GEORGIAN MATHEMATICAL JOURNAL: Vol. 2, No. 3, 1995, 259-276

BASIC BOUNDARY VALUE PROBLEMS OF THERMOELASTICITY FOR ANISOTROPIC BODIES WITH CUTS. II

R. DUDUCHAVA, D. NATROSHVILI, AND E. SHARGORODSKY

ABSTRACT. In the first part [1] of the paper the basic boundary value problems of the mathematical theory of elasticity for three-dimensional anisotropic bodies with cuts were formulated. It is assumed that the two-dimensional surface of a cut is a smooth manifold of an arbitrary configuration with a smooth boundary. The existence and uniqueness theorems for boundary value problems were formulated in the Besov ($\mathbb{B}^s_{p,q}$) and Bessel-potential ($\mathbb{H}^s_p$) spaces. In the present part we give the proofs of the main results (Theorems 7 and 8) using the classical potential theory and the nonclassical theory of pseudodifferential equations on manifolds with a boundary.

This paper continues [1]. After recalling some auxiliary results, we prove Theorems 7 and 8 formulated in §3.

§ 4. AUXILIARY RESULTS

4.1. Convolution Operators. $\mathbb{S}(\mathbb{R}^n)$ denotes the space of C^∞-smooth fast decaying functions, while $\mathbb{S}'(\mathbb{R}^n)$ stands for the dual space of tempered distributions. The Fourier transform and its inverse

$$\mathcal{F}\varphi(x) = \int\limits_{\mathbb{R}^n} e^{ix\xi}\varphi(\xi)d\xi, \quad \mathcal{F}^{-1}\varphi(\xi) = (2\pi)^{-n}\int\limits_{\mathbb{R}^n} e^{-ix\xi}\psi(x)dx$$

are continuous operators in both spaces $\mathbb{S}(\mathbb{R}^n)$ and $\mathbb{S}'(\mathbb{R}^n)$. Hence the convolution operator

$$\mathbf{a}(D)\varphi = \mathcal{F}^{-1}a\mathcal{F}\varphi, \quad a \in \mathbb{S}'(\mathbb{R}^n), \quad \varphi \in \mathbb{S}(\mathbb{R}^n) \tag{4.1}$$

1991 *Mathematics Subject Classification.* 35C15, 35S15, 73M25.
Key words and phrases. Thermoelasticity, anisotropic bodies, cuts, potentials, pseudodifferential equations, boundary integral equations..

1072-947X/95/0500-0259$07.50/0

is a continuous transformation

$$\mathbf{a}(D) : \mathbb{S}(\mathbb{R}^n) \to \mathbb{S}'(\mathbb{R}^n)$$

(cf. [2], [3]).

If operator (4.1) has a bounded extension

$$\mathbf{a}(D) : \mathbb{L}_p(\mathbb{R}^n) \to \mathbb{L}_p(\mathbb{R}^n), \quad 1 \le p \le \infty,$$

we write $a \in M_p(\mathbb{R}^n)$ and $a(\xi)$ is called the (Fourier) L_p-multiplier. Let

$$M_p^{(r)}(\mathbb{R}^n) = \big\{(1+|\xi|^2)^{r/2} a(\xi) : a \in M_p(\mathbb{R}^n)\big\}.$$

Recall that the Bessel potential space $\mathbb{H}_p^s(\mathbb{R}^n)$ is defined as a subset of $\mathbb{S}'(\mathbb{R}^n)$ endowed with the norm

$$\begin{gathered} \|u|\mathbb{H}_p^s(\mathbb{R}^n)\| = \|\mathcal{I}^s(D)u|\mathbb{L}_p(\mathbb{R}^n)\|, \\ \mathcal{I}^s(\xi) := (1+|\xi|^2)^{s/2}. \end{gathered} \tag{4.2}$$

Therefore due to the obvious property

$$\mathbf{a}_1(D)\mathbf{a}_2(D) = (\mathbf{a}_1\mathbf{a}_2)(D), \quad a_j \in M_p^{(r_j)}(\mathbb{R}^n) \tag{4.3}$$

we easily find that the operator

$$\mathbf{a}(D) : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}^n), \quad s, r \in \mathbb{R}, \quad 1 \le p \le \infty, \tag{4.4}$$

is bounded if and only if $a \in M_p^{(r)}(\mathbb{R}^n)$.

The interpolation property

$$\begin{gathered} \mathbb{B}_{p,q}^s(\mathbb{R}^n) = \big[\mathbb{H}_p^{s_1}(\mathbb{R}^n),\ \mathbb{H}_p^{s_2}(\mathbb{R}^n)\big]_{\theta,q}, \\ 1 < p < \infty, \quad 1 \le p \le \infty, \quad s_1, s_2 \in \mathbb{R}, \\ s = (1-\theta)s_1 + \theta s_2, \quad 0 \le \theta \le 1 \end{gathered} \tag{4.5}$$

(see [4], [5]) for $a \in M_p^{(r)}(\mathbb{R}^n)$ ensures the boundedness of the operator

$$\mathbf{a}(D) : \mathbb{B}_{p,q}^s(\mathbb{R}^n) \to \mathbb{B}_{p,q}^{s-r}(\mathbb{R}^n), \quad 1 \le q \le \infty. \tag{4.6}$$

Equality (4.2) and boundedness (4.4) imply that the operator

$$\mathcal{I}^r : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}^n) \tag{4.7}$$

arranges an isometric isomorphism.

Further, it is well known that the operators

$$\begin{gathered} \mathcal{I}_+^r : \widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^n) \to \widetilde{\mathbb{H}}_p^{s-r}(\mathbb{R}_+^n), \\ \mathcal{I}_-^r : \mathbb{H}_p^s(\mathbb{R}_+^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}_+^n), \quad \mathcal{I}_\pm^r(\xi) = (\xi_n \pm i|\xi'| \pm i)^r, \\ \mathbb{R}_+^n := \mathbb{R}^{n-1} \times \mathbb{R}^+, \quad \mathbb{R}^+ := [0, +\infty), \quad \xi = (\xi', \xi^n) \in \mathbb{R}^n, \quad \xi' \in \mathbb{R}^{n-1}, \end{gathered} \tag{4.8}$$

also arrange isomorphisms (though not isometric ones; see, for example, [3], [6]). Isomorphisms similar to (4.8) exist for any smooth manifold with a Lipschitz boundary (for details see [3], [7]).

The equality $M_2(\mathbb{R}^n) = \mathbb{L}_\infty(\mathbb{R}^n)$ is well known and trivial. A reasonable description of the class $M_p^r(\mathbb{R}^n)$ for $p \neq 2$ is less trivial and the problem still remains unsolved.

Theorem 12 (see [8], Theorem 7.9.5; [9]). *Let* $1 < p < \infty$ *and*

$$\sum_{\substack{|\beta|<[n/2]+1\\ 0\leq\beta\leq 1}} \sup\left\{|\xi^\beta D^\beta a(\xi)|,\ \xi \in \mathbb{R}^n\right\} \leq M < \infty,$$

where for the multi-index $\beta = (\beta_1, \dots, \beta_n)$ *the inequality* $0 \leq \beta \leq 1$ *reads as* $0 \leq \beta_j \leq 1$, $j = 1, \dots, n$. *Then* $a \in \bigcap\limits_{1<p<\infty} M_p(\mathbb{R}^n)$.

If $a \in M_p^{(r)}(\mathbb{R}^n)$, the operators

$$\begin{aligned} \mathbf{r}_+\mathbf{a}(D) &: \widetilde{\mathbb{H}}_p^s(\mathbb{R}^n_+) \to \mathbb{H}_p^{s-r}(\mathbb{R}^n_+) \\ &: \widetilde{\mathbb{B}}_{p,q}^s(\mathbb{R}^n_+) \to \mathbb{B}_{p,q}^{s-r}(\mathbb{R}^n_+) \end{aligned} \tag{4.9}$$

are bounded ($1 < p < \infty$, $s, r \in \mathbb{R}$, $1 \leq q \leq \infty$); here $\mathbf{r}_+\varphi = \varphi\big|_{\mathbb{R}^n_+}$ denotes the restriction operator.

