GEORGIAN MATHEMATICAL JOURNAL: Vol. 4, No. 5, 1997, 477-500

ON ITO–NISIO TYPE THEOREMS FOR DS-GROUPS

V. TARIELADZE

Abstract. It is shown that the convergence of convolution products of probability measures on certain non-locally compact topological abelian groups can be verified by means of characteristic functionals. Analogous results are obtained also for almost everywhere convergence of series of independent random elements in the considered groups. A connection with the Sazonov property of the groups is discussed.

1. Preliminaries. Throughout the paper $\mathbb{N}$ denotes the set of natural numbers; $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ are, respectively, the fields of rational, real, and complex numbers with the ordinary (Euclidean) metric, and $\mathbb{T}$ denotes the multiplicative group of complex numbers of modulus 1 with the metric induced from $\mathbb{C}$.

For a topological abelian group X we denote by X' the topological dual group which consists of all continuous (unitary) characters $h : X \to \mathbb{T}$; the group operation in X' is the natural pointwise multiplication. No topology in X' is specified.

A topological abelian group X is called *dually separated* or *DS-group* if X' separates the points of X; in other words, X is a DS-group if for any different $x_1, x_2 \in X$ there is a character $h \in X'$ such that $h(x_1) \neq h(x_2)$. Hausdorff locally compact abelian (LCA-) groups and any additive subgroup of any Hausdorff locally convex space are examples of DS-groups. Below we shall see another type of examples too.

Let X be a completely regular Hausdorff topological space. Denote by $\mathcal{M}_t(X)$ the set of all Radon probability measures μ defined on the Borel σ-algebra of X. In $\mathcal{M}_t(X)$ we consider only the weak topology (for all the notions unexplained here the reader is referred to [1] and [2]). For fixed $x \in X$ we denote by δ_x the Dirac measure concentrated at x. The Prokhorov theorem says that a subset $M \subset \mathcal{M}_t(X)$ is relatively compact if

1991 *Mathematics Subject Classification.* Primary 60B10, 28C10; Secondary 43A25.
Key words and phrases. Radon probability measure, convolution, DS-group, characteristic functional, random element.

1072-947X/97/0900-0477$12.50/0

it is *tight*, i.e., for any $\varepsilon > 0$ there is a compact subset $K \subset X$ such that $\mu(K) \geq 1 - \varepsilon$ for all $\mu \in M$; tightness is also a necessary condition for the relative compactness of M in the following important cases: X is locally compact (see, e.g., [1]), X is completely metrizable [2, Theorem 1.3.6], X is metrizable, and M is compact and countable (L.LeCam, see [3, Proposition 6]). A sequence of Radon probability measures is called tight if its range is tight. In general, a sequence, convergent in $\mathcal{M}_t(X)$, may not be tight.

Let $(X, +)$ be a Hausdorff topological group. For $\mu_1, \mu_2 \in \mathcal{M}_t(X)$ we denote by $\mu_1 * \mu_2$ their convolution. $(\mathcal{M}_t(X), *)$ is a Hausdorff topological semigroup with the neutral element δ_θ, where θ is the neutral element of X. If P is a property concerning a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_t(X)$, then we say that (μ_n) has *shift-P* if there is a sequence (x_n) in X such that the sequence $(\delta_{x_n} * \mu_n)$ has P. As a rule, the tightness or convergence plays the role of P. For $\mu \in \mathcal{M}_t(X)$ we denote by $\widetilde{\mu}$ the image of μ under the map $x \to -x$; μ is called *symmetric* if $\mu = \widetilde{\mu}$. A measure $\mu \in \mathcal{M}_t(X)$ is called *symmetrized* if $\mu = \nu * \widetilde{\nu}$ for some $\nu \in \mathcal{M}_t(X)$. If X is abelian then any symmetrized measure is symmetric. If $x \in X$ then δ_x is symmetric iff x has the second order, i.e., $2x := x + x = \theta$; δ_x is symmetrized iff $x = \theta$. Let (μ_n) and (μ'_m) be sequences of Radon probability measures on X. We shall frequently use the following well-known assertions (see, e.g., [4] or [2]) :

If (μ_n) and (μ'_n) are tight then $(\mu_n * \mu'_n)$ is tight.

If (μ_n) and $(\mu_n * \mu'_n)$ are tight then (μ'_n) is tight.

If $(\mu_n * \mu'_n)$ is tight then (μ'_n) is shift-tight.

A Radon probability measure μ on a Hausdorff topological group X is called *idempotent* if $\mu * \mu = \mu$. Idempotent measures are explicitly the normalized Haar measures of compact subgroups of X [5]. For $\mu, \nu \in \mathcal{M}_t(X)$ we say that ν is *a factor* of μ and write $\nu \prec \mu$ if $\nu * \nu' = \mu$ for some $\nu' \in \mathcal{M}_t(X)$. A measure $\mu \in \mathcal{M}_t(X)$ is called *idempotent-free* if δ_θ is the only idempotent factor of μ. The group X is called *aperiodic* [6] if $\{\theta\}$ is the unique compact subgroup of X. It follows that X is aperiodic iff δ_θ is the unique idempotent measure in $\mathcal{M}_t(X)$. Additive subgroups of a Hausdorff topological vector space over $\mathbb{Q}$ are typical examples of aperiodic topological groups.

Lemma 1. *Let (X,+) be a Hausdorff topological abelian group, $\mu \in \mathcal{M}_t(X)$ be a symmetric measure, and $x \in X$ be an element such that $\mu * \delta_x$ is again symmetric. Then:*

(a) *$\mu * \delta_{2x} = \mu$;*

(b) *if μ is idempotent-free then $2x = \theta$;*

(c) *if X is aperiodic then $x = \theta$.*

Proof. (a). Since μ and $\mu * \delta_x$ are both symmetric measures, we have

$$\mu * \delta_{-x} = \widetilde{\mu} * \widetilde{\delta}_x = \widetilde{\mu * \delta_x} = \mu * \delta_x.$$

This equality implies (a).

(b). Consider the set

$$H = \{y \in X : \mu * \delta_y = \mu\}.$$

It is easy to see that H is a compact subgroup of X and if ν is the normalized Haar measure of H then $\mu * \nu = \mu$. In particular, $\nu \prec \mu$ and so, by assumption, $\nu = \delta_\theta$, i.e., $H = \{\theta\}$. But according to (a), $2x \in H$, i.e., $2x = \theta$.

(c). Since X is aperiodic, μ is idempotent-free and according to (b) $2x = \theta$. This implies that $\{\theta, x\}$ is a compact subgroup of X. Using again the aperiodicity of X, we obtain $x = \theta$. $\square$

Remark. It is easy to see that Lemma 1 is not valid for nonabelian groups.

In the sequel we need one more notion. A Hausdorff topological group $(X, +)$ is called *wide sense-root compact (ws-root compact)* if for any sequence (x_n) in X, for which the set $\{2x_n : n \in \mathbb{N}\}$ is relatively compact in X, the set $\{x_n : n \in \mathbb{N}\}$ itself is relatively compact in X. A related notion of the 2-root compact group is considered in [6]. It is clear that any (additive) subgroup of a Hausdorff topological vector space over $\mathbb{Q}$ is ws-root compact. Also, any *strong Corwin group* X (i.e., the group X having the property that the map $x \to 2x$ is a homeomorphism) is ws-root compact. Not every LCA-group is ws-root compact. Any aperiodic LCA-group is ws-root compact, since any of such groups is, in fact, a subgroup of a finite-dimensional vector space over $\mathbb{R}$ (see [6]). We do not know whether this is so in general.

Below, if $(X, +)$ is a group then $u : X \to X$ denotes the map defined by the equality $ux = 2x$ for all $x \in X$.

Lemma 2. *Let (μ_n) be a sequence of symmetric Radon probability measures on a Hausdorff topological abelian group X and (x_n) be a sequence in X such that the sequence $(\mu_n * \delta_{x_n})$ is tight.*

Then:

(a) *the set $\{2x_n : n \in \mathbb{N}\}$ is relatively compact in X;*

(b) *the sequence $(\mu_n \circ u^{-1})$ is tight;*

(c) *if either $\mu_n, n \in \mathbb{N}$, are symmetrized measures or X is ws-root compact, then (μ_n) itself is tight.*

Proof.

(a). Clearly, the sequence $(\mu_n * \mu_n * \delta_{2x_n})$ is also tight. The symmetry easily implies that the sequence $(\mu_n * \mu_n)$ is tight too. The tightness of these two sequences gives that (δ_{2x_n}) is a tight sequence and this is equivalent to (a).

(b) immediately follows from the assumption and (a).

