)[\alpha\beta z(H''(z))+(\alpha+\beta(1-\alpha))z(H'(z)-G'(z))$$

$$+(1-\alpha)(1-\beta)(H(z)-G(z))/z]\}\ >0$$

for $z\in U,\ A_n=\overline{A}_n$ *and* $B_n=\overline{B}_n, n=1,2,...$

Corollar 2 $F = H + \overline{G} \ \in T_H(1,1)$ *if and only if* $zF_z(z) + \overline{z}F_{\overline{z}}(z) \ \in \ \overline{T}_H(1)$.

Proof. If $F \in T_H(1,1)$, then by Theorem 1,

$$z^2H''(z) + \overline{z^2G''(z)} + zH'(z) + \overline{zG'(z)} \ \in T_H.$$

Hence, we have

$$(z^2H''(z) + zH'(z) + \overline{z^2G''(z) + zG'(z)}) * (t_1(z) + \overline{t_1(z)}) \ \in \widetilde{T}_H(1),$$

$$(z^2H''(z) + zH'(z)) * t_1(z) = zH'(z),$$

$$(z^2G''(z) + zG'(z)) * t_1(z) = zG'(z)$$

and $zH'(z) + \overline{zG'(z)} \ \in \widetilde{T}_H(1)$.

Conversely, if $zF_z(z) + \overline{z}F_{\overline{z}}(z) \in \widetilde{T}_H(1)$, then by [4, 8]

$$z^2H''(z) + zH'(z) + \overline{z^aG''(z) + zG'(z)} \in T_H.$$

Thus, from Theorem 1, $F \ \in T_H(1,1)$.

Theorem 2 *Each of the classes $T_H^0(\alpha,\beta)$ is convex and compact. Moreover, each member of $T_H^0(\alpha,\beta)$ is sense preserving.*

Proof. Since T_H^0 is convex and compact [1], $T_H^0(\alpha,\beta)$ is convex and compact.As each member of T_H^0 is serving, it follows that each member of $T_H^0(\alpha,\beta)$ is also sense preserving.

Theorem 3 *If $F \in T_H(\alpha, \beta);\ \alpha, \beta > 0$, then there exists $f \in T_H$ such that*

$$F(z) = \frac{1}{\alpha\beta}\int_0^1 \zeta^{\frac{1}{\alpha}+\frac{1}{\beta}-4} f(\zeta^2 z)d\zeta, z \in U \tag{8}$$

and the converse is also true.

Proof. In an elementary way, the following integral formulas are obtained:

$$t_\alpha(z) = \frac{1}{\alpha}\int_0^1 \zeta^{\frac{1}{\alpha}-1}\frac{z}{1-z\zeta}d\zeta \quad t_\beta(z) = \frac{1}{\beta}\int_0^1 \zeta^{\frac{1}{\beta}-1}\frac{z}{1-z\zeta}d\zeta$$

for $z \in U$, $|\zeta| \leq 1$ and $\alpha, \beta > 0$. Thus, if $F \in T_H(\alpha, \beta)$, then there exists $f \in T_H$ such that

$$F(z) = f(z) * (t_\alpha(z) + \overline{t_\alpha(z)}) * (t_\beta(z) + \overline{t_\beta(z)}).$$

Since

$$f(z) * \frac{z}{1-z\zeta} * \frac{z}{1-z\zeta} = \frac{f(\zeta^2 z)}{\zeta^2},$$

the proof is complete.

Theorem 4 *If $F \in T_\alpha^0$ and $R \in T$, then $F*(R+\overline{R})$ belongs to $T_\alpha^0(\alpha, \beta)$in the disk $|z| < 2-\sqrt{3}$ and this bound is sharp, where T stands for the well known class of typically real analytic functions.*

Proof. Let $F = H + \overline{G} = (h * t_\alpha * t_\beta) + (\overline{h * t_\alpha * t_\beta})$ $\in T_H^0(\alpha,\beta)$, $f = h + \overline{g} \in T_H^0$ and $R \in T$. From [5, Theorem 3.2], $h - g \in T$. Moreover, by [2, II, Theorem 1.3], $(h-g)*R \in T$ in the disk $|z| < 2-\sqrt{3}$ and this bound is sharp. By [5, Theorem 3.2],we get $(h + \overline{g} * (R + \overline{R}) \in T_H^0)$. Hence,

$$F*(R+\overline{R}) = [(h+\overline{g})*(R+\overline{R})]*(t_\alpha+\overline{t}_\alpha)*(t_\beta+\overline{t}_(\beta)) \in T_H^0(\alpha,\beta)$$

in the disk $|z| < 2-\sqrt{3}$.

Theorem 5 *If*

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B_n z^n} \in T_H^0(\alpha,\beta),$$

then for $n = 1, 2, ...$

$$|A_n| \leq \frac{(n+1)(2n+1)}{6|1+(n-1)\alpha||1+(n-1)\beta|}$$

and

$$|B_n| \leq \frac{(n-1)(2n-1)}{6|1+(n-1)\alpha||1+(n-1)\beta|}.$$

Equality occurs for the function $k = k_0 * (t_\alpha + \overline{t}_\alpha) * (t_\beta + \overline{t}_\beta)$ *with Koebe harmonic function*

$$k_0(z) = Re\left(\frac{z+(1/3)z^3}{(1-z)^3}\right) + iIm\left(\frac{z}{(1-z)^2}\right).$$

Proof. It is known [1, Theorem 6.4] that sharp inequalities

$$|a_n| \leq \frac{1}{6}(n+1)(2n+1) \ \ and \ \ |b_n| \leq \frac{1}{6}(n-1)(2n-1); n \geq 2$$

are true for

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \in T_H^0.$$

Hence, by

$$|A_n| = \frac{|a_n|}{|1 + (n-1)\alpha||1 + (n-1)\beta|}$$

and

$$|B_n| = \frac{|b_n|}{|1 + (n-1)\alpha||1 + (n-1)\beta|} \ ,$$

we have the proof of the theorem.

Remark 1 *Evidently, one can easily repeat the above reasoning for a finite number of times and introduce, respectively, the families* $T_H(\alpha_1, \alpha_2, ..., \alpha_n)$.

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