An equality similar to (4.3)

$$\mathbf{r}_+\mathbf{a}_1(D)\ell_0\mathbf{r}_+\mathbf{a}_2(D) = \mathbf{r}_+(\mathbf{a}_1\mathbf{a}_2)(D), \quad a_j \in M_p^{(r_j)}(\mathbb{R}^n), \tag{4.10}$$

where ℓ_0 is extension by 0 from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, fails to be fulfilled in general. However, (4.10) holds if there is an analytic extension either $a_1(\xi', \xi_n - i\lambda)$ or $a_2(\xi', \xi_n + i\lambda)$, which can be estimated from above by $C(1 + |\xi| + \lambda)^N$ with $N > 0$, $\lambda > 0$, $C = const$.

4.2. Pseudodifferential operators. If the symbol $a(x, \xi)$ depends on the variable x, the corresponding convolution (cf. (4.1))

$$\mathbf{a}(x, D)\varphi(x) := \mathcal{F}^{-1}_{\xi\to x} a(x, \cdot)\mathcal{F}_{y\to\xi}\varphi(\xi) \tag{4.11}$$

is called the pseudodifferential operator ($\varphi \in \mathbb{S}(\mathbb{R}^n)$, $|a(x, \xi)| < C(1+|\xi|)^N$, $N > 0$, $C = const$).

Let $M_p^{(s,s-r)}(\mathbb{R}^n \times \mathbb{R}^n)$ denote a class of symbols $a(x, \xi)$ for which operator (4.11) can be extended to the bounded mapping

$$\mathbf{a}(x, D) : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}^n). \tag{4.12}$$

By $S^r(\Omega\times\mathbb{R}^n)$ ($\Omega\subset\mathbb{R}^n$, $r\in\mathbb{R}$) is denoted the Hörmander class of symbols $a(x,\xi)$ if

$$\left|D_x^\alpha D_\xi^\beta a(x,\xi)\right|\le M_{\alpha,\beta}\big(1+|\xi|\big)^{r-|\beta|},\quad \forall x\in\Omega,\quad \forall\xi\in\mathbb{R}^n, \tag{4.13}$$

where $M_{\alpha,\beta}$ is independent of x and ξ.

By $S_r^{l,m}(\Omega\times\mathbb{R}^n)$ ($\Omega\subset\mathbb{R}^n$, $l,m\in\mathbb{Z}^+$, $r\in\mathbb{R}$) we denote the class of symbols $a(x,\xi)$ satisfying the estimates

$$\int\limits_\Omega\left|D_x^\alpha(\xi D_\xi)^\beta a(x,\xi)\right|dx\le M'_{\alpha,\beta}\big(1+|\xi|\big)^r$$
$$\forall\xi\in\mathbb{R}^n,\quad |\alpha|\le l,\quad |\beta|\le m,$$

where

$$(\xi D_\xi)^\beta:=(\xi_1D_{\xi_1})^{\beta_1}\dots(\xi_nD_{\xi_n})^{\beta_n}.$$

If $\Omega\subset\mathbb{R}^n$ is compact, then $S^r(\Omega\times\mathbb{R}^n)\subset S_r^{l,m}(\Omega\times\mathbb{R}^n)$. Such an inclusion does not hold for non-compact Ω.

Theorem 13. *Let $s,r\in\mathbb{R}$, $l,m\in\mathbb{Z}^+$, $m>[n/2]+1$; then*

$$S^r(\mathbb{R}^n\times\mathbb{R}^n)\subset M_p^{(s,s-r)}(\mathbb{R}^n\times\mathbb{R}^n).$$

If, additionally, $-l+1+1/p<s-r<l+1/p$, then

$$S_r^{l+n,m}(\mathbb{R}^n\times\mathbb{R}^n)\subset M_p^{(s,s-r)}(\mathbb{R}^n\times\mathbb{R}^n).$$

Proof. When a symbol $a\in S^0(\mathbb{R}^n\times\mathbb{R}^n)$ has a compact support with respect to x, then the continuity of $\mathbf{a}(x,D)$ in $\mathbb{L}_p(\mathbb{R}^n)$ follows from Theorem 12, as shown in [10].

For an arbitrary $a\in S^0(\mathbb{R}^n\times\mathbb{R}^n)$ the above statement is proved for $\mathbb{L}_p(\mathbb{R}^n)$ using the arguments involved in the proof of Theorem 3.5 from [12]. In the general case the continuity of the mapping $\mathbb{H}_p^s(\mathbb{R}^n)\to\mathbb{H}_p^{s-r}(\mathbb{R}^n)$ is established with the aid of the order reduction operator (4.7) (see [4], [10]), while the continuity of the mapping $\mathbf{a}(x,D):\mathbb{B}_{p,q}^s(\mathbb{R}^n)\to\mathbb{B}_{p,q}^{s-r}(\mathbb{R}^n)$ is proved by interpolation (see [4]).

For a different proof of the first claim see [11].

To prove the second claim we shall introduce some notation. For a multi-index $\mu=(\mu_1,\dots,\mu_n)$, $0\le\mu\le1$ we define

$$dx^\mu:=\prod_{\substack{\mu_j=1\\ j=1,2,\dots,n}}dx_j,\quad (x,h)_\mu:=(z_1,\dots,z_n),$$
$$z_j=\begin{cases}x_j, & \text{if}\quad \mu_j=1,\\ h_j, & \text{if}\quad \mu_j=0,\end{cases}\qquad x,h\in\mathbb{R}^n.$$

Let

$$a_{(\alpha)}(x,\xi) := D_x^\alpha a(x,\xi).$$

By virtue of Theorem 12 the inclusion $a \in S_r^{l,m}(\mathbb{R}^n \times \mathbb{R}^n)$ implies

$$\int\limits_{\mathbb{R}^n} \big\| D_x^\alpha a(x,\cdot) \big| M_p^{(r)}(\mathbb{R}^n) \big\| dx < \infty, \quad |\alpha| \le l+n.$$

From this finiteness and Fubini's theorem we get

$$\operatorname{mes}_{\mathbb{R}^n} \Delta_{\mu,\gamma} = 0 \quad \text{for any} \quad 0 \le \mu \le 1, \quad |\gamma| \le l,$$

where

$$\Delta_{\mu,\gamma} := \Big\{ h \in \mathbb{R}^n : \int\limits_{\mathbb{R}^{|\mu|}} \big\| a_{(\mu+\gamma)}\big((y,h)_\mu,\cdot\big) \big| M_p^{(r)}(\mathbb{R}^n) \big\| dy^\mu = \infty \Big\}.$$

If now

$$\Delta = \bigcup_{\substack{0 \le \mu \le 1 \\ |\gamma| \le l}} \Delta_{\mu,\gamma}$$

then, obviously, $\operatorname{mes}_{\mathbb{R}^n} \Delta = 0$. There exists a vector $h_0 \in \mathbb{R}^n \backslash \Delta$. Then we have

$$\int\limits_{\mathbb{R}^n} \big\| a_{(\mu+\gamma)}\big((y,h_0)_\mu,\cdot\big) \big| M_p^{(r)}(\mathbb{R}^n) \big\| dy^\mu < \infty.$$

With these conditions we can use Theorem 5.1 and Remark 5.5 from [20] where the claimed inclusion $a \in M_p^{(s,s-r)}(\mathbb{R}^n \times \mathbb{R}^n)$ is proved. $\square$

Let

$$\mathbf{A}, \mathbf{B} : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}^n)$$

be the bounded operators; they are called locally equivalent at $x_0 \in \mathbb{R}^n$ (see [3], [13]) if

$$\inf\big\{ \|\chi(\mathbf{A}-\mathbf{B})\| : \chi \in C_{x_0}(\mathbb{R}^n) \big\} = \inf\big\{ \|(\mathbf{A}-\mathbf{B})\chi\mathbf{I}\| : \chi \in C_{x_0}(\mathbb{R}^n) \big\} = 0,$$

where $\mathbf{I}$ is the identity operator and $C_{x_0}(\mathbb{R}^n) = \{\chi \in C_0^\infty(\mathbb{R}^n) : \chi(x) = 1$ in some neighborhood of $x_0\}$. In such a case we write $\mathbf{A} \overset{x_0}{\sim} \mathbf{B}$. In a similar manner we define the equivalence $\mathbf{A}_0 \overset{x_0}{\sim} \mathbf{B}_0$ for operators

$$\mathbf{A}_0, \mathbf{B}_0 : \widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}_+^n).$$