(c). First, let our measures be symmetrized, i.e., we have $(\mu_n) = (\beta_n * \widetilde{\beta}_n)$, where (β_n) is a sequence of Radon probability measures on X. So, by assumption, $(\beta_n * \widetilde{\beta}_n * \delta_{x_n})$ is tight. Consequently, (β_n) is shift-tight and this easily implies that (μ_n) is tight. Now if X is ws-root compact, then (a) implies that (δ_{-x_n}) is tight. This and the assumption imply that (μ_n) is tight. □

Let X be a DS-group and μ be a Radon probability measure on X. The *Fourier transform* or the *characreristic functional* of μ is the functional $\widehat{\mu} : X' \to \mathbb{C}$ defined by the equality

$$\widehat{\mu}(h) = \int\limits_X h(x) d\mu(x), \qquad h \in X'.$$

The characteristic functional uniquely defines the measure. More precisely, the following version of the *uniqueness theorem* is valid: if $\mu_1, \mu_2 \in \mathcal{M}_t(X)$ and the set $\{h \in X' : \widehat{\mu}_1(h) = \widehat{\mu}_2(h)\}$ contains a separating subgroup of X', then $\mu_1 = \mu_2$ [7, p. 629]. It is easy to see that if $\mu, \nu, \mu_1, \mu_2 \in \mathcal{M}_t(X)$, then the following assertions hold:

$\mu = \mu_1 * \mu_2$ iff $\widehat{\mu} = \widehat{\mu}_1 \widehat{\mu}_2$;

$\mu = \widetilde{\nu}$ iff $\widehat{\mu} = \bar{\widehat{\nu}}$, where the bar means complex conjugation;

μ is symmetric iff $\widehat{\mu}$ is a real-valued functional;

μ is an idempotent measure iff $\widehat{\mu}(h) \in \{0, 1\}, \forall h \in X'$;

$\mu = \delta_x$ for some $x \in X$ iff $|\widehat{\mu}| = 1$, where $|.|$ means modulus.

Let $(Y, .)$ be a multiplicative group with neutral element 1. A functional $\chi : Y \to \mathbb{C}$ is called *positive definite* if the inequality

$$\sum_{k,j=1}^{n} c_k \bar{c}_j \chi(y_k . y_j^{-1}) \geq 0$$

holds for any $n \in \mathbb{N}, c_1, ..., c_n \in \mathbb{C}$, and $y_1, ..., y_n \in Y$. Positive definite χ is called *normalized* if $\chi(1) = 1$.

It is easy to see that if X is a DS-group and $\chi : X' \to \mathbb{C}$ is a characteristic functional of a Radon probability measure on X, then it is positive definite, normalized, and continuous on X' in the topology $\mathrm{comp}(X', X)$ of uniform convergence on compact subsets of X. When X is a LCA-group, the *Bochner*(*–Weyl–Raikov*) theorem says that the converse assertion is also valid. In general, the Bochner theorem fails, but if instead of $\mathrm{comp}(X', X)$ one uses a different topology, its analogue remains valid for certain non-locally compact DS-groups; see Section 4 for more information. We shall use the following assertion, which, in fact, says that a positive definite quotient of two arbitrary characreristic functionals is always a characteristic functional.

Proposition 1. *Let X be a DS-group, $\Gamma \subset X'$ be a subgroup, and χ : $X' \to \mathbb{C}$ be a positive definite normalized functional. Let us also assume* *that there are Radon probability measures μ and μ_1 on X such that*

$$\widehat{\mu}_1(h)\chi(h) = \widehat{\mu}(h), \qquad h \in \Gamma.$$

Then there is a Radon probability measure β on X such that

$$\widehat{\beta}(h) = \chi(h) \qquad \forall h \in \Gamma.$$

Moreover, if Γ is separating, then the measure β is unique and $\mu_1 * \beta = \mu$.

Proof. Consider in Γ the discrete topology and denote

$$G = (\Gamma', \mathrm{comp}(\Gamma', \Gamma)).$$

Then G is a compact abelian group. Let $b : X \to G$ be the canonical embedding and let ν, respectively, ν_1, be the image of μ, respectively, μ_1, under b. According to the Pontrjagin duality theorem Γ is identified with $(G', \mathrm{comp}(G', G))$ and we can write $\widehat{\nu}_1\chi = \widehat{\nu}$. By the Bochner theorem there is $\gamma \in \mathcal{M}_t(G)$ such that $\widehat{\gamma} = \chi$. Evidently, $\nu_1 * \gamma = \nu$. This equality and the assumption that μ and μ_1 are Radon probability measures on X, imply that for a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X we have $\gamma(\bigcup_n b(K_n)) = 1$. This equality implies that there is a Radon probability measure β on X such that γ is the image of β under b (see [1, p. 39]). Clearly, we have $\widehat{\beta}(h) = \chi(h)$ for all $h \in \Gamma$. The rest of the proposition follows immediately from the uniqueness theorem. □

Remark. A similar result for $\Gamma = X'$ is formulated in [7, p. 642], but in its proof there an incorrect proposition from [7, p. 635] is used.

Lemma 3. *Let X be a DS-group and $\mu \in \mathcal{M}_t(X)$. Denote*

$$\mathcal{U}(\widehat{\mu}) = \{h \in X' : \widehat{\mu}(h) \neq 0\}.$$

The following assertions are equivalent:

(i) *μ is idempotent-free;*
(ii) *$\mathcal{U}(\widehat{\mu})$ generates algebraically X';*
(iii) *$\mathcal{U}(\widehat{\mu})$ separates the points of X.*

Proof. (i)$\Longrightarrow$(ii). Suppose (i) is not valid. Let $\mathcal{T}$ be the topology of convergence in measure μ in X'. Then $Y = (X', \mathcal{T})$ is a topological abelian group and $\mathcal{U}(\widehat{\mu})$ is an open neighbourhood of 1 in Y. Also, let H be the subgroup generated by $\mathcal{U}(\widehat{\mu})$. Then H is an open subgroup of Y and $H \neq Y$. This implies that there is a character $\chi : Y \to \mathbb{T}$ such that $\chi \neq 1$ and $\chi(h) = 1$ for all $h \in H$. Clearly, we have $\widehat{\mu}\chi = \widehat{\mu}$. Since χ is positive definite, by Proposition 1 this equality implies that there is $\beta \in \mathcal{M}_t(X)$ with $\widehat{\beta} = \chi$. Since $|\chi| = 1$, there is $x \in X$ such that $\beta = \delta_x$. Therefore for this x we have $\mu * \delta_x = \mu$. Since $x \neq 0$, from the later equality we derive, as in the proof of Lemma 1, that μ has an idempotent factor which is different from δ_θ. A contradiction.

(ii)$\Longrightarrow$(iii) is evident.

(iii)$\Longrightarrow$(i). Let $\nu \in \mathcal{M}_t(X)$ be an idempotent factor of μ. Evidently,

$$\mathcal{U}(\widehat{\mu}) \subset \mathcal{U}(\widehat{\nu}) = \{h \in X' : \widehat{\nu}(h) = 1\}.$$

This and (iii) imply that $\mathcal{U}(\widehat{\nu})$ is a separating subgroup of X'. By the uniqueness theorem $\nu = \delta_\theta$. $\square$

Corollary. *Let X be a DS-group and $\mu, \nu_n, n \in \mathbb{N}$ be Radon probabiliry measures on X. Suppose that*

$$\widehat{\mu}(h) = \prod_n \widehat{\nu}_n(h) := \lim_n \prod_{k=1}^{n} \widehat{\nu}_k(h) \quad \forall h \in X'$$

and $\mathcal{U}(\widehat{\mu})$ separates the points of X (i.e., μ is idempotent-free). Then the following Cauchy condition is satisfied:

$$\lim_{n,m} \prod_{k=n+1}^{n+m} \widehat{\nu}_k(h) = 1 \quad \forall h \in X'.$$

Proof. Consider the set

$$H = \left\{ h \in X' : \lim_{n,m} \prod_{k=n+1}^{n+m} \widehat{\nu}_k(h) = 1 \right\}.$$

It is easy to see that H is a subgroup of X' and $\mathcal{U}(\widehat{\mu}) \subset H$. Now by the implication (iii)$\Longrightarrow$(ii) of the lemma we obtain $H = X'$. $\square$

We shall also make use of

Proposition 2. *Let X be a DS-group and $(\mu_\alpha)_{\alpha \in A}$ be a net with a relatively compact range in $\mathcal{M}_t(X)$ and let the set $\{h \in X' : \lim_\alpha \widehat{\mu}_\alpha(h)$ exists$\}$ contain a separating subgroup of X'. Then $(\mu_\alpha)_{\alpha \in A}$ converges in $\mathcal{M}_t(X)$.*

The proof employs the uniqueness theorem and proceeds as the proof of Theorem 4.3.1 in [2].

