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# On the $n$-uniformly close to convex functions with respect to a convex domain

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**Abstract**

Using the differential operator $D^n f$ introduced by G. Sălăgean and the results given in [2] and [3] on the several types of close-to-convex functions, in this paper I define new sets of univalent functions called $n$–uniformly close-to-convex with respect to a convex domain. In the definition of this functions it is specified which is the convex domain, symmetrical to the real axis, for example: elliptic, parabolic, hyperbolic region, or half plane. A certain analogy with the uniformly convex functions and with the functions of the class $S_p$ defined by Goodman and by Frode Ronning respectively, justifies the name "uniformly close-to-convex".

In the second part, the intermediate classes of Mocanu type are also being defined.

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# 1 Introduction

Let

$$A = \{f / f \in \mathcal{H}(U)\ ,\ f(0) = 0\ ,\ f'(0) = 1\}$$
$$S = \{f \in A\ ,\ \text{and } f \text{ is univalent}\}\ .$$

Let $D^n$ the Sălăgean differential operator defined as:

**Definition 1 .** *$D^n : \mathcal{H}(U) \longrightarrow \mathcal{H}(U)$ and*

*(i)* $D^0 f(z) = f(z)$

*(ii)* $D^1 f(z) = Df(z) = zf(z)$

*(iii)* $D^n f(z) = D(D^{n-1} f(z))$.

We denote by $C(\alpha)$, $S^*(\alpha)$ and $CC(\alpha)$ the well-known subclass of $S$: convex, starlike and close-to-convex functions of order $\alpha$, in other words, with respect to a half-plane ($Re\, w > \alpha$). For example,

$$CC(\alpha) = \left\{ f \in A\, /\, Re \frac{f'(z)}{g'(z)} > \alpha\ ,\ g \in C(0)\, ,\, \alpha > 0\ z \in U \right\}\ .$$

Using the operator $D^n$, Gr. Sălăgean [15, 16] defines the set of $n$–starlike of order $\alpha$ functions noted $S_n(\alpha)$

$$S_n(\alpha) = \left\{ f \in A\, /\, Re \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha\ ,\ z \in U\ ,\ \alpha \in [0,1)\ ,\ n \in N_0 \right\}$$

where $N_0 = \{0, 1, 2, \ldots\}$.

**Remark 1 .** *If $f \in S_n(\alpha)$ then according to the Definition 1 we can write*

$$Re \frac{z(D^n f(z))'}{D^n f(z)} > \alpha \quad , \quad z \in U$$

*therefore the function $F(z) = D^n f(z)$ belongs to $S^*(\alpha)$, $\alpha \in [0,1)$.*

The main results of this paper have been obtained by using the well-known "admissible functions method" introduced by S.S. Miller and P.T. Mocanu. I need the following special cases included in the theorems:

**Theorem A** [11, 12]. *Let $q$ be the convex in $U$ and let $P : U \longrightarrow C$ with $Re\, P(z) > 0$. If $p$ is analytic in $U$, then*

$$p(z) + P(z) z p'(z) \prec q(z) \Longrightarrow p(z) \prec q(z)\ .$$

**Theorem B** [4]. *Let $q$ be convex in $U$ with $Re\,[\beta q(z)+\gamma]>0$. If $p$ is analytic in $U$ with $p(0)=q(0)$ then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec q(z)\Longrightarrow p(z)\prec q(z)\ .$$

## 2 Preliminary results

In [1, 2, 3] the results are being given on so-called $n$-close-to-convex functions of order $\alpha$ with respect to a half-plane (or of Kaplan type) $CCK_n(\alpha)$ and $n$-close-to-convex functions of order $\alpha$ with respect to a sector (or of Rény type, or named later, strongly close-to-convex functions) noted $CCR_n(\alpha)$.

**Definition 2.** *The function $f\in A$ belongs to set $CCK_n(\alpha)$ if the differential expression $D^{n+1}f(z)/D^ng(z)$ take values in the halfplane $Re\,w>0$, that is*

$$CCK_n(\alpha)=\left\{f\in A/Re\frac{D^{n+1}f(z)}{D^ng(z)}>\alpha, if\ g\in S_n(0), n\in N_0, \alpha\in[0,1), z\in U\right\}.$$

**Remark 2.** *For $n=0$, $CCK_0(\alpha)=CC(\alpha)$.*

**Definition 3.** *The function $f$ is called $n$-close-to-convex of order $\gamma$ with respect to a sector if it verifies the following conditions*

$$CCR_n(\gamma)=\left\{f\in A\ /\ \left|arg\frac{D^{n+1}f(z)}{D^ng(z)}\right|\le\frac{\pi}{2}\gamma\ ,\ \gamma\in[0,1),\ g\in SR_n(0)\right\}\ .$$

*For $n=0$ the above definition can by expressed also in the form: $f(z)\in CCR_0(\gamma)$ if for every $0\le\theta_1<\theta_2\le 2\pi$ we have*

$$\int_{\theta_1}^{\theta_2}\left(1+Re\frac{zf''(z)}{f'(z)}\right)d\theta>-\pi\gamma\quad,\quad z=re^{i\theta}\ ,\ r\in(0,1)\ ,\ \gamma\in[0,1)\ ,$$

*which has got the well-known geometric characterization.*

Let now have an univalent function $q(z)$, $q(0)=1$, $q'(0)>0$ which maps the unit disc $U$ into a symmetrical domain with respect to real axis.