Assume now that $\overline{S} = S \cup \partial S$ is a compact n-dimensional C^∞-smooth manifold with a C^∞-smooth boundary ∂S and

$$S = \overset{N}{\underset{j=1}{\cup}} V_j, \quad \varkappa_j : X_j \to V_j, \quad X_j \subset \mathbb{R}_+^n \tag{4.14}$$

are coordinate diffeomorphisms. Let $\{\chi_j\}_1^N \subset C_0^\infty(S)$ be a partition of the unity subordinated to the covering of S in (4.14); also let

$$\varkappa_{j_*}\varphi(t) = \chi_j^0\varphi\big(\chi_j(t)\big), \quad \varkappa_{j^*}^{-1}\psi(x) = \chi_j\psi\big(\varkappa_j^{-1}(x)\big),$$

where $\chi_j^0(t) := \chi_j(\varkappa_j(t))$, $t \in \mathbb{R}^n_+$, $x \in S$. The following mapping properties

$$\begin{aligned} \varkappa_{j_*} &: \mathbb{H}_p^r(S) \to \mathbb{H}_p^r(\mathbb{R}^n_+), \quad \operatorname{supp}\varkappa_j^{-1} \cap \partial S \neq \varnothing, \\ \varkappa_{j_*} &: \widetilde{\mathbb{H}}_p^r(S) \to \widetilde{\mathbb{H}}_p^r(\mathbb{R}^n_+), \quad \operatorname{supp}\varkappa_j^{-1} \cap \partial S \neq \varnothing, \\ \varkappa_{j_*} &: \mathbb{H}_p^r(S) \to \mathbb{H}_p^r(\mathbb{R}^n), \quad \operatorname{supp}\varkappa_j^{-1} \cap \partial S = \varnothing. \end{aligned} \tag{4.15}$$

are almost evident.

A bounded operator

$$\mathbf{A} : \widetilde{\mathbb{H}}_p^\nu(S) \to \mathbb{H}_p^{\nu-r}(S) \tag{4.16}$$

is called pseudodifferential (of order r) if:

(i) $\chi_1\mathbf{A}\chi_2\mathbf{I}$ is a compact operator in $\widetilde{\mathbb{H}}_p^r(S) \to \mathbb{H}_p^{\nu-r}(S)$ for any $\chi_1, \chi_2 \in C_0^\infty(S)$ with disjoint supports $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \varnothing$;

(ii)

$$\begin{aligned} \varkappa_{j^*}\mathbf{A}\varkappa_{j^*}^{-1} &\overset{x_0}{\sim} \mathbf{a}(x_0, D), \quad x_0 \in S, \\ \varkappa_{j^*}\mathbf{A}\varkappa_{j^*}^{-1} &\overset{x_0}{\sim} \mathbf{r}_+\mathbf{a}(x_0, D), \quad x_0 \in \partial S, \end{aligned} \tag{4.17}$$

where $a(x_0, \cdot) \in M_p^{(r)}(\mathbb{R}^n)$ for any $x_0 \in \overline{S}$.

Example 14 (see [3], Example 3.19]). . Let $\overline{\Omega} \subset \mathbb{R}^n$ be a compact domain with a smooth boundary $\partial\Omega \neq \varnothing$.

The operator $\mathbf{r}_\Omega\mathbf{a}(x, D)$, where $a(x,\xi) \in S^r(\Omega \times \mathbb{R}^n)$ and $\mathbf{r}_\Omega\varphi = \varphi\big|_\Omega$ denotes the restriction, is a pseudodifferential one of order r and

$$\begin{aligned} \mathbf{r}_\Omega\mathbf{a}(x, D) &\overset{x_0}{\sim} \mathbf{a}(x_0, D), \quad x_0 \notin \partial\Omega, \\ \mathbf{r}_\Omega\mathbf{a}(x, D) &\overset{x_0}{\sim} \mathbf{r}_+\mathbf{a}(x_0, D), \quad x_0 \in \partial\Omega. \end{aligned} \tag{4.18}$$

If $a(x_0, \xi)$ has the radial limits

$$a^\infty(x_0, \xi) = \lim_{\lambda\to\infty} \lambda^{-r}a(x_0, \lambda\xi) \tag{4.19}$$

which are nontrivial bounded functions of ξ, then $a^\infty(x_0, \xi)$ is a homogeneous function of order r with respect to ξ:

$$a^\infty(x_0, \lambda\xi) = \lambda^r a^\infty(x_0, \xi), \quad \lambda > 0.$$

Let

$$a^0(x_0, \xi) = a^\infty\big(x_0, (1 + |\xi'|)|\xi'|^{-1}\xi', \xi_n\big) \tag{4.20}$$

represent the modified symbol (see [6], Section 3). Assume that $a^0 \in M_p^{(r)}(\mathbb{R}^n)$; then using (4.17) and the relation

$$\lim_{R\to\infty} \sup_{|\xi|\geq R} |\xi|^{-r} \big| a(x_0,\xi) - a^0(x_0,\xi) \big| = 0$$

we obtain

$$\begin{aligned} \varkappa_{j_*} \mathbf{A} \varkappa_{j^*}^{-1} &\overset{x_0}{\sim} \mathbf{a}^0(x_0, D), \quad x_0 \notin \Omega, \\ \varkappa_{j_*} \mathbf{A} \varkappa_{j^*}^{-1} &\overset{x_0}{\sim} \mathbf{r}_+ \mathbf{a}^0(x_0, D), \quad x_0 \in \Omega. \end{aligned} \tag{4.21}$$

Thus the operators $\chi[\mathbf{a}(x_0,D) - \mathbf{a}^0(x_0,D)]$, $[\mathbf{a}(x_0,D) - \mathbf{a}^0(x_0,D)]\chi\mathbf{I}$ with $\chi \in C_0^\infty(\mathbb{R}^n)$ are compact in $\mathbb{H}_p^\nu(\mathbb{R}^n) \to \mathbb{H}_p^{\nu-r}(\mathbb{R}^n)$ (see [3]). As for the compact operator $\mathbf{T} : \mathbb{H}_p^\nu(\mathbb{R}^n) \to \mathbb{H}_p^{\nu-r}(\mathbb{R}^n)$, the equivalence $\mathbf{T} \overset{x_0}{\sim} \mathbf{0}$ holds automatically.

The functions $a^\infty(x_0,\xi)$ (see (4.19)) and $a^0(x_0,\xi)$ (see (4.20)) are respectively called the homogeneous principal symbol and the modified principal symbol of the operator $\mathbf{A}$.

Theorem 15 (see [3])). *Let* (4.16) *be a pseudodifferential operator* $(r,\nu \in \mathbb{R}, 1 < p < \infty)$. $\mathbf{A}$ *is a Fredholm operator if and only if the following conditions are fulfilled:*

(i) $\inf\{|\det a^\infty(x_0,\xi)| : x_0 \in \overline{S},\ \xi \in \mathbb{R}^n\} > 0$;

(ii) $\mathbf{r}_+ \mathbf{a}_{\nu,r}(x_0,D)$ *is a Fredholm operator in the space* $\mathbb{L}_p(\mathbb{R}^n_+)$ *for any* $x_0 \in \partial S$, *where*

$$a_{\nu,r}(x_0,\xi) = \big(\xi_n - i|\xi'| - i\big)^{\nu-r} a^0(x_0,\xi) \big(\xi_n + i|\xi'| + i\big)^{-\nu}, \\ \xi = (\xi', \xi_n), \quad \xi' \in \mathbb{R}^{n-1}.$$

Theorem 16 (see [3]). *Let* $\mathbf{a}(x,D)$ *be a pseudodifferential operator of the order* $r \in \mathbb{R}$ *with the* $N \times N$ *matrix symbol* $a(x,\cdot) \in S^r(\mathbb{R}^n)$ *for any* $x \in \overline{S}$. *If* $a(x,\xi)$ *is positive definite, i.e.,*

$$\begin{gathered} \big(a(x,\xi)\eta,\eta\big) \geq \delta_0 |\xi|^r |\eta|^2 \quad \textit{for some} \quad \delta_0 > 0 \\ \textit{and any} \quad \xi \in \mathbb{R}^n, \quad x \in \overline{S}, \quad \eta \in \mathbb{C}^N, \end{gathered} \tag{4.22}$$

then

$$\mathbf{a}(x,D) : \widetilde{\mathbb{H}}_2^{\frac{r}{2}+\nu}(S) \to \mathbb{H}_2^{-\frac{r}{2}+\nu}(S) \tag{4.23}$$

is a Fredholm operator for any $|\nu| < \frac{1}{2}$ *and*

$$\operatorname{Ind} \mathbf{a}(x,D) = 0. \tag{4.24}$$

4.3. Further Auxiliary Results. Let $\mathcal{H}^r(\mathbb{R}^n)$ denote the class of functions with the properties

(i) $a(\lambda\xi)=\lambda^r a(\xi)$, $\lambda>0$, $\xi\in\mathbb{R}^n$;

(ii) $a\in C^\infty(S^{n-1})$, $S^{n-1}:=\{\omega\in\mathbb{R}^n : |\omega|=1\}$;

(iii) if $a(\xi)=a_0(\omega',t,\xi_n)$, where $\omega'=|\xi'|^{-1}\xi'$, $t=|\xi'|$, $\xi=(\xi',\xi_n)\in\mathbb{R}^n$, then

$$\begin{gathered}\lim_{t\to 0} D_t^k a_0(\omega',t,-1)=(-1)^k\lim_{t\to 0} D_t^k a_0(\omega',t,1),\\ \omega'\in S^{n-2},\quad k=0,1,2,\dots .\end{gathered}\tag{4.25}$$

For $r=0$ condition (4.25) coincides with the well-known transmission property (see [6,14]).