This proposition shows that the characteristic functionals can be used for establishing the convergence of relatively compact nets. Much more can be said in the case of LCA-groups. Namely, *P. Levy's theorem* says that if $\mu, \mu_n, n \in \mathbb{N}$ are Radon probability measures on a LCA-group X and

$$\lim_n \widehat{\mu}_n(h) = \widehat{\mu}(h) \qquad \forall h \in X',$$

then

$$\lim_n \mu_n = \mu.$$

An analogous assertion remains also valid if X is a nuclear locally convex space [8]. But if X is an infinite-dimensional normed space over $\mathbb{R}$, then P. Levy's theorem is false (see, e.g., [2, Proposition 1.3.2]. Nevertheless, as shown in [9] (see also [2, Corollary 1 of Theorem 5.2.4]), if X is a Banach space over $\mathbb{R}$ and $\mu_n = \nu_1 * ... * \nu_n, n \in \mathbb{N}$, where $\nu_n, n \in \mathbb{N}$ are symmetric Radon probability measures on X, then the conclusion of P. Levy's theorem remains valid. In the next section we shall present the analogues of this remarkable result for general DS-groups. The results are obtained as a rule without the assumptions of metrizability, completeness, and separability. In the third section these results are applied to the study of almost everywhere convergence of series of independent random elements in metrizable DS-groups. From this point of view our approach differs from that of [2, 9, 10].

2. Convergence of Convolution Products. In this section we shall use the following notation. $(X, +)$ denotes a Hausdorff topological abelian group with a neutral element θ; $(\nu_n)_{n\in\mathbb{N}}$ denotes an arbitrary sequence of Radon probability measures on X,

$$\mu_n := \nu_1 * \cdots * \nu_n \qquad \forall n \in \mathbb{N}.$$

Recall also that $u : X \to X$ is the mapping defined by the equality $ux = 2x := x + x$.

The first two assertions of the following theorem are known.

Theorem 1. *Suppose that the sequence (μ_n) is tight and $\mu \in \mathcal{M}_t(X)$ is its limit point.*

Then:

(a) *$\mu_n \prec \mu$ for all n and any limit point of (μ_n) has the form $\delta_x * \mu$ for some $x \in X$;*

(b) *if X is metrizable, then (μ_n) is shift-convergent to μ;*

(c) *if $\nu_n, n \in \mathbb{N}$ are symmetrized measures, then $\lim_n \mu_n = \mu$;*

(d) *$\lim_n \mu_n * \widetilde{\mu}_n = \mu * \widetilde{\mu}$; in particular, if $\nu_n, n \in \mathbb{N}$ are symmetric measures, then $\lim_n \mu_n * \mu_n = \mu * \mu$.*

(e) *if $\nu_n, n \in \mathbb{N}$ are symmetric measures and μ is idempotent-free, then $\lim_n \mu_n \circ u^{-1} = \mu \circ u^{-1}$;*

(f) *if $\nu_n, n \in \mathbb{N}$ are symmetric measures, μ is idempotent-free, and X contains no second-order elements, then $\lim_n \mu_n = \mu$;*

(g) *if $\nu_n, n \in \mathbb{N}$ are symmetric measures and X is aperiodic, then $\lim_n \mu_n = \mu$.*

Proof. (a) is a particular case of a more general assertion of [11, p. 38]. (b) follows from (a) and the metrizability of X; see, e.g., [4, p. 71], or [12, p. 347].

(c). We have $\nu_n = \alpha_n * \widetilde{\alpha}_n$, where $\alpha_n \in \mathcal{M}_t(X)$ for all $n \in \mathbb{N}$. Denote $\beta_n = \alpha_1 * \cdots * \alpha_n$; then $\mu_n = \beta_n * \widetilde{\beta}_n$ for all $n \in \mathbb{N}$. This equality implies that (β_n) is shift-tight, i.e., for a sequence (x_n) in X, the sequence $(\beta_n * \delta_{x_n})$ is tight. Now, since any limit point of (μ_n) has the form $\beta * \widetilde{\beta}$, where β is a limit point of $(\beta_n * \delta_{x_n})$, the application of (a) to the sequence $(\alpha_n * \delta_{(x_n - x_{n-1})})$, where $x_0 := \theta$, gives that (μ_n) has only one limit point.

(d) is a particular case of (c); also, (d) follows directly from (a).

(e). Clearly, the range of the sequence $(\mu_n \circ u^{-1})$ is relatively compact and any of its limit point has the form $\beta \circ u^{-1}$, where β is a limit point of (μ_n). Now the assumptions, Lemma 1(b), and (a) easily imply that $\beta \circ u^{-1} = \mu \circ u^{-1}$, i.e., $\mu \circ u^{-1}$ is the only limit point of (μ_n).

(f) follows from the assumptions, Lemma 1(b), and (a).

(g) follows from the assumptions and (f). $\square$

Remark 1. A more general version of the second part of Theorem 1(a) even without the assumption of tightness is obtained in [11, p. 38].

Remark 2. For separable Banach spaces Theorem 1(g) was proved in [9] (see also [2, Corollary 2 of Theorem 5.2.3]).

Remark 3.

(1) If μ is not idempotent-free, then Theorem 1(e) is not valid: put $X := \mathbb{T}, \nu_n := \frac{1}{2}(\delta_i + \delta_{(-i)})$ for all n, where $i^2 = -1$; in this case we shall have that $(\mu_n \circ u^{-1})_{n\in\mathbb{N}}$ has two limit points δ_1 and $\delta_{(-1)}$, i.e., is not convergent.

(2) Theorem 1(g) is not valid without the assumption of aperiodicity of X. Example: let X be the multiplicative group $\{-1, 1\}$ and $\nu_n := \delta_{(-1)}$ for all n.

Theorem 1′. *Suppose that the sequence (μ_n) is shift-tight, i.e., for a sequence (x_n) in X the sequence $(\mu_n * \delta_{x_n})$ is tight and $\mu \in \mathcal{M}_t(X)$ is its limit point.*

Then:

(a) *$\mu_n * \delta_{x_n} \prec \mu$ for all n and any limit point of $(\mu_n * \delta_{x_n})$ has the form $\delta_x * \mu$ for some $x \in X$;*

(b) *if X is metrizable, then (μ_n) is shift-convergent to μ;*

(c) *if $\nu_n, n \in \mathbb{N}$, are symmetrized measures, then (μ_n) is tight and convergent in $\mathcal{M}_t(X)$;*

(d) *the sequence $\mu_n * \widetilde{\mu}_n$ is tight and $\lim_n \mu_n * \widetilde{\mu}_n = \mu * \widetilde{\mu}$; in particular, if $\nu_n, n \in \mathbb{N}$, are symmetric measures, then $\lim_n \mu_n * \mu_n = \mu * \mu$;*

(e) *if $\nu_n, n \in \mathbb{N}$, are symmetric measures, then the sequence $(\mu_n \circ u^{-1})$ is tight.*

Assume, in addition, that X is ws-root compact and $\nu_n, n \in \mathbb{N}$ are symmetric measures. Then the following assertions are also valid:

(f) *(μ_n) is tight;*

(g) *if* μ *is idempotent-free, then* $(\mu_n \circ u^{-1})$ *is convergent in* $\mathcal{M}_t(X)$*;*

(h) *if* μ *is idempotent-free and* X *contains no second-order elements, then* (μ_n) *is tight and convergent in* $\mathcal{M}_t(X)$*;*

(i) *if* X *is aperiodic, then* (μ_n) *is tight and convergent to in* $\mathcal{M}_t(X)$.

Proof. The assertions (a) and (b) are particular cases of the corresponding assertions of Theorem 1.

(c). Evidently, μ_n, $n \in \mathbb{N}$, are also symmetrized measures. So by Lemma 2(c) the sequence (μ_n) is tight. It remains to apply Theorem 1(c).

(d). Since the sequence $(\mu_n * \widetilde{\mu}_n)$ is tight and $\mu_n * \widetilde{\mu}_n = \nu_1 * \widetilde{\nu}_n * \ldots * \nu_n * \widetilde{\nu}_n$ for all n, we can again apply Theorem 1(c).

(e) follows from Lemma 2(b).

(f) follows from Lemma 2(c).