Let $\mathcal{P}(q)$ the family of holomorphic functions $p$ in $U$ so that $p(0)=1$ and $p(U)\subseteq q(U)$, in other words, $p\prec q$.

We note by $S^C(q)$ and $S^*(q)$ the classes of all univalent functions for which we have got

$$1+\frac{zf''(z)}{f'(z)}\in\mathcal{P}(q) \quad \text{respectively} \quad \frac{zf'(z)}{f(z)}\in\mathcal{P}(q)\ .$$

The connection between the functions of $S^C(q)$ and $S^*(q)$ is given by a theorem of Alexander-type.

**Theorem 1.** *The function $f(z)\in S^C(q)$ if and only if $zf'(z)\in S^*(q)$ where $q(z)$, $S^C(q)$ and $S^*(q)$ verify the above conditions.*

**Proof.** By a simple classical calculus the conclusion of Theorem 1 follows.

**Definition 4.** *The function $f\in A$ is $n$–starlike with respect to convex domain $\mathcal{D}$ if the differential expression $D^{n+1}f(z)\,/\,D^nf(z)$ takes values in the domain $\mathcal{D}$ or*

$$\frac{D^{n+1}f(z)}{D^nf(z)}\prec q(z) \quad ; \quad q(U)=\mathcal{D}\ .$$

We can note by $S_n^*(q)$ the set of all these functions.

**Remark 3.** The special set $S_n^*\left(\dfrac{1+(1-2\alpha)z}{1-z}\right)$ called $n$–starlike of order $\alpha$ was studied by Gr. Sălăgean [15, 16]. The set $S_n^*\left[\left(\dfrac{1+z}{1-z}\right)^\alpha\right]$ $n$–starlike of Rény type was defined in [1]. Let be the set noted $S_n^*(Q(z))$ where $Q(z)=1+\dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$ which maps the unit disk $U$ in the domain $\Omega$ bounded by a parabola

$$\Omega=\{w:|w-1|<Re\,W\}=\{W=u+iv\,,\ v^2=2u-1\}\ .$$

Frode Ronning [13, 14], Ma and Minda [9] have independently introduced the class $S_p$, where $S_p = S_0^*[Q(z)]$. For $n = 1$ $S_1^*[Q(z)] = UCV$ the well-known set of uniformly convex functions, which was introduced by Goodman [5]. I.C. Magdaş [10] has studied the classes $S_n^*[Q(z)]$ for general $n$.

S. Kanas and A. Wisniowska [6, 7] have introduced and studied in detail the classes of $K$–uniformly convex and related classes of $K$–starlike functions $(0 \leq K < \infty)$ denoted $K - UCV$ and $K - ST$ for which the values of expressions $\dfrac{zf'(z)}{f(z)}$ and $1 + \dfrac{zf''(z)}{f'(z)}$ lie inside the conic regions respectively. Using the Sălăgean differential operator $D^n f$, mentioned before, S. Kanas and Teuro Yaguchi have then introduced and extensively studied some subclasses of $K - UCV$ and $K - ST$ [8].

## 3 Main results

**1.** A general family of close-to-convex functions.

**Definition 5.** *Let $q(z)$ be an univalent function $q(0) = 1$, $Re\, q(z) > 0$, $q'(0) > 0$ which maps the unit disc $U$ into convex domain $\mathcal{D}$ symmetrical with respect to the real axis.*

*Let be $f \in A$, we say that $f$ is $n$–close-to-convex with respect to $\mathcal{D}$, or $n$–close-to-convex subordinated to function $q$, if there exists a function $q \in S_n^*(q)$ such that*

$$\frac{D^{n+1}f(z)}{D^n g(z)} \prec q(z) \quad , \quad z \in U \ , \ n \in \mathbb{N}$$

We can note by $CC_n(q)$ the set of all these functions.