Lemma 17. *Let $a\in\mathcal{H}^r(\mathbb{R}^n)$ be a positive definite $N\times N$ matrix-function (cf. (4.22))*

$$\begin{gathered}\big(a(\xi)\eta,\eta\big)\geq\delta_0|\xi|^r|\eta|^2\quad \textit{for some}\quad \delta_0>0\\ \textit{and any}\quad \xi\in\mathbb{R}^n,\quad \eta\in\mathbb{C}^N.\end{gathered}\tag{4.26}$$

Then $a(\xi)$ admits the factorization

$$a(\xi)=a_-(\xi)a_+(\xi),\quad a_\pm(\xi)=\big(\xi_n\pm i|\xi'|\big)^{-\frac{r}{2}}b_\pm(\xi),\tag{4.27}$$

where $b_+^{\pm1}(\xi',\xi_n+i\lambda)$, $b_-^{\pm1}(\xi',\xi_n-i\lambda)$ have uniformly bounded analytic extensions for $\lambda>0$, $\xi'\in\mathbb{R}^{n-1}$, $\xi_n\in\mathbb{R}$ and

$$\sum_{|\alpha|\leq m}\sup\big\{|\xi^\alpha D^\alpha b_\pm^{\pm1}(\xi)| : \xi\in\mathbb{R}^n\big\}\leq M_m<\infty,\ \ m=0,1,2,\dots .\tag{4.28}$$

Proof. For the proof of this lemma see [2,9,15]. □

Remark 18. A lemma similar to the above one but for a general elliptic symbol was proved in [2,9] (see [6] for the scalar case $N=1$). In [15, §2] a similar but more general assertion is proved when $a(x,\xi)$ depends smoothly on a parameter $x\in S$.

A pair of Banach spaces $\{\mathbb{X}_0,\mathbb{X}_1\}$ embedded in some topological space $\mathbb{E}$ is called an interpolation pair. For such a pair we can introduce the following two spaces: $\mathbb{X}_{\min}=\mathbb{X}_0\cap\mathbb{X}_1$ and $\mathbb{X}_{\max}=\mathbb{X}_0+\mathbb{X}_1:=\big\{x\in\mathbb{E} : x=x_0+x_1,\ x_j\in\mathbb{X}_j,\ j=0,1\big\}$; $\mathbb{X}_{\min}$ and $\mathbb{X}_{\max}$ become Banach spaces if they are endowed with the norms

$$\begin{gathered}\|x|\mathbb{X}_{\min}\|=\max\big\{\|x|\mathbb{X}_0\|,\ \|x|\mathbb{X}_1\|\big\},\\ \|x|\mathbb{X}_{\max}\|=\inf\big\{\|x_0|\mathbb{X}_0\|+\|x_1|\mathbb{X}_1\| : x=x_0+x_1,\ x_j\in\mathbb{X}_j,\ j=0,1\big\},\end{gathered}$$

respectively.

Moreover, we have the continuous embeddings

$$\mathbb{X}_{\min} \subset \mathbb{X}_0,\ \mathbb{X}_1 \subset \mathbb{X}_{\max}. \tag{4.29}$$

For any interpolation pairs $\{\mathbb{X}_0, \mathbb{X}_1\}$ and $\{\mathbb{Y}_0, \mathbb{Y}_1\}$ the space $\mathcal{L}(\{\mathbb{X}_0\mathbb{X}_1\}, \{\mathbb{Y}_0\mathbb{Y}_1\})$ consists of all linear operators from $\mathbb{X}_{\max}$ into $\mathbb{Y}_{\max}$ whose restrictions to $\mathbb{X}_j$ belong to $\mathcal{L}(\mathbb{X}_j, \mathbb{Y}_j)$ $(j = 0, 1)$. The notation $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ is used for the space of all linear bounded operators $\mathbf{A} : \mathbb{X} \to \mathbb{Y}$.

Lemma 19. *Assume $\{\mathbb{X}_0, \mathbb{X}_1\}$ and $\{\mathbb{Y}_0, \mathbb{Y}_1\}$ to be interpolation pairs and the embeddings $\mathbb{X}_{\min} \subset \mathbb{X}_{\max}$, $\mathbb{Y}_{\min} \subset \mathbb{Y}_{\max}$ to be dense. Let an operator $\mathbf{A} \in \mathcal{L}(\mathbb{X}_0, \mathbb{Y}_0) \cap \mathcal{L}(\mathbb{X}_1, \mathbb{Y}_1)$ have a common regularizer: let $\mathbf{R} \in \mathcal{L}(\mathbb{Y}_0, \mathbb{X}_0) \cap \mathcal{L}(\mathbb{Y}_1, \mathbb{X}_1)$ and $\mathbf{RA} - \mathbf{I} \in \mathcal{L}(\mathbb{X}_0\mathbb{X}_0) \cap \mathcal{L}(\mathbb{X}_1, \mathbb{X}_1)$ be compact. Then*

$$\mathbf{A} : \mathbb{X}_{\min} \to \mathbb{Y}_{\min}, \quad \mathbf{A} : \mathbb{X}_{max} \to \mathbb{Y}_{\max}$$

are Fredholm operators and

$$\operatorname{Ind}_{\mathbb{X}_{\min}\to\mathbb{Y}_{\min}} \mathbf{A} = \operatorname{Ind}_{\mathbb{X}_{\max}\to\mathbb{Y}_{\max}} \mathbf{A} = \operatorname{Ind}_{\mathbb{X}_j\to\mathbb{Y}_j} \mathbf{A}, \quad j = 0, 1. \tag{4.30}$$

If $y \in \mathbb{Y}_j$, then any solution $x \in \mathbb{X}_{\max}$ of the equation $\mathbf{A}x = y$ belongs to $\mathbb{X}_j$. In particular,

$$\ker_{\mathbb{X}_{\min}} \mathbf{A} = \ker_{\mathbb{X}_j} \mathbf{A} = \ker_{\mathbb{X}_{\max}} \mathbf{A}, \quad j = 0, 1. \tag{4.31}$$

Proof. We begin by noting that the definition of a norm in $\mathbb{X}_{\min}, \dots, \mathbb{Y}_{\max}$ implies

$$\begin{aligned} \big\|\mathbf{A}|\mathcal{L}(\mathbb{X}_{\min}, \mathbb{Y}_{\min})\big\| &\le \max\big\{\|\mathbf{A}|\mathcal{L}(\mathbb{X}_j, \mathbb{Y}_j)\| : j = 0, 1\big\}, \\ \big\|\mathbf{A}|\mathcal{L}(\mathbb{X}_{\max}, \mathbb{Y}_{\max})\big\| &\le \max\big\{\|\mathbf{A}|\mathcal{L}(\mathbb{X}_j, \mathbb{Y}_j)\| : j = 0, 1\big\}. \end{aligned}$$

Whence we find

$$\mathcal{L}(\mathbb{X}_0, \mathbb{Y}_0) \cap \mathcal{L}(\mathbb{X}_1, \mathbb{Y}_1) \subset \mathcal{L}(\mathbb{X}_{\min}, \mathbb{Y}_{\min}) \cap \mathcal{L}(\mathbb{X}_{\max}, \mathbb{Y}_{\max}).$$

Next we shall prove that $\mathbf{A}$ is a Fredholm operator in the spaces $\mathbb{X}_{\min} \to \mathbb{Y}_{\min}$ and $\mathbb{X}_{\max} \to \mathbb{Y}_{\max}$. For this it suffices to show that $\mathbf{AR}-\mathbf{I}$, $\mathbf{RA}-\mathbf{I}$ are compact in the spaces $\mathbb{X}_{\min}$ and $\mathbb{X}_{\max}$, since by the conditions of the lemma they are compact in $\mathbb{X}_0$ and $\mathbb{X}_1$. Let us prove a more general inclusion

$$\operatorname{Com}(\mathbb{X}_0, \mathbb{Y}_0) \cap \operatorname{Com}(\mathbb{X}_1, \mathbb{Y}_1) \subset \operatorname{Com}(\mathbb{X}_{\min}, \mathbb{Y}_{\min}) \cap \operatorname{Com}(\mathbb{X}_{\max}, \mathbb{Y}_{\max}),$$

that implies the claimed assertion.