(g). According to (f) the sequence (μ_n) is tight. Let μ' be its limit point. It is easy to see that for some $z \in X$ the measure $\mu' * \delta_z$ is a limit point of the sequence $(\mu_n * \delta_{x_n})$. This and (a) imply that μ' is idempotent-free, since μ is idempotent-free by the supposition. Consequently Theorem 1 (e) is applicable.

(h) follows in the same manner from (f) and Theorem 1(f).

(i) follows from Lemma 2(c) and Theorem 1(g). $\square$

Lemma 4. *Suppose that there is a compact subset* K *of* X *such that*

$$\limsup_n \mu_n(K) > 0.$$

Then the sequence (μ_n) *is shift-tight.*

Proof. Our assumption evidently implies that the sequence (μ_n) is increasing (i.e., $\mu_n \prec \mu_{n+1}$ for all n) and not dispersing in the sense of [12], and thus Lemma 4 is a particular case of Lemma 6.3 from [12, p. 343]. $\square$

To formulate a refinement of Theorem 1(g) let us recall first that a Hausdorff topological space is called a *Suslin space* if it is a continuous surjective image of a complete separable metric space.

Proposition 3. *Let* X *be a Suslin group, and* $\mathcal{T}$ *a Hausdorff group topology in* X *which is weaker than the initial topology* $\mathcal{T}_X$ *of* X*. Suppose that there is a subset* $K \subset X$ *which is compact in topology* $\mathcal{T}$ *and for which*

$$\limsup_n \mu_n(K) > 0.$$

Then:

(a) (μ_n) *is shift-tight (with respect to* $\mathcal{T}_X$*-compacts);*

(b) *if either* ν_n, $n \in \mathbb{N}$, *are symmetrized measures or* X *is a ws-root compact aperiodic topological group and* ν_n, $n \in \mathbb{N}$, *are symmetric measures, then* (μ_n) *is tight and convergent in* $\mathcal{M}_t(X)$.

Proof. Denote $Y = (X, \mathcal{T})$. Then, according to Lemma 4, (μ_n) is shift-tight with respect to $\mathcal{T}$-compact subsets of X, i.e., for a sequence (x_n) the sequence $(\mu_n * \delta_{x_n})$ is $\mathcal{T}$-tight. So, by Prokhorov's theorem, $\{\mu_n * \delta_{x_n} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{M}_t(Y)$. Let $\mu \in \mathcal{M}_t(Y)$ be a limit point of $(\mu_n * \delta_{x_n})$ in the topology of $\mathcal{M}_t(Y)$. Theorem 1(a) implies that for all $n \in \mathbb{N}$ there is a measure $\mu_n' \in \mathcal{M}_t(Y)$ such that $\mu_n * \mu_n' = \mu$. Now, since X is a Suslin space, the Borel σ-algebras of X and of Y are the same (see [1, p. 101]) and any Radon probability measure on Y is also a Radon measure on X (see [1, p. 124]). So we have that $\mu, \mu_n', n \in \mathbb{N}$, are Radon probability measures on X and the equality $\mu_n * {\mu_n}' = \mu$ also holds in $\mathcal{M}_t(X)$ for all n. The last equality implies (a), since the one-element set $\{\mu\}$ is tight.

(b) follows from (a) according to Theorem $1'$(c,i). □

Remark 1. The assumption of ws-root compactness is essential for Proposition 3(b). This follows from Proposition 4(f) below.

Remark 2. Proposition 3(b) is an essential refinement of Theorem 3.4.4 from [8, p. 107], where the case of a complete separable metrizable topological vector space X with a separating dual space is considered and it is required that $\mathcal{T}$ be a vector topology in X such that the space $(X, \mathcal{T})$ should also have a separating dual space.

Now we proceed to the study of convergence of convolution products by using characteristic functionals.

Theorem 2. *Let X be a DS-group. The following assertions are equivalent:*

(i) *the sequence $(\mu_n * \widetilde{\mu}_n)$ is convergent in $\mathcal{M}_t(X)$;*

(ii) *there is a Radon probability measure ρ on X such that the set*

$$\{h \in X' : \lim_n |\widehat{\mu}_n(h)|^2 = \widehat{\rho}(h)\}$$

contains a separating subgroup Γ of X';

(iii) *the sequence (μ_n) is shift-tight.*

Proof. The implication (i)$\Longrightarrow$(ii) is evident. (iii) implies (i) according to Theorem $1'$(d). It remains to prove (ii)$\Longrightarrow$(iii). Fix any $n \in \mathbb{N}$ and define the functional $\chi_n : X' \to \mathbb{R}_+$ by the equality

$$\chi_n(h) = \lim_m \prod_{k=n+1}^{n+m} \widehat{\nu}_k(h)|^2, \quad h \in X'.$$

Evidently, χ_n is a positive definite normalized functional on X' and (ii) readily implies that

$$\widehat{\rho}(h) = |\widehat{\mu}_n(h)|^2 \chi_n(h), \quad h \in \Gamma.$$

Therefore by Proposition 1 there is $\beta_n \in \mathcal{M}_t(X)$ such that $\widehat{\beta}_n(h) = \chi_n(h)$ for all $h \in \Gamma$, and so

$$\rho = \mu_m * \widetilde{\mu}_n * \beta_n.$$

Since this equality holds for all n and the one-element set $\{\rho\}$ is tight, we obtain that (iii) is valid. □

The following theorem is the promised generalization of the mentioned result of [9].

Theorem 3. *Let X be a DS-group, $\nu_n, n \in \mathbb{N}$, be symmetric measures, and suppose that there is a measure $\mu \in \mathcal{M}_t(X)$ such that the set*

$$\{h \in X' : \lim_n \widehat{\mu_n}(h) = \widehat{\mu}(h)\}$$

contains a separating subgroup Γ of X'.

Then:

(a) *the sequence $(\mu_n \circ u^{-1})$ is tight and* $\lim_n \mu_n \circ u^{-1} = \mu \circ u^{-1}$*;*

(b) *if either $\nu_n, n \in \mathbb{N}$, are symmetrized measures or X is ws-root compact, then the sequence (μ_n) is itself tight and* $\lim_n \mu_n = \mu$.

Proof.

(a). By Theorem 2 the sequence (μ_n) is shift-tight, which by Theorem $1'$(e) implies that the sequence $(\mu_n \circ u^{-1})$ is tight. This and Proposition 2 lead to (a).

(b), in the case of symmetrized measures, follows from Theorem 2 and Theorem $1'$(c); the case of ws-root compact X follows from Theorem 2, Theorem $1'$(f), and Proposition 2. □

Remark. Below (see Proposition 4(f)) we shall see that Theorem 3 cannot be improved.

To formulate a version of Theorem 3 for not necessarily symmetric measures we need some more notions.

Let X be a topological abelian group and $\Gamma \subset X'$ be a nonempty subset. Denote by $\sigma(X, \Gamma)$ the weakest topology in X with respect to which all members of Γ are continuous. If X is a topological vector space over $\mathbb{R}$ with the non-trivial topological dual space X^*, then the usual weakened topology $\sigma(X, X^*)$ is finer than $\sigma(X, X')$, but any sequence in X which is convergent in the last topology also converges in the weakened topology. That is why when the underlying group is a Banach space the following notion coincides with the known one. We say that a DS-group X has the *Schur property* if any sequence (x_n) that is convergent in X with respect to $\sigma(X, X')$ converges also in the initial topology of X.

Theorem 4. *Let X be a metrizable DS-group and suppose that there is an idempotent-free $\mu \in \mathcal{M}_t(X)$ such that*

$$\lim_n \widehat{\mu}_n(h) = \widehat{\mu}(h), \qquad \forall h \in X'.$$

Then:

(a) *there is a sequence (x_n) in X and $x \in X$, such that (x_n) converges to x in the topology $\sigma(X, X')$ and*

$$\lim_n \mu_n * \delta_{x_n} = \mu * \delta_x;$$

(b) *if X has the Schur property, then $\lim_n \mu_n = \mu$.*

Proof.

(a). By Theorem 2 and Theorem 1$'$(b) there are a sequence (x_n) and $\mu' \in \mathcal{M}_t(X)$ such that $\lim_n \mu_n * \delta_{x_n} = \mu'$. We want to show that (x_n) converges in the topology $\sigma(X, X')$ to an element $x \in X$. Evidently, we have

$$\lim_n \widehat{\mu}_n(h)h(x_n) = \widehat{\mu'}(h), \qquad \forall h \in X'.$$

Denote

$$H := \{h \in X' : \lim_n h(x_n) \quad \text{exists}\}.$$

We have

$$\{h \in X' : \widehat{\mu'}(h) \neq 0\} \subset H.$$

Since $|\widehat{\mu'}| = |\widehat{\mu}|$, by Lemma 3 μ' is also idempotent-free so that $H = X'$. Define now $\chi : X' \to \mathbb{T}$ by the equality

$$\chi(h) = \lim_n h(x_n), \qquad h \in X'.$$

Clearly, χ is a character so that it is positive definite and the equality

$$\widehat{\mu}(h)\chi(h) = \widehat{\mu'}(h), \qquad h \in X,'$$

holds. By Proposition 1 there is $\beta \in \mathcal{M}_t(X)$ such that $\widehat{\beta} = \chi$. Since $|\chi| = 1$, we have $\beta = \delta_x$ for some $x \in X$. Consequently we find that (x_n) converges to x in the topology $\sigma(X, X')$ and $\mu * \delta_x = \mu'$.