**Remark 4.** *From the above definition it easily results that $q_1(z) \prec q_2(z)$ implies $CC_n(q_1) \subset CC_n(q_2)$.*

**Theorem 2.** *If $n \in \mathbb{N}_0$ and $f \in CC_{n+1}(q)$ then $f \in CC_n(q)$. That is*

$$CC_{n+1}(q) \subset CC_n(q) \ .$$

**Proof.** With the notation $\dfrac{D^{n+1}f(z)}{D^n g(z)} = p(z)$ we have $\dfrac{D^{n+2}f(z)}{D^{n+1}g(z)} = p(z) + \dfrac{1}{h(z)} zp'(z)$ where $h(z) = \dfrac{D^{n+1}g(z)}{D^n g(z)}$.

According to the Definition 4 and 5 it comes that $Re\, h(z) \geq 0$. It is easy to observe that

$$\psi(r,s) = r + \frac{1}{h(z)} s = p(z) + \frac{1}{h(z)} zp'(z)$$

is an admissible function according to the definition given by P.T. Mocanu and S.S. Miller [11, 12]. By dint of admissible function theory, see Theorem A, it follows that

$$p(z) + \frac{1}{h(z)} zp'(z) \prec q(z) \quad \text{implies} \quad p(z) \prec q(z)$$

that means the conclusion of Theorem 2.

**Corollary 2.** *The following inclusions are true:*

$$CC_{n+1}(q) \subset CC_n(q) \subset CC_0(q) \subset CC$$

*where $CC$ - is the set of classical close-to-convex functions defined by Kaplan which are univalent. Hence we can assert that the sets $CC_n(q)$ contains only univalent functions.*

**2.** If we choose for $q(z)$ some functions which maps the unit disc $U$ into a domain bounded by a parabola, or elipsa, or hyperbola respectively, we obtain the special close-to-convex function which improves several previous results.

Since these functions have connections with the uniformly convex functions $USC$ and with the uniformly starlike functions $UST$ and with the set $SP$, we call them $n$–uniformly close-to-convex functions.

**Definition 6.** *A function $f \in A$ is $n$-uniformly close-to-convex of order $\gamma$ and type $\alpha$ where $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, and $n \in \mathbb{N}_0$ if there exists* *a function* $g \in US_n(\alpha, \gamma)$ *so that*

$$Re\, \frac{D^{n+1}f(z)}{D^n g(z)} \geq \alpha \left| \frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right| + \gamma \qquad \forall\, z \in U\ .$$

We can note by $UCC_n(\alpha, \gamma)$ the sets of these functions.

**Remark 5.** *A simple geometric interpretation of the relation from Definition 6 is that* $f \in UCC_n(\alpha, \gamma)$ *if and only if the differential expression* $D^{n+1}f(z) \,/\, D^n g(z)$ *takes all values into the region* $D_{\alpha,\gamma}$*, when it is:*

(i) *elliptic region*

$$\frac{\left(u - \dfrac{\alpha^2 - \gamma}{\alpha^2 - 1}\right)^2}{\left[\dfrac{\alpha(1-\gamma)^2}{\alpha^2 - 1}\right]^2} + \frac{v^2}{\left(\dfrac{1-\gamma}{\sqrt{\alpha^2 - 1}}\right)^2} < 1 \quad for \quad \alpha > 1\ ;$$

(ii) *parabolic region*

$$v^2 < 2(1-\gamma)u - (1-\gamma^2) \quad for \quad \alpha = 1\ ;$$

(iii) *hiperbolic region*

$$\frac{\left(u - \dfrac{\gamma - \alpha^2}{1 - \alpha^2}\right)^2}{\left[\dfrac{\alpha(1-\gamma)}{1 - \alpha^2}\right]^2} - \frac{v^2}{\left(\dfrac{1-\gamma}{\sqrt{1 - \alpha^2}}\right)^2} > 1 \quad \begin{array}{l} and \quad u > 0 \\ for \quad \alpha \in (0,1) \end{array}\ ;$$

(iv) *half plane*

$$u > \gamma \quad for \quad \alpha = 0.$$

*In all these cases we have*

$$Re\, \left\{ \frac{D^{n+1}f(z)}{D^n g(z)} \right\} > \frac{\alpha + \gamma}{\alpha + 1}\ .$$

*That is* $UCC_n(\alpha, \gamma) \subset CCK_n\left(\dfrac{\alpha + \gamma}{\alpha + 1}\right) \subset CC.$

Special cases:

a) *If* $g(z) \equiv f(z) \quad UCC_n(\alpha,\gamma) = US_n(\alpha,\gamma)$ *defined by I.C. Magdaş* [9].

b) *If* $g(z) \equiv f(z)$ *and* $\alpha = k,\ \gamma = 0, \qquad UCC_n(k,0) = (k,n) - UCV$ *the subclasses introduced and studied by Stansislawa Kanas and Teuro Yaguchi* [8].

c) *For* $n = 1$, $g(z) = f(z)$, $\alpha = 1$, $\gamma = 0$ *we obtain the classical definition of uniformly convex functions* $USC$ *introduced by Goodman* [5].

d) *For* $n = 0$, $g(z) = f(z)$, $\alpha = 1$, $\gamma = 0$ *we rediscover the definition for the function which belongs to the set* $SP$*, introduced by Frode Ronning* [13, 14].