Assume $\mathbf{T} : \mathbb{X}_j \to \mathbb{Y}_j$ $(j = 0, 1)$ to be compact and $\{x_k\}_{k\in\mathbb{N}}$ to be an arbitrary bounded sequence in $\mathbb{X}_{\min}$. Then $\{x_k\}_{k\in\mathbb{N}}$ is bounded in both spaces $\mathbb{X}_0$ and $\mathbb{X}_1$. It can be assumed without loss of generality that the sequences $\{\mathbf{T}x_k\}_{k\in\mathbb{N}}$ are convergent in both $\mathbb{Y}_0$ and $\mathbb{Y}_1$ (otherwise we can select subsequences). Then $\{\mathbf{T}x_k\}_{k\in\mathbb{N}}$ is convergent in $\mathbb{Y}_{\min}$ and therefore $\mathbf{T} \in \operatorname{Com}(\mathbb{X}_{\min}, \mathbb{Y}_{\min})$.

If S_0, S_1, and $S_{\max}$ denote the unit balls in $\mathbb{X}_0$, $\mathbb{X}_1$, and $\mathbb{X}_{\max}$, respectively, then $S_{\max} \subset S_0 + S_1$. Due to the compactness of $\mathbf{T} : \mathbb{X}_j \to \mathbb{Y}_j$ $(j = 0,1)$, there exist $\varepsilon/2$-grids $\{y_k^{(j)}\}_{k=1}^{m_j} \subset \mathbf{T}(S_j)$ $(j = 0,1)$, $\varepsilon > 0$. Then $\{y_k^{(0)} + y_n^{(1)}\}_{k,n} \subset \mathbf{T}(S_0) + \mathbf{T}(S_1)$ defines an ε-grid in $\mathbf{T}(S_{\max})(\subset \mathbf{T}(S_0) + \mathbf{T}(S_1))$. Since $\varepsilon > 0$ is arbitrary, $\mathbf{T} : \mathbb{X}_{\max} \to \mathbb{Y}_{\max}$ is compact.

Now we shall show that the density of the embedding $\mathbb{Y}_{\min} \subset \mathbb{Y}_{\max}$ implies the density of $\mathbb{Y}_{\min} \subset \mathbb{Y}_j$ $(j = 0,1)$. For the sake of definiteness assume that $j = 0$. By the condition of the lemma for any $\varepsilon > 0$ and $a \in \mathbb{Y}_0$ there exists $b \in \mathbb{Y}_{\min}$ with the property

$$\|(a - b)|\mathbb{Y}_{\max}\| < \varepsilon;$$

i.e., there exist $a_0 \in \mathbb{Y}_0$, $a_1 \in \mathbb{Y}_1$ such that $a - b = a_0 + a_1$,

$$\|a_0|\mathbb{Y}_0\| + \|a_1|\mathbb{Y}_1\| < \varepsilon.$$

Since $a \in \mathbb{Y}_0$ and $b \in \mathbb{Y}_{\min} \subset \mathbb{Y}_0$, we obtain $a - b \in \mathbb{Y}_0$ and $a_1 = (a-b) - a_0 \in \mathbb{Y}_0$, so that $a_1 \in \mathbb{Y}_0 \cap \mathbb{Y}_1 = \mathbb{Y}_{\min}$ and $a_1 + b \in \mathbb{Y}_{\min}$. Therefore

$$\big\|[a - (a_1 + b)]|\mathbb{Y}_0\big\| = \|a_0|\mathbb{Y}_0\| < \varepsilon,$$

which proves that the embedding $\mathbb{Y}_{\min} \subset \mathbb{Y}_0$ is dense.

The density of the embeddings $\mathbb{Y}_{\min} \subset \mathbb{Y}_j \subset \mathbb{Y}_{\max}$, $j = 0,1$, yields

$$\mathbb{Y}^*_{\max} \subset \mathbb{Y}^*_j \subset \mathbb{Y}^*_{\min}, \quad j = 0,1.$$

Since $\mathbb{X}_{\min} \subset \mathbb{X}_j \subset \mathbb{X}_{\max}$ and $\mathbf{A}^* : \mathbb{Y}^*_j \to \mathbb{X}^*_j$ $(j = 0,1)$, $\mathbf{A}^* : \mathbb{Y}^*_{\min} \to \mathbb{X}^*_{\min}$, $\mathbf{A}^* : \mathbb{Y}^*_{\max} \to \mathbb{X}^*_{\max}$ are Fredholm, we have

$$\ker_{\mathbb{X}_{\min}} \mathbf{A} \subset \ker_{\mathbb{X}_j} \mathbf{A} \subset \ker_{\mathbb{X}_{\max}} \mathbf{A}, \tag{4.32}$$

$$\ker_{\mathbb{Y}^*_{\max}} \mathbf{A}^* \subset \ker_{\mathbb{Y}^*_j} \mathbf{A}^* \subset \ker_{\mathbb{Y}^*_{\min}} \mathbf{A}^*. \tag{4.33}$$

The dimensions of the kernels $(\dim\ker \mathbf{A})$ in appropriate spaces will be denoted by $\alpha_{\min}$, α_j, $\alpha_{\max}$, while the notation $\beta_{\min}$, β_j, $\beta_{\max}$ will be used for the dimensions of cokernels $(\dim \operatorname{Coker} \mathbf{A})$. Note that for a Fredholm operator we have

$$\dim \operatorname{Coker} \mathbf{A} = \dim \ker \mathbf{A}^*.$$

Embeddings (4.32) and (4.33) imply

$$\alpha_{\min} \le \alpha_j \le \alpha_{\max}, \quad j = 0,1, \tag{4.34}$$

$$\beta_{\max} \le \beta_j \le \beta_{\min}, \quad j = 0,1. \tag{4.35}$$

By the definition of $\operatorname{Ind} \mathbf{A}$ we obtain

$$\operatorname{Ind}_{\mathbb{X}_{\min}\to\mathbb{Y}_{\min}} \mathbf{A} \le \operatorname{Ind}_{\mathbb{X}_j\to\mathbb{Y}_j} \mathbf{A} \le \operatorname{Ind}_{\mathbb{X}_{\max}\to\mathbb{Y}_{\max}} \mathbf{A}. \tag{4.36}$$

A similar inequality for indices of the regularizer $\mathbf{R}$ is proved just in the same manner. Since $\operatorname{Ind} \mathbf{R} = -\operatorname{Ind} \mathbf{A}$, the inequalities inverse to (4.36) are valid and therefore (4.30) holds. Now from (4.34) and (4.35) we obtain $\alpha_{\min} = \alpha_j = \alpha_{\max}$. The latter equality and (4.32) give (4.31). $\square$

Remark 20. Similar statements under different conditions on spaces and operators are well known (see, for example, [16], [17], [18]).

§ 5. Proofs of Theorems

5.1. *Proof of Theorem* 7. In the first place we shall prove that $\mathbf{P}^1_S$ (see (3.2), (3.6), (3.7)) is a pseudodifferential operator according to the definition given in Subsection 4.2.