(b) follows immediately from (a). $\square$

Remarks.

(1) Evidently, the Schur property is also necessary for the validity of the conclusion of Theorem 4(b). In this sense the result is final.

(2) We do not know how essential the metrizability assumption and the assumption that μ is idempotent-free are for the validity of Theorem 4.

Now we are going to show that Theorem 3 cannot be improved in the whole class of DS-groups. For this we need an example of a DS-group which is not a LCA-group and which is not ws-root compact. In particular, such a group will not be topologically isomorphic to a subgroup of any topological vector space over $\mathbb{Q}$.

As usual, $\mathbb{R}^{\mathbb{N}}$ denotes the vector space of all real sequences $x : \mathbb{N} \to \mathbb{R}$, and e_n, $n \in \mathbb{N}$, is its natural basis. Let also Θ be the subgroup of $\mathbb{R}^{\mathbb{N}}$ consisting of all integer-valued sequences with only a finite number of non-zero terms. l_p, where $0 < p < \infty$, is the ordinary metric linear space of all p-summable real sequences.

Proposition 4. *Let Y_p be the quotient group l_p/Θ with the quotient metric, $\kappa : l_p \to Y_p$ be the canonical surjection map, and $y_n := \kappa(\frac{1}{2}e_n), n \in \mathbb{N}$. Then:*

(a) *Y_p is a complete separable metrizable DS-group;*

(b) *$2y_n = \Theta$, $\forall n \in \mathbb{N}$;*

(c) *the set $\{y_n : n \in \mathbb{N}\}$ is not relatively compact in Y_p;*

(d) *Y_p is not ws-root compact;*

(e) *$p > 1 \Longrightarrow \lim_n h(y_n) = 1$, $\forall h \in (Y_p)'$;*

(f) *if $p > 1, \nu_1 := \delta_{y_1}, \nu_{n+1} := \delta_{(y_n + y_{n+1})}$ and $\mu_n := \nu_1 * ... * \nu_n$ for all $n \in \mathbb{N}$, then $\nu_n, n \in \mathbb{N}$ are symmetric measures,*

$$\lim_n \widehat{\mu}_n(h) = 1 \qquad \forall h \in (Y_p)',$$

$\mu_n * \mu_n = \delta_\Theta$ for all n, but the sequence $(\mu_n)_{n\in\mathbb{N}}$ does not converge in $\mathcal{M}_t(Y_p)$.

Proof.

(a). It is needed only to verify that Y_p is a DS-group. For a fixed $n \in \mathbb{N}$ define the map $h_n : Y_p \to \mathbb{T}$ by

$$h_n(\kappa(x)) = \exp(2\pi i \kappa(x)), \qquad x \in l_p.$$

Then h_n, $n \in \mathbb{N}$, are continuous characters separating the points of Y_p.

(b) is evident.

(c). Let d_p be the quotient metric on Y_p. It is easy to see that

$$\inf\{d_p(y_n, y_m) : n, m \in \mathbb{N}, n \neq m\} > 0,$$

i.e., the set $\{y_n : n \in \mathbb{N}\}$ cannot be relatively compact.

(d) follows from (b) and (c).

(e). Since $p > 1$, the sequence $(\frac{1}{2}e_n)_{n\in\mathbb{N}}$ converges to 0 in the weakened topology of l_p, and since κ is a continuous homomorphism, this easily implies that (y_n) tends to zero in the topology $\sigma(Y_p, (Y_p)')$.

(f) follows immediately from (b),(c), and (e). $\square$

Let us say that a DS-group X is an *Ito–Nisio group* if for any sequence $\mu, \nu_n, n \in \mathbb{N}$, of symmetric Radon probability measures on X for which

$$\lim_n (\widehat{\nu_1}(h)...\widehat{\nu_n}(h)) = \widehat{\mu}(h) \qquad \forall h \in X',$$

the sequence of convolution products $\nu_1 * ... * \nu_n$, $n \in \mathbb{N}$, converges to μ. Theorem 3 says that all ws-root compact DS-groups are Ito–Nisio groups. According to Proposition 4(f) the DS-group Y_p is not a Ito–Nisio group when $p > 1$; we do not know what happens when $p \leq 1$. It seems interesting to give an internal description of the class of all Ito-Nisio groups.

We also remark that the DS-group Y_p from Proposition 3 is topologically isomorphic to the DS-group $l_p(\mathbb{N}, \mathbb{T})$ considered in [13].

3. Convergence of Sums of Independent Summands. In this section we assume that $(X, +)$ is a metrizable topological abelian group and $(\Omega, \mathcal{A}, \mathbb{P})$ is a fixed probability space. A Borel measurable mapping $\xi : \Omega \to X$ for which the range $\xi(\Omega)$ is separable and the *distribution* $\mathbb{P}_\xi := \mathbb{P} \circ \xi^{-1}$ is a Radon probability measure on X is called a *random element* in X. If ξ_1, ξ_2 are random elements, then the pair (ξ_1, ξ_2) as a mapping from Ω to $X \times X$ is a random element in $X \times X$; this implies that the sum $\xi_1 + \xi_2$ is also a random element in X. If ξ_1, ξ_2 are (stochastically) independent random elements in X, then $\mathbb{P}_{\xi_1+\xi_2} = \mathbb{P}_{\xi_1} * \mathbb{P}_{\xi_2}$. A random element ξ in X is called *symmetric* if $\mathbb{P}_\xi$ is a symmetric measure. *Convergence almost surely* (*a.s.*), respectively, *convergence in probability*, for a sequence of random elements means convergence of $\mathbb{P}$-almost everywhere, respectively, convergence in measure $\mathbb{P}$.

We emphasize that the above definition of a random element is not standard: besides the measurability and separability of the range it is also required that its distribution be a Radon measure. The last requirement is essential when it is not assumed that the range-group is complete.

Throughout the section the following notation is used:

$(\xi_n)_{n\in\mathbb{N}}$ is a fixed (stochastically) independent sequence of random elements in X;

$\nu_n := \mathbb{P}_{\xi_n} \quad \forall n \in \mathbb{N}$;

$\eta_n := \sum_{k=1}^{n} \xi_k \quad \forall n \in \mathbb{N}$;

$\mu_n := \nu_1 * ... * \nu_n \quad \forall n \in \mathbb{N}$;

Evidently, the independence implies that the distribution of η_n is μ_n for all $n \in \mathbb{N}$.

To obtain analogues of the theorems of the previous section for a.s. convergence we need the following known generalization of another result of P. Levy (see, e.g., [5], or [2, Ch.V.2, Corollary 1 of Theorem 3 and Exercises 3(b) and (c)]).

Theorem 5. *Let X be a metrizable topological abelian group and the sequence (μ_n) converge to an idempotent-free Radon probability measure μ on X. Then the sequence (η_n) converges a.s. to a random element η in X whose distribution is μ.*

Corollary 1. *Let X be aperiodic and (μ_n) converge to a Radon probability measure μ on X. Then (η_n) converges a.s. to a random element η in X with $\mathbb{P}_\eta = \mu$.*

Proof. Since X is aperiodic and μ is idempotent-free, we see that Theorem 5 is applicable. $\square$

Remark. Aperiodicity is essential here. An analogous assertion is also valid in the nonabelian case; see [6, Theorem 2.2.19].

Corollary 2. *Let X be aperiodic, $\xi_n, n \in \mathbb{N}$, be symmetric random elements, and the sequence (μ_n) be tight.*

Then (η_n) converges a.s. to a random element in X.

Proof. Apply Theorem 1(g) and the previous corollary. $\square$

Remark. This corollary generalizes an analogous assertion of [9] (see also Corollary 2 of Theorem 5.2.3 in [2]).

Corollary 3. *Let X be a metrizable, ws-root compact, aperiodic, Suslin group, $\mathcal{T}$ be a Hausdorff group topology in X which is weaker than the initial topology of X; let further $\xi_n, n \in \mathbb{N}$, be symmetric random elements and suppose that there is a $\mathcal{T}$-compact $K \subset X$ such that*

$$\limsup_n \mu_n(K) > 0.$$

Then (η_n) converges a.s. in the topology of X.