**Remark 6.** *All the functions of the sets* $UCC_n(\alpha,\gamma)$ *verify the conclusions of the Remark 4, Theorem 2 and Corollary 2.*

# 4 Intermediate classes

**Definition 7.** *For* $\beta \in \mathbb{R}$, $n \in \mathbb{N}_0$, $g \in US_n(\alpha)$ *we denote by*

$$J(n,\beta,g;f(z)) = (1-\beta)\frac{D^{n+1}f(z)}{D^n g(z)} + \beta\frac{D^{n+2}f(z)}{D^{n+1}g(z)} \qquad z \in U \ .$$

*We say that* $f$ *is* $n$*–uniformly* $\beta$ *close-to-convex Mocanu function iff*

$$J(n,\beta,g,f) \prec Q(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \qquad z \in U$$

*and we denote by* $UCCM_n(\beta)$ *the set of all these functions.*

**Theorem 4.** *If* $n \in \mathbb{N}_0$, $\beta > 0$, $UCCM_n \subset UCC_n$.

**Proof.** If we denote $D^{n+1}f(z)\ /\ D^n g(z) = p(z)$ we obtain

$$J(n,\beta,g,f) = p(z) + \frac{\beta}{h(z)}\cdot zp'(z)\ .$$

But $Re\,\beta\,/\,h(z) > 0$ follows that

$$\psi(p(z), zp'(z)) = p(z) + \frac{\beta}{h(z)} zp'(z)$$

is an "admissible function" and according to the "admissible functions method" it follows that

$$\beta(z) + \frac{\beta}{h(z)} zp'(z) \prec Q(z) \;\longrightarrow\; p(z) \prec Q(z)$$

is the conclusion of the last theorem.

**Remark 7.** *The function* $id\,z = z$ *belongs to all these classes.*

# References


[1] Blezu D., *Doctoral Thesis*, Univ. Babeş Bolyai, Cluj-Napoca (1984).

[2] Blezu D., *On the n-close-to-convex functions with respect to a convex set (I)*, Mathematica, Cluj-Napoca, Tome 28(5), Nr.1 1986, pp.9-19.

[3] Blezu D., *On the n-close-to-convex functions with respect to a convex set (II)*, Mathematica, Cluj-Napoca, Tome 31(54), Nr.1 1989, pp.15-25.

[4] Enigenburg P.J., Miller S.S., Mocanu P. and Reade M.D., *On a Briot-Bouquet differential subordination*, General Inequalities, 3, ISNM, vol.64 Birkhäuser Verlag, Basel (1983), 339-348.

[5] Goodman, *On uniformly convex functions*, Ann. Polon. Math. LVIII (1991), 86-92.

[6] Kanas S. and Wisniowska, *Conic region and k-uniform convexity I*, Journal of Math. Anal. and Appl. 105(1999), 327-336.

[7] Kanas S. and Wisniowska, *Conic region and k-uniform convexity II*, Folia Sci.Univ.Technic Resov 170(1998), 65-78.

[8] Kanas S. and Yaguchi Teuro, *Subclasses of $k$-uniformly convex and starlike functions defined by generalized derrivative I*, Indian J.Pure Appl.Math. 32, Nr.9, 1275-1282(2001)

[9] Ma W. and Minda D., *Uniformly convex function*, Ann. Polon. Math. 57(2), 1992, 165-175.

[10] Magdaş I.C., *Doctoral Thesis*, Univ. Babeş Bolyai, Cluj-Napoca, Romania (1998).

[11] Miller S.S. and Mocanu P.T., *Differential subordinations and univalent functions*, Michigan Math. J. 28(1981), 157-171.

[12] Miller S.S. and Mocanu P.T., *Differential subordinations and inequalities in the complex plane*, J. of Diff. Equs. 67, 2(1987), 199-211.

[13] Ronning F., *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., 118, No.1, 1993, 189-196.

[14] Ronning F., *On starlike functions associated with parabolic regions*, Ann. Univ. Marie Curie - Sklodowska, Sect.A, 45(14), 1991, 117-122..

[15] Sălăgean G.S., *Properties of starlikeness and convexity preserved by some integral operators*, Lect. Notes in Math., 743, Springer-Verlag, Berlin (1979), 367-372.

[16] Sălăgean G.S., *Subclasses of univalent functions*, Complex Analysis, Fifth Romanian - Finish Seminar, Lect. Notes in Math. 1013, Springer-Verlag 1983, 362-372.

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