Let $U_1, \dots, U_N$ be a covering of $S \subset \mathbb{R}^3$ (see (4.14), where $n = 2$), $\varkappa_1, \dots, \varkappa_N$ be coordinate diffeomorphisms, and

$$\begin{gathered}\widetilde{\varkappa}_j : \widetilde{X}_j \to \widetilde{U}_j, \quad \widetilde{X}_j, \widetilde{U}_j \subset \mathbb{R}^3, \quad \widetilde{U}_j \cap S = V_j, \\ \widetilde{X}_j = (-\varepsilon, \varepsilon) \times X_j, \quad \widetilde{\varkappa}_j|_{X_j} = \varkappa_j, \quad j = 1, \dots, N,\end{gathered} \tag{5.1}$$

be extensions of diffeomorphisms (4.14). By $d\varkappa_j(t) = \varkappa_j'(t)$ and $d\widetilde{\varkappa}_j(\widetilde{t}) = \widetilde{\varkappa}_j'(\widetilde{t})$ $(t = (t_1, t_2) \in \mathbb{R}^2_+,\ \widetilde{t} = (t_0, t_1, t_2) \in \mathbb{R}^3_+)$ we denote the corresponding Jacobian matrices of orders 3×2 and 3×3. $\varkappa_j'(t)$ will coincide with $\widetilde{\varkappa}_j'(0, t) (t \in X_j \subset \mathbb{R}^2_+)$ if the first column in these matrices is deleted.

Let further

$$\Gamma_{\chi_j}(t) = \big(\det \|(\partial_k \varkappa_j, \partial_l \varkappa_j)\|_{2\times 2}\big)^{1/2}, \quad \partial_k \varkappa_j = (\partial_k \varkappa_{j1}, \partial_k \varkappa_{j2}, \partial_k \varkappa_{j3})$$

denote the square root of the Gramm determinant of the vector-function $\varkappa_j = (\varkappa_{j1}, \varkappa_{j2}, \varkappa_{j3})$.

If the operator $\mathbf{P}^1_S$ is lifted locally from the manifold S onto the half-space $\mathbb{R}^2_+$ by means of operators (4.15), then we obtain the operator (cf. (4.17))

$$\begin{gathered}\mathbf{P}^1_{s,\varkappa_j} v(t) = \varkappa_{j*} \mathbf{P}^1_s \varkappa_{j*}^{-1} v(t) = \chi^0_j(t) \int\limits_{\mathbb{R}^2_+} \Phi\big((\varkappa_j(t) - \\ - \varkappa_j(\theta), \tau\big) \chi^0_j(\theta) \Gamma_{\varkappa_j}(\theta) v(\theta) d\theta, \quad t \in \mathbb{R}^2_+, \quad \chi^0_j \in C^\infty_0(\mathbb{R}^2_+).\end{gathered}$$

From the last equality it follows that operator (3.7) is bounded. Moreover,

$$\begin{gathered}\mathbf{K}_j v(t) := \chi^0_j(t) \int\limits_{\mathbb{R}^2_+} \big[\Phi(\varkappa_j(t) - \varkappa_j(\theta), \tau) \Gamma_{\varkappa_j}(\theta) - \\ - \Phi(\varkappa_j'(t)(t - \theta), \tau) \Gamma_{\varkappa_j}(t)\big] \chi^0_j(\theta) v(\theta) d\theta\end{gathered}$$

has the order -2, i.e., the operator

$$\mathbf{K}_j : \widetilde{\mathbb{H}}^\nu_p(\mathbb{R}^2_+) \to \mathbb{H}^{\nu+2}_p(\mathbb{R}^2_+) \tag{5.2}$$

is bounded for any $\nu \in \mathbb{R}$ (see [19, Section 33.2 and Theorem 13]). Due to (5.2) the operator

$$\mathbf{K}_j : \widetilde{\mathbb{H}}_p^{\nu}(\mathbb{R}^2_+) \to \mathbb{H}_p^{\nu+1}(\mathbb{R}^2_+) \tag{5.3}$$

is compact, since $\chi_j^0 \in C_0^\infty(\mathbb{R}^2_+)$ [see (4.19)]. From (5.3), Example 14, and (2.1), it follows that the symbol of the pseudodifferential operator $\mathbf{P}_S^1$ reads $(x \in \overline{S}, \xi \in \mathbb{R}^2)$

$$\begin{aligned}
\mathcal{P}_S^1(x,\xi) &= \Gamma_{\varkappa_j}(t) \int\limits_{\mathbb{R}^2} e^{i\xi\eta}\Phi\big(\varkappa_j'(t)\eta,\tau\big)d\eta = \\
&= \Gamma_{\varkappa_j}(t) \int\limits_{\mathbb{R}^2} e^{i\xi\eta}\Phi\big(\widetilde{\varkappa}_j'(0,t)(0,\eta),\tau\big)d\eta = \\
&= \frac{\Gamma_{\varkappa_j}(t)}{(2\pi)^3} \int\limits_{\mathbb{R}^2} e^{i\xi\eta} \int\limits_{\mathbb{R}^3} e^{-i(\widetilde{\varkappa}_j'(0,t)(0,\eta),\widetilde{y})}\mathcal{A}^{-1}(\widetilde{y},\tau)d\widetilde{y}d\eta = \\
&= \frac{\Gamma_{\varkappa_j}(t)}{(2\pi)^3 \det \widetilde{\varkappa}_j'(0,t)} \int\limits_{\mathbb{R}^2} e^{i\xi\eta} \int\limits_{\mathbb{R}^2} e^{-i\eta y} \int\limits_{-\infty}^{\infty} \mathcal{A}^{-1}\Big(\big[(\widetilde{\varkappa}_j'(0,t))^T\big]^{-1}\widetilde{y},\tau\Big)dy_0dyd\eta = \\
&= \frac{\Gamma_{\varkappa_j}(t)}{2\pi \det \widetilde{\varkappa}_j'(0,t)} \int\limits_{-\infty}^{\infty} \mathcal{A}^{-1}\Big(\big[(\widetilde{\varkappa}_j'(0,t))^T\big]^{-1}\zeta,\tau\Big)dy_0.
\end{aligned} \tag{5.4}$$

for $t = \varkappa_j^{-1}(x)$, $x \in S$, $t \in \mathbb{R}^2_+$, $\xi \in \mathbb{R}^2$, $\widetilde{y} = (y_0, y) \in \mathbb{R}^3$, $\zeta = (y_0, \xi)$. By (2.3) the principal homogeneous symbol of $\mathbf{P}_S^1$ (see (2.18)) is written in the form

$$(\mathcal{P}_S^1)^\infty(x,\xi) = \frac{\Gamma_{\varkappa_j}(t)}{2\pi \det \widetilde{\varkappa}_j'(0,t)} \int\limits_{-\infty}^{\infty} \mathcal{A}_0^{-1}\Big(\big[(\widetilde{\varkappa}_j'(0,t))^T\big]^{-1}\zeta\Big)dy_0, \tag{5.5}$$

$$x \in \overline{S}, \quad \xi \in \mathbb{R}^2, \quad t = \varkappa_j^{-1}(x) \in \mathbb{R}^2_+, \quad \zeta = (y_0, \xi),$$

$$\mathcal{A}_0^{-1}(\widetilde{\xi}) = \begin{Vmatrix} \mathcal{C}^{-1}(\widetilde{\xi}) & 0 \\ 0 & \mathbf{\Lambda}^{-1}(-i\widetilde{\xi}) \end{Vmatrix}, \quad \widetilde{\xi} \in \mathbb{R}^3, \tag{5.6}$$

where $\mathcal{C}(\widetilde{\xi})$ and $\mathbf{\Lambda}(\widetilde{\xi})$ are defined by (2.4). Since $-\mathcal{C}(\widetilde{\xi})$ and $-\mathbf{\Lambda}(-i\widetilde{\xi})$ are positive-definite (see (1.12) and (1.14)), the same is true for $-\mathcal{A}_0^{-1}(\widetilde{\xi})$:

$$\big(- \mathcal{A}_0^{-1}(\widetilde{\xi})\eta,\eta\big) \geq \delta_2|\eta|^2|\widetilde{\xi}|^{-2}, \quad \delta_2 > 0, \;\; \eta \in \mathbb{C}^4, \;\; \widetilde{\xi} \in \mathbb{R}^3.$$