Proof. Apply Proposition 3(b) and Corollary 2. $\square$

Remark. This corollary is a refinement of the corresponding result of [10]; see Remark 2 to Proposition 3.

In the case of DS-groups it is possible to formulate the following criterion of a.s. convergence.

Theorem 5′. *Let X be a metrizable DS-group. The following statements are equivalent:*

(i) *(η_n) converges a.s. to a random element η in X;*

(ii) *(μ_n) converges to a Radon probability measure μ on X and*

$$\lim_{m,n} \prod_{k=n+1}^{n+m} \widehat{\nu}_k(h) = 1 \qquad \forall h \in X';$$

(iii) *(μ_n) converges to a Radon probability measure μ on X and the set*

$$H = \left\{ h \in X' : \lim_{m,n} \prod_{k=n+1}^{n+m} \widehat{\nu}_k(h) = 1 \right\}$$

separates the points of X;

(iv) (μ_n) *is a tight sequence and the set*

$$H = \left\{ h \in X' : \lim_{m,n} \prod_{k=n+1}^{n+m} \widehat{\nu}_k(h) = 1 \right\}$$

separates the points of X.

Proof. It is well known that (i) implies the first part of (ii) with $\mu = \mathbb{P}_\eta$. Fix now a character $h \in X'$. Then again (i) implies

$$\lim_{n,m} \prod_{k=n+1}^{n+m} h \circ \xi_k = 1$$

in probability and by Lebesgue's theorem on convergence of integrals this gives the second part of (ii).

The implications (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) $\Longrightarrow$(iv) are evident.

(iv)$\Longrightarrow$(iii). It is easy to see that H is a subgroup of X' and

$$H \subset \left\{ h \in X' : \lim_n \widehat{\mu}_n(h) \ \text{ exists} \right\}.$$

This, by Proposition 2, implies the first part of (iii).

(iii)$\Longrightarrow$(i). Denote $\mu_{n,m} = \nu_{n+1} * \ldots * \nu_{n+m}$ for all $n, m \in \mathbb{N}$. The tightness of (μ_n) implies that $\mu_{n,m} : n, m \in \mathbb{N}$ is a tight family of Radon probability measures on X; we have $\lim_{n,m} \widehat{\mu}_{n,m}(h) = 1$ for all $h \in H$. Therefore by Proposition 2 we obtain $\lim_{n,m} \mu_{n,m} = \delta_0$. This implies that the sequence (η_n) is a Cauchy sequence in probability. Consequently this sequence converges in probability and thus almost surely to a random element η^1 in the completion of X. From this, since μ is a Radon probability measure on X, we can easily derive that η^1 equals a.s. a random element η in X (cf. [2, p. 269, the proof of Corollary 1]). □

Remark. The implication (ii)$\Longrightarrow$(i) of Theorem 5′ and the Corollary to Lemma 3 immediately imply an alternative proof of Theorem 5 in the case of DS-groups.

Theorem 6. *Let* X *be a metrizable DS-group and suppose that there is an idempotent-free Radon probability measure* ρ *on* X *such that the set*

$$\{h \in X' : \lim_n |\widehat{\mu_n}(h)|^2 = \widehat{\rho}(h)\}$$

contains a separating subgroup Γ *of* X'.

Then:

(a) *there is a sequence* (x_n) *in* X *such that* $(\eta_n + x_n)$ *converges a.s. to a random element in* X*;*

(b) *if* $\nu_n, n \in \mathbb{N}$, *are symmetrized measures, then* (η_n) *converges a.s. to a random element in* X*;*

(c) *if $\xi_n, n \in \mathbb{N}$, are symmetric random elements, then $(2\eta_n)$ converges a.s. to a random element in X;*

(d) *if X is a ws-root compact group containing no second-order elements and $\xi_n, n \in \mathbb{N}$, are symmetric random elements, then (η_n) itself converges to a random element in X.*

Proof.

(a). By Theorem 2 and Theorem 1′(b) there is a sequence (x_n) in X such that $\lim_n \mu_n * \delta_{x_n} = \mu \in \mathcal{M}_t(X)$ and $\mu * \widetilde{\mu} = \rho$. Evidently, μ is also idempotent-free and it remains to apply Theorem 5.

(b) follows directly from Theorem 2, Theorem 1′(c), and Theorem 5.

(c). An application of (a), symmetry, and Theorem 5 give that the sequences $(\eta_n + x_n)$ and $(\eta_n - x_n)$ are convergent to their limit random elements in X a.s. This clearly implies the needed conclusion.

(d) follows directly from Theorem 2, Theorem 1′(h), and Theorem 5. □

Corollary. *Let X be a metrizable, ws-root compact, and aperiodic DS-group, $\xi_n, n \in \mathbb{N}$, be symmetric, and suppose that there is a Radon probability measure ρ on X such that the set*

$$\{h \in X' : \lim_n |\widehat{\mu_n}(h)|^2 = \widehat{\rho}(h)\}$$

contains a separating subgroup of X'.

Then (η_n) converges a.s. to a random element in X.

Proof. Since X is aperiodic, ρ is idempotent-free, and X contains no second-order elements, we conclude that Theorem 6(d) is applicable. □

The following assertion is a version of Theorem 3 for a.s. convergence.

Theorem 7. *Let X be a metrizable DS-group, $\xi_n, n \in \mathbb{N}$, be symmetric random elements, and suppose that there is an idempotent-free Radon probability measure μ on X such that the set*

$$\{h \in X' : \lim_n \widehat{\mu_n}(h) = \widehat{\mu}(h)\}$$

contains a separating subgroup Γ of X'.

Then:

(a) *the sequence $(2\eta_n)$ converges a.s. to a random element in X;*

(b) *if either $\nu_n, n \in \mathbb{N}$, are symmetrized measures or X is ws-root compact, then (η_n) itself converges a.s. to a randon element in X.*

Proof. (a) follows from Theorem 6(c); (b) follows from Theorem 3(b) and Theorem 5. □

Remarks.

(1) Proposition 4(f) shows that Theorem 7(b) is not valid without the assumption of ws-root compactness even for arbitrary complete separable metrizable DS-groups.

(2) Theorem 7(b) for Banach spaces and $\Gamma = X'$ is proved in [9]; the case of complete separable metrizable topological vector spaces and arbitrary Γ is considered in [10] (see also [2, Theorem 5.2.4]).

The following assertion, which is a consequence of Theorem 7 and the Borel–Cantelli lemma, gives a necessary condition for the convergence of an infinite product of characteristic functionals to a characteristic functional.

Proposition 5. *Let X be a metrizable DS-group and $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of symmetric Radon probability measures on X. Also suppose that there is a Radon probability measure μ on X such that the set $\{h \in X' : \widehat{\mu}(h) \neq 0\}$ separates the points of X and the set*

$$\{h \in X' : \lim_n \prod_{k=1}^{n} \widehat{\nu_k}(h) = \widehat{\mu}(h)\}$$

contains a separating subgroup of X'. Also let V be any open neighborhood of zero in X.

Then:

(a) $\sum_n \nu_n(\{x \in X : 2x \notin V\}) < \infty$;

(b) *if either $\nu_n, n \in \mathbb{N}$, are symmetrized measures or X is ws-root compact, then*

$$\sum_n \nu_n(X \setminus V) < \infty.$$

Proof. (a). Let d be an invariant metric which is compatible with the topology of X. Then there is $\varepsilon_V > 0$ such that

$$\{x \in X : d(x, 0) < \varepsilon_V\} \subset V.$$

Choose now a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and an independent sequence $\xi_n : \Omega \to X$, $n \in \mathbb{N}$ of random elements such that $\mathbb{P} \circ \xi_n^{-1} = \nu_n$ for all $n \in \mathbb{N}$. The assumption according to Theorem 7(a) implies that the series $\sum_n 2\xi_n$ is a.s. convergent in X. In particular, $d(2\xi_n(.), \theta) \to 0$ a.s. when $n \to \infty$. This and the independence according to the Borel–Cantelli lemma (see, e.g., [2, Corollary (b) to Proposition 5.1.3]) imply

$$\sum_n \nu_n(\{x \in X : d(2x, \theta) \geq \varepsilon\}) = \sum_n \mathbb{P}(\{\omega \in \Omega : d(2\xi_n(\omega), \theta) \geq \varepsilon\}) < \infty$$