Applying this fact, we proceed as follows:

$$\big((-\mathcal{P}_S^1)^\infty(x,\xi)\eta,\eta\big) =$$

$$
\begin{aligned}
&= \frac{\Gamma_{\varkappa_j}(t)}{2\pi \det \varkappa_j'(t)} \int\limits_{-\infty}^{+\infty} \Big(-\mathcal{A}_0^{-1}\big(\big[(\widetilde{\varkappa}_j'(0,t))^T\big]^{-1}\zeta\big)\eta,\eta\Big) dy_0 \geq \\
&\qquad \geq \delta_2|\eta|^2 \int\limits_{-\infty}^{+\infty} \big|\widetilde{\varkappa}_j'(0,t)\zeta\big|^{-2} dy_0 \geq \\
&\qquad \geq \delta_3|\eta|^2 \int\limits_{-\infty}^{+\infty} \frac{dy_0}{y_0^2+|\xi|^2} = \delta_4|\eta|^2|\xi|^{-1}, \qquad (5.7) \\
&\eta\in\mathbb{C}^4, \quad \xi\in\mathbb{R}^2, \quad \zeta=(y_0,\xi), \quad \delta_k = const>0, \quad k=2,3,4.
\end{aligned}
$$

Formulas (1.6), (5.5) and (5.6) also imply

$$
\begin{gathered}
D_x^\alpha D_{\xi_1}^m(\mathcal{P}_S^1)^\infty(x,\lambda\xi) = |\lambda|^{-1}\lambda^{-m} D_x^\alpha D_{\xi_1}^m(\mathcal{P}_S^1)^\infty(x,\xi), \\
|\alpha|<\infty, \quad m=0,1,\dots, \quad \xi\in\mathbb{R}^2, \quad \lambda\in\mathbb{R}.
\end{gathered} \qquad (5.8)
$$

Hence we have the equivalences (see (4.18), (4.21), (5.1), (5.2))

$$
\begin{gathered}
\varkappa_{j*}\mathbf{P}_S^1\varkappa_{j*}^{-1} \overset{x_0}{\approx} (\mathbf{P}_S^1)^0(x_0,D), \quad x_0\in U_j\subset S, \quad x_0\notin\partial S, \\
\varkappa_{j*}\mathbf{P}_S^1\varkappa_{j*}^{-1} \overset{x_0}{\approx} \mathbf{r}_+(\mathbf{P}_S^1)^0(x_0,D), \quad x_0\in U_j\cap\partial S,
\end{gathered}
$$

where (see (4.20))

$$
(\mathcal{P}_S^1)^0(x,\xi) := (\mathcal{P}_S^1)^\infty\big(x,(1+|\xi_1|)|\xi_1|^{-1}\xi_1,\xi_2\big).
$$

Due to (5.7) the symbol $(\mathcal{P}_S^1)^0(x,\xi)$ is an elliptic one,

$$
\inf\{|\det(\mathcal{P}_S^1)^\infty(x,\xi)| : x\in\overline{S}, \quad |\xi|=1\}>0.
$$

Since condition (5.8) implies the continuity property (4.25) for the symbol $(\mathcal{P}_S^1)^\infty(x,\xi)$, by virtue of Lemma 17 it admits the factorization

$$
\begin{gathered}
(\mathcal{P}_S^1)^0(x,\xi) = \big[(\xi_2-i|\xi_1|-i)^{-1/2}\mathcal{P}_-(x,\xi)\big]\big[(\xi_2+i|\xi'|+i)^{-1/2}\mathcal{P}_+(x,\xi)\big], \\
\mathcal{P}_-^{\pm1}(x,\cdot),\ \mathcal{P}_+^{\pm1}(x,\cdot)\in M_p(\mathbb{R}^2), \quad x\in\partial S,
\end{gathered}
$$

where $\mathcal{P}_-^{\pm1}(x,\xi_1-i\lambda)$, $\mathcal{P}_+^{\pm1}(x,\xi_1+i\lambda)$ have bounded analytic extensions for $\lambda>0$. According to Theorem 15 operator (3.7) is a Fredholm one if and only if the operators $\mathbf{r}_+(\mathbf{P}_S^1)_{\nu,-1}(x_0,D)$ are Fredholm ones in $\mathbb{L}_p(\mathbb{R}_+^2)$ for all $x_0\in\partial S$, where

$$
\begin{aligned}
(\mathcal{P}_S^1)_{\nu,-1}(x_0,\xi) &= \frac{(\xi_2-i|\xi_1|-i)^{\nu+1}}{(\xi_2+i|\xi_1|+i)^\nu}(\mathcal{P}_S^1)^0(x_0,\xi) = \\
&= \Big(\frac{\xi_2-i|\xi_1|-i}{\xi_2+i|\xi_1|+i}\Big)^{\nu+1/2}\mathcal{P}_-(x_0,\xi)\mathcal{P}_+(x_0,\xi), \quad x_0\in\partial S. \qquad (5.9)
\end{aligned}
$$

Therefore (see (4.10), (5.9))

$$\mathbf{r}_+(\mathbf{P}_S^1)_{\nu,-1}(x_0,D) = \mathbf{r}_+\mathbf{P}_-(x_0,D)\ell_0\mathbf{r}_+\mathbf{G}_\nu(D)\mathbf{P}_+(x_0,D), \quad (5.10)$$

with

$$\mathcal{G}_\nu(\xi) = \Big(\frac{\xi_2 - i|\xi_1| - i}{\xi_2 + i|\xi_1| + i}\Big)^{\nu+1/2} \quad (5.11)$$

and since $\mathbf{r}_+\mathbf{P}_\pm(x_0,D)$ are invertible (according to (4.10) the inverses read $\mathbf{r}_+\mathbf{P}_\pm^{-1}(x,D)$). The proof will be completed if we find invertibility conditions for $\mathbf{r}_+\mathbf{G}_\nu(D)$ in $\mathbb{L}_p(\mathbb{R}^2_+)$; the latter is invertible if and only if

$$1/p - 1 < \nu + 1/2 < 1/p \quad (5.12)$$

and the inverse reads $(\mathbf{r}_+\mathbf{G}_\nu(D))^{-1} = \mathcal{I}_+^{\nu+1/2}(D)\ell_0\mathbf{r}_+\mathcal{I}_-^{-\nu-1/2}(D)$ (see [2], §2). Conditions (5.12) coincide with (3.8).

The local inverses to $\mathbf{P}_S^1 : \widetilde{\mathbb{H}}_p^\nu(S) \to \mathbb{H}_p^{\nu+1}(S)$ are, therefore, independent of the parameters p and ν if conditions (3.8) are fulfilled.

In fact, the operator

$$(\mathbf{r}_+\mathbf{P}_S^1)_{\nu,-1}^{-1}(x_0,D) := \mathbf{P}_+^{-1}(x_0,D)\mathcal{I}_+^{\nu+1/2}(D)\ell_0\mathbf{r}_+\mathcal{I}_-^{-\nu-1/2}(D)\mathbf{P}_+^{-1}(x_0,D)$$

is inverse to $(\mathbf{r}_+\mathbf{P}_S^1)_{\nu,-1}(x_0,D)$ in $L_p(\mathbb{R}^2_+)$; if we "lift" these operators from the space $L_p(\mathbb{R}^2_+)$ to the Bessel potential spaces by means of the Bessel potentials $\mathcal{I}_\pm^\mu(D)$ defined by (4.8), we shall come to the following conclusion: if (3.8) holds, the operator

$$\begin{aligned}&\mathcal{I}_+^{-\nu}(D)\ell_0(\mathbf{r}_+\mathbf{P}_S^1)_{\nu,-1}^{-1}(x_0,D)\mathcal{I}_-^{\nu+1}(D) =\\ &= (D)\mathbf{P}_+^{-1}(x_0,D)\mathcal{I}_+^{1/2}(D)\ell_0\mathbf{r}_+\mathcal{I}_-^{1/2}\mathbf{P}_-^{-1}(x_0,D)\end{aligned}$$

inverts the operator

$$\begin{aligned}&\mathcal{I}_+^{-\nu}(D)\ell_0(\mathbf{r}_+\mathbf{P}_S^1)_{\nu,-1}(x_0,D)\mathcal{I}_-^{\nu+1}(D) =\\ &= \mathbf{P}_S^1(x_0,D) : \widetilde{\mathbb{H}}_p^\nu(\mathbb{R}^2_+) \to \mathbb{H}_p^{\nu+1}(\mathbb{R}^2_+), \quad x_0 \in \partial S\end{aligned}$$

which is a local representation of $\mathbf{P}_S^1 = \mathbf{P}_S^1(x,D)$ $(x \in S,\ x_0 \in \partial S)$.