for all $\varepsilon > 0$. This implies (a), since

$$\{x \in X : d(2x, \theta) < \varepsilon_V\} \subset \{x \in X : 2x \in V\}.$$

(b) follows in a similar way from Theorem 7(b). □

4. Connection with the Sazonov Property. A DS-group X is said to possess the *Sazonov property* if in X' there exists an *admissible topology* $\mathcal{T}$, which is a topology in X' with the following two properties: $\mathcal{T}$ is a *necessary topology*, i.e., the characteristic functional of any Radon probability measure on X is $\mathcal{T}$-continuous and $\mathcal{T}$ is also a *sufficient topology*, i.e., any positive definite normalized $\mathcal{T}$-continuous $\chi : X' \to \mathbb{C}$ is the characteristic functional of a Radon probability measure on X. The Bochner theorem in this terminology means that if X is a LCA-group, then $\mathrm{comp}(X', X)$ is an admissible topology; the Minlos theorem (see [2, Theorem 6.4.3]) implies that the same is also true when X is a nuclear Frechet space or even if X is a complete metrizable nuclear group (see [14], where the notion is introduced and the Bochner theorem for such groups is obtained). But if X is an infinite-dimensional Banach space, then $\mathrm{comp}(X', X)$ is not an admissible topology. The existence of an admissible topology for an infinite-dimensional Hilbert space was established by Sazonov and, independently, by L. Gross; the existence and characterization of a class of (non-Hilbertian) Banach spaces which possess the Sazonov property were obtained by Mushtari (see [2, Ch.VI] or [15] for further information and comments). For DS-groups the question was studied in [13]. The spaces $l_p, 0 < p \leq 2; L_p, 1 \leq p \leq 2$, possess the Sazonov property (Mushtari). The same is true for the DS-groups Y_p, $0 < p \leq 2$, considered in Section 2, see [13].

Let X be a DS-group. Denote by $Fo(X', X)$ the weakest topology in X with respect to which the functionals $\widehat{\mu}, \mu \in \mathcal{M}_t(X)$ are continuous. Evidently, $Fo(X', X)$ is the weakest necessary topology in X'. This topology is a group topology, i.e., it is compatible with the group structure of X' (cf. [16], where a similar assertion is proved for topological vector spaces). We can say that a DS-group X possesses the Sazonov property iff $Fo(X', X)$ is a sufficient topology.

For DS-groups which possess the Sazonov property the convergence of convolution products by means of characteristic functionals can be verified in a more traditional way.

Proposition 6. *Let X be a DS-group which possesses the Sazonov property, $\mathcal{T}$ be an admissible topology in X', $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of Radon probability measures on X, and $\mu_n := \nu_1 * ... * \nu_n$ for all n.*

The following statements are equivalent:

(a) *the sequence $(\mu_n * \widetilde{\mu}_n)_{n\in\mathbb{N}}$ converges in $\mathcal{M}_t(X)$;*

(b) *$\chi := \lim_n |\widehat{\mu}_n|^2$ is a $\mathcal{T}$-continuous functional on X'.*

Proof. The implication (a)$\Longrightarrow$(b) is evident.

(b)$\Longrightarrow$(a). Evidently, χ is a positive definite normalized functional on X' and, by assumption, is continuous in an admissible topology. Therefore

$\chi = \widehat{\rho}$ for a Radon probability measure ρ on X. Now it remains to apply Theorem 2(c). $\square$

Let us say that a DS-group X possesses *the Sazonov property in a wide sense* if in X' there is a necessary topology $\mathcal{T}$ such that the implication $(b) \Longrightarrow (a)$ of Proposition 6 remains valid. Thus, Proposition 6 says that if a DS-group possesses the Sazonov property then it possesses the Sazonov property in a wide sense too. It is known that if a Banach space X possesses the Sazonov property then X is of cotype 2 [2, Proposition 6.2.4, Corollary]. We shall show that a similar assertion is also valid for metrizable DS-groups possessing the Sazonov property in a wide sense. For this we present a definition of cotype 2 in the case of general topological abelian groups.

Let $(X,+)$ be a group and $V \subset X$ be a nonempty subset. Define the functional $m_V : X \to \mathbb{N} \cup \{\infty\}$ by the equality ($\inf \varnothing := \infty$)

$$m_V(x) = \inf\{n \in \mathbb{N} : nx \notin V\}, \qquad x \in X.$$

A closely related functional n_V^x is used in [14, p. 8]. It is easy to see that if X is a vector space over $\mathbb{R}$ and V is a radial subset (i.e., $rV \subset V\ \forall r \in [0,1]$) and $||.||_V$ is its Minkowski functional, then

$$\frac{1}{m_V(x)} \le ||x||_V \le 2\frac{1}{m_V(x)-1} \qquad \forall x \in X.$$

Here and below we put $\frac{1}{\infty} = 0, \frac{1}{0} = \infty$.

A probability measure π on the additive group $\mathbb{Z}$ of integers is called a *symmetrized Poisson measure* if $\pi = \alpha * \widetilde{\alpha}$, where $\alpha = \exp(-\frac{1}{2})\sum_{j=0}^{\infty} \frac{1}{2^j j!}\delta_j$ is the ordinary Poisson measure with parameter $\frac{1}{2}$. An independent sequence $(\zeta_n)_{n\in\mathbb{N}}$ of integer-valued random elements on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is called a *Poisson sequence*; respectively, a *Rademacher sequence* if $P_{\zeta_n} = \pi$; respectively, if $P_{\zeta_n} = \frac{1}{2}\delta_{(-1)} + \frac{1}{2}\delta_1$ for all n. It is easy to see that if (ζ_n) is a Poisson or Rademacher sequence and (t_n) is a sequence in $\mathbb{T}$, then the infinite product $\prod_n t_n^{\zeta_n}$ converges a.s. in $\mathbb{T}$ iff $\sum_n (1 - \Re t_n) < \infty$.

Lemma 5. *Let X be a metrizable DS-group, (x_n) be a sequence in X, and (ζ_n) be a Poisson sequence. The following assertions are equivalent:*

(i) *the series $\sum_n \zeta_n x_n$ converges a.s. to a random element η in X;*

(ii) *$\sum_n (1 - \Re h(x_n)) < \infty$ for all $h \in X'$ and the functional χ defined by the equality*

$$\chi(h) = \exp(\sum_n (\Re h(x_n) - 1)), \qquad h \in X',$$

is the characteristic functional of a Radon probability measure μ on X.

Proof. Let us denote by ν_n the distribution of $\zeta_n x_n$. Then we shall have

$$\widehat{\nu}_n(h) = \exp(\Re h(x_n) - 1) \qquad \forall n \in \mathbb{N}, \forall h \in X'.$$

Now the implication (i)$\Longrightarrow$(ii) is easy: the first part of (ii) follows at once from the assertion mentioned before the lemma and the second part follows from the independence if we put $P_\eta = \mu$. The implication (ii)$\Longrightarrow$(i) follows from Theorem 7(b), since ν_n, $n \in \mathbb{N}$, are symmetrized measures and $\widehat{\mu}(h) > 0$ for all $h \in X'$. $\square$

Remark. If in Lemma 5 instead of a Poisson sequence we consider a Rademacher sequence and in (ii) we take

$$\chi(h) = \prod_n \Re h(x_n), \qquad h \in X',$$

then the implication (i)$\Longrightarrow$(ii) remains valid. According to Theorem 7(b), if X is ws-root compact, the implication (ii)$\Longrightarrow$(i) also holds. Proposition 4(f) shows that ws-root compactness is essential here.

Let $(X, +)$ be a metrizable topological group, and $p, 0 < p < \infty$, be a number. X is said to be of *Poisson*, respectively, *Rademacher, cotype p* if for any sequence (x_n) in X for which the series $\sum_n \zeta_n x_n$ a.s. converges to a random element in X, where (ζ_n) is a Poisson, respectively, Rademacher, sequence, the condition

$$\sum_n \frac{1}{m_V^p(x_n)} < \infty$$

is satisfied for any neighborhood V of zero in X.

It is easy to see that in the case of Banach spaces the given definition agrees with the known definitions. Also, it is known that in this case the notions of Poisson and Rademacher cotypes coincide (see [17]).

Proposition 7. *Let X be a metrizable DS-group which possesses the Sazonov property in a wide sense; in particular, suppose that X possesses the Sazonov property. Then X is of Poisson cotype* 2.