Thus the regularizer constructed by means of the local inverses (see, for example, [2], [3], [13]) can be chosen independent of p and ν if (3.8) holds. Now we can take $p = 2$ and by Theorem 16 and Lemma 19 we get $\operatorname{Ind}\mathbf{P}_S^1 = 0$.

To complete the proof for the space $H_p^\nu(S)$ it remains to check that $\ker\mathbf{P}_S^1 = 0$. We need to do this only for $\nu = -1/2$ and $p = 2$, since $\ker\mathbf{P}_S^1$ is also independent of the parameters p and ν (see Lemma 19).

The equality $\ker \mathbf{P}_S^1 = 0$, in turn, follows from the triviality of a solution of the homogeneous Problem D. Actually, formula (1.17) implies that for any solution $U = (u_1, \ldots, u_4)$ of the homogeneous Problem D we have

$$\int\limits_{\mathbb{R}_S^3} \Big\{ c_{ijkl} D_l u_k D_j \overline{u}_i + \rho\tau^2 u_k \overline{u}_k + \frac{1}{\overline{\tau} T_0} \lambda_{ij} D_j u_4 D_i \overline{u}_4 + \frac{c_0}{T_0} u_4 \overline{u}_4 \Big\} dx = 0;$$

recalling that $\tau = \sigma + i\omega$ and separating the real and the imaginary part, we obtain

$$\begin{gathered} \int\limits_{\mathbb{R}_S^3} \Big\{ c_{ijkl} D_l u_k D_j \overline{u}_i + \rho(\sigma^2 - \omega^2) u_k \overline{u}_k + \\ + \frac{\sigma}{|\tau|^2 T_0} \lambda_{ij} D_j u_4 D_i \overline{u}_4 + \frac{c_0}{T_0} u_4 \overline{u}_4 \Big\} dx = 0, \\ \frac{\omega}{T_0} \int\limits_{\mathbb{R}_S^3} \Big\{ 2\sigma T_0 u_k \overline{u}_k + \lambda_{ij} D_j u_4 D_i \overline{u}_4 \Big\} dx = 0. \end{gathered} \tag{5.13}$$

Whence by (1.12) and (1.14) we find $U = 0$ for an arbitrary τ with $\operatorname{Re} \tau > 0$. For $\tau = 0$ we obtain

$$D_j u_k(x) + D_k u_j(x) = 0, \quad u_4 = 0, \quad k, j = 1, 3, \quad x \in \mathbb{R}_S^3. \tag{5.14}$$

The general solution of this system is (see [1])

$$U = [a \times x] + b,$$

where a and b are the constant three-dimensional vectors with complex entries and $[\cdot \times \cdot]$ denotes the vector product of two vectors. From conditions (1.10) and (5.14) it follows that $U = 0$.

Thus the homogeneous Problem D has only a trivial solution and $\ker \mathbf{P}_S^1 = \{0\}$.

To prove the theorem for the Besov space $\mathbb{B}_{p,p}^\nu(S)$ recall the following interpolation property from (4.5):

If $\mathbf{A} : \widetilde{\mathbb{H}}_p^\nu(S) \to \mathbb{H}_p^{\nu+r}(S)$ is bounded for any $\nu_0 < \nu < \nu_1$ and some $1 < p < \infty$, then the operator $\mathbf{A} : \widetilde{\mathbb{B}}_{p,q}^\nu(S) \to \mathbb{B}_{p,q}^{\nu+r}(S)$ is also bounded for any $\nu_0 < \nu < \nu_1$, $1 \leq q \leq \infty$.

Let conditions (3.8) be fulfilled. Then the operator $\mathbf{P}_S^1 : \widetilde{\mathbb{H}}_p^\nu(S) \to \mathbb{H}_p^{\nu+1}(S)$ has the bounded inverse $(\mathbf{P}_S^1)^{-1} : \mathbb{H}_p^{\nu+1}(S) \to \widetilde{\mathbb{H}}_p^\nu(S)$; due to the above-mentioned interpolation property the operator $(\mathbf{P}_S^1)^{-1} : \mathbb{B}_{p,q}^{\nu+1}(S) \to \widetilde{\mathbb{B}}_{p,q}^\nu(S)$ will also be bounded and therefore the operator $\mathbf{P}_S^1$ in (3.6) has the bounded inverse.

5.2. *Proof of Theorem* 8. After the localization and local transformation of variables (see (5.1)–(5.9)) we obtain the equivalences

$$\varkappa_{j*}\mathbf{P}_S^4\varkappa_{j*}^{-1} \overset{x_0}{\sim} (\mathbf{P}_S^4)^0(x_0,D), \quad x_0\in U_j\subset S, \quad x_0\notin\partial S,$$
$$\varkappa_{j*}\mathbf{P}_S^4\varkappa_{j*}^{-1} \overset{x_0}{\sim} \mathbf{r}_+(\mathbf{P}_S^4)^0(x_0,D), \quad x_0\in U_j\cap\partial S,$$

where

$$(\mathcal{P}_S^4)^0(x_0,\xi)=\mathcal{B}^0(x_0,\xi)(\mathcal{P}_S^1)^0(x_0,\xi)(\mathcal{B}^0)^T(x_0,\xi) \tag{5.15}$$

and $\mathcal{B}^0(x_0,\xi)$ represents the modified principal symbol of the operators $\mathbf{B}(D_x,n(x))$ and $\mathbf{Q}(D_x,n(x))$ (whose principal symbols coincide). The order of $\mathcal{B}^0(x_0,\xi)$ is 1 and therefore (5.15), (5.7) yield

$$((\mathcal{P}_S^4)^\infty(x_0,\xi)\eta,\eta)\geq\delta_5|\xi||\eta|^2, \quad \xi\in\mathbb{R}^2, \quad \eta\in\mathbb{C}^4, \quad \delta_5>0.$$

The homogeneity property

$$D_{\xi_1}^m D_x^\alpha(\mathcal{P}_S^4)^\infty(x,\lambda\xi)=|\lambda|\lambda^{-m}D_{\xi_1}^m D_x^\alpha(\mathcal{P}_S^4)^\infty(x,\xi),$$
$$|\alpha|<\infty, \quad m=0,1,\dots, \quad \xi\in\mathbb{R}^2, \quad \lambda\in\mathbb{R}$$

holds as well (see (5.8)).

Thus the symbol $(\mathcal{P}_S^4)^\infty(x,\xi)$ is elliptic

$$\inf\{|\det(\mathcal{P}_S^4)^\infty(x,\xi)|: x\in\overline{S}, \quad |\xi|=1\}>0$$

and operator (3.10) is a Fredholm one if and only if the operators $\mathbf{r}_+(\mathbf{P}_S^4)^0_{\nu+1,1}(x_0,D)$ are Fredholm in $\mathbb{L}_p(\mathbb{R}^2_+)$ for all $x_0\in\partial S$; here

$$(\mathcal{P}_S^4)^0_{\nu+1,1}(x_0,\xi)=\frac{(\xi_2-i|\xi_1|-i)^\nu}{(\xi_2+i|\xi_1|+i)^{\nu+1}}(\mathcal{P}_S^4)^0(x_0,\xi)=$$
$$=\Big(\frac{\xi_2-i|\xi_1|-i}{\xi_2+i|\xi_1|+i}\Big)^{\nu+1/2}\mathcal{P}_-^4(x_0,\xi)\mathcal{P}_+^4(x_0,\xi),$$
$$(\mathcal{P}_+^4)^{\pm1}(x,\cdot),\ (\mathcal{P}_-^4)^{\pm1}(x,\cdot)\in M_p(\mathbb{R}^2), \quad x_0\in\partial S,$$

and $(\mathcal{P}_+^4)^{\pm1}(x_0,\xi_1,\xi_2+i\lambda)$, $(\mathcal{P}_-^4)^{\pm1}(x_0,\xi_1,\xi_2-i\lambda)$ have bounded analytic extensions for $\lambda>0$. The proof is completed similarly to that of Theorem 7. □

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(Received 01.10.1993)

Authors' addresses:

R. Duduchava
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Z. Rukhadze Str., 380093 Tbilisi
Republic of Georgia

D. Natroshvili
Georgian Technical University
Department of Mathematics (99)
77, M. Kostava Str., 380075 Tbilisi
Republic of Georgia

E. Shargorodsky
Tbilisi State University
Department of Mechanics and Mathematics
2, University Str., 380043 Tbilisi
Republic of Georgia