Proof. Fix a symmetric open neighborhood V of zero and a sequence (x_n) in X such that the series $\sum_n \zeta_n x_n$ converges a.s. to a r.e. in X for a Poisson sequence (ζ_n) . We want to show that $\frac{1}{m_V^2(x_n)} < \infty$. Denote $k_n = m_V(x_n)$. Since, evidently, $\lim_n x_n = \theta$, we can suppose that $k_n > 1$ for all n. Take now q_n such that $0 < q_n < 1$ and $q_n(1 - q_n) = \frac{1}{2k_n^2}$ for all n and define

$$\nu'_n := q_n \delta_{k_n x_n} + (1 - q_n)\delta_\theta \qquad \forall n \in \mathbb{N}.$$

Then we shall have

$$\nu'_n * \widetilde{\nu}'_n = \frac{1}{2k_n^2}\delta_{k_n x_n} + \frac{1}{2k_n^2}\delta_{(-k_n x_n)} + (1 - \frac{1}{k_n^2})\delta_\theta \qquad \forall n \in \mathbb{N}$$

and

$$|\widehat{\nu'_n}(h)|^2 = 1 - \frac{1}{k_n^2}(1 - \Re(h(k_n x_n))) \qquad \forall n \in \mathbb{N} \qquad \forall h \in X'.$$

Now by the first part of Lemma 5(ii) it is easy to see that

$$\chi'(h) := \prod_n |\widehat{\nu'_n}(h)|^2 > 0 \qquad \forall h \in X'.$$

Clearly, χ' is a positive definite normalized functional on X' and

$$1 - \chi'(h) \leq \ln \widehat{\mu}(h) \qquad \forall h \in X',$$

where μ is the measure from Lemma 5(ii). This inequality implies that χ is continuous in the topology $Fo(X', X)$ (here the fact that the latter is a group topology is used). Therefore it is also continuous with respect to any necessary topology in X'. Since by our assumption X possesses the wide sense Sazonov property, we conclude that the sequence $(\nu'_1 * \widetilde{\nu}'_1 * \cdots * \nu'_n * \widetilde{\nu}'_n)_{n\in\mathbb{N}}$ converges to a Radon probability measure μ' on X. This immediately implies

$$\prod_n |\widehat{\nu'_n}(h)|^2 = \widehat{\mu'}(h) \qquad \forall h \in X'.$$

According to Proposition 5 this equality gives

$$\sum_n \nu'_n * \widetilde{\nu}'_n(X \setminus V) < \infty.$$

Since evidently

$$\nu'_n * \widetilde{\nu}'_n(X \setminus V) = \frac{1}{k_n^2}$$

for all n, we now obtain $\sum_n \frac{1}{k_n^2} < \infty$. $\square$

Remark. If X is a Banach space possessing the Sazonov property in a wide sense, then by using the same method and Theorem 6.2.3 from [2] it can be proved that X is isomorphic to a closed subspace of the space $L_0(\Omega, \mathcal{A}, \mathbb{P})$. From this it follows that if a Banach space X possesses the Sazonov property in a wide sense and has the approximation property (or even the measure approximation property; see [16]) too, then X possesses the Sazonov property. For the general case a similar assertion is unknown.

Proposition 8. *For $2 < p < \infty$, the group Y_p (see Proposition 4) is not of Poisson cotype 2. Therefore it does not possess the Sazonov property in a wide sense, in particular, it does not possess the Sazonov property.*

Proof. We use the same notations as in Proposition 4. Take $k_n = [(n+1)^{\frac{1}{2}}]$ for all $n \in \mathbb{N}$ (here [.] means the integer part); put $x_n := \kappa(\frac{1}{2k_n}e_n)$ for all n and $V_0 := \{y \in Y_p : d_p(y, \theta) < \frac{1}{4}\}$. Let us show that the series $\sum_n \zeta_n x_n$, where (ζ_n) is a Poisson sequence, converges a.s. in Y_p. In fact, we have

$$\sum_n \int_\Omega \left|\frac{1}{k_n}\zeta_n\right|^p d\mathbb{P} = \Big(\sum_n \frac{1}{k_n^p}\Big) \int_\Omega |\zeta_1|^p d\mathbb{P} < \infty.$$

Consequently, the series $\sum_n \frac{\zeta_n}{2k_n} e_n$ is a.s. convergent in l_p. Therefore the series $\sum_n \zeta_n x_n$ is also a.s. convergent in Y_p. Now, since $d_p(k_n x_n, \theta) = \frac{1}{2}$, we obtain $m_V(x_n) \le k_n$ for all n and so

$$\sum_n \frac{1}{m_V^2(x_n)} \ge \sum_n \frac{1}{k_n^2} = \infty,$$

i.e., Y_p is not of Poisson cotype 2. The rest follows from Proposition 7. □

Remark. It is well known that for $2 < p < \infty$, the Banach space l_p is not of cotype 2, but from this fact it does not follow automatically that its quotient $Y_p := l_p/\Theta$ is not of cotype 2 either; there is a Banach space E that is not of cotype 2 and its closed subspace F such that both spaces F and E/F are linearly isometric to a Hilbert space (Lindenstrauss, see [18]).

The above corollary shows that the DS-groups $Y_p, 0 < p < \infty$, inherit the properties of the spaces $l_p, 0 < p < \infty$: when $0 < p \le 2$ in both cases we have the Sazonov property, and when $2 < p < \infty$ we do not have it. This completes the corresponding result of [13], where the case of groups $Y_p, 2 < p < \infty$, was left open. This circumstance served as the motivation for this paper.

Aknowledgment

The research described in this publication was made possible in part by Grant No. MXC 000 from the International Science Foundation.

References

1. L. Schwartz, Radon measures on arbitrary topological spaces and cylindrical measures. *Tata Institute of Fundamental Research Studies in Mathematics,* 6, *Oxford University Press* (*published for the Tata Institute of Fundamental Research*), *London*, 1973.

2. N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan, Probability distributions on Banach spaces. *D. Reidel, Dordrecht*, 1987.

3. D. H. Fremlin, D. G. H. Garling, and R. G. Heydon, Bounded measures on topological spaces. *Proc. London Math. Soc.* **25**(1972), 115–136.

4. K. P. Parthasarathy, Probability measures on metric spaces. *Academic Press, New York and London*, 1967.

5. A. Tortrat, Lois de probabilite sur une espace topologique complement regulier et produits infinis a termes independants dans un group topologique. *Ann. Inst. H. Poincaré* **12**(1965), 217–237.

6. H. Heyer, Probability measures on locally compact groups. *Springer-Verlag, Berlin–Heidelberg–New York,* 1977.

7. V. I. Tarieladze, Characteristic functionals and cylindrical measures in DS-groups. *Prob. Theory and Math. Stat., Prohorov et al.* (*eds*)*, vol.* 2, 625–648, *VNU Science Press, Utrecht,* 1986.

8. P. Boulicaut, Convergence etroite et convergence cylindrique d'une suite des probabilites de Radon. *Z. Wahrshenlichkeitstheor. Verw. Geb.* **28**(1973), 43–52.

9. K. Ito and M. Nisio, On convergence of sums of independent Banach space valued random variables. *Osaka Math. J.* **5**(1968), 35–48.

10. V. V. Buldygin, Convergence of random series in topological spaces. (Russian) *Naukova Dumka, Kiev,* 1980.

11. I. Cziszar, On the *weak** continuity of convolution in a convolution algebra over an arbitrary topological group. *Studia Sci. Math. Hungarica* **6**(1971), 27–40.

12. I. Z. Ruzsa, Infinite convolutions and shift-convergence of measures on topological groups. *Probability measures on groups* (*Oberwolfach,* 1981), 337–361, *Lecture Notes in Math.* 928, *Springer-Verlag, Berlin–New York,* 1982.

13. V. I. Tarieladze, A topological description of characteristic functionals in some groups. *Theor. Prob. Appl.* **34**(1989), 658–667.

14. W. Banaszczyk, Additive subgroups of topological vector spaces. *Lecture Notes in Math.* 1466, *Springer-Verlag, Berlin,* 1991.

15. D. Mushtari, Probabilities and topologies in Banach spaces. (Russian). *Kazan University Press, Kazan*, 1989.

16. V. Tarieladze, On the topological description of characteristic functionals in infinite-dimensional spaces. *Probability in Banach Spaces,* 9*,* 223–238, *Progr. Probab.* 35, *Birkhäuser Boston, Boston,* 1994.

17. B. Maurey and G. Pisier, Series de variables aleatoires vectorielles independantes et geometrie des espaces de Banach. *Studia Math.* **58**(1976), 45–90.

18. P. Enflo, J. Lindenstrauss, and G. Pisier, On the "three space problem." *Math. Scand.* **38**(1975), 199–210.

(Received 29.06.1995)

Author's address:
Muskhelishvili Institute of Computational Mathematics
Georgian Academy of Sciences
8, Akuri St., Tbilisi 380093
Georgia