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ON THE REPRESENTATION OF NUMBERS BY THE DIRECT SUMS OF SOME QUATERNARY QUADRATIC FORMS

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abstract. The systems of bases are constructed for the spaces of cusp forms $S_{2m}(\Gamma_0(5),\chi^m)$ and $S_{2m}(\Gamma_0(13),\chi^m)$ for an arbitrary integer $m\geq 2$. Formulas are obtained for the number of representations of a positive integers by the direct sums of some quaternary quadratic forms.

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Let F_{2m} denote a direct sum of m quaternary quadratic forms F_2 with the same positive discriminant q (q is prime $\equiv 1 \pmod 4$). We shall use the notions, notation and some results from [1].

In what follows let

$$Q^{(q)}_{\frac{q-1}{2}m}=Q^{(q)}_{\frac{q-1}{2}m}(y)=Q^{(q)}_{\frac{q-1}{2}m}\Big(y_1,y_2,\dots,y_{(q-1)m}\Big),$$

where

$$Q^{(q)}_{\frac{q-1}{2}m}=\sum_{k=1}^{m}Q^{(q)}_{\frac{q-1}{2}}\Big(y_{(k-1)(q-1)+1},\dots,y_{k(q-1)}\Big)$$

with

$$Q^{(q)}_{\frac{q-1}{2}}(u_1,\dots,u_{q-1})=q\sum_{1\leq i\leq j\leq q-2}u_iu_j+q\sum_{1\leq i\leq q-2}u_iu_{q-1}+\frac{q-1}{2}\,u_{q-1}^2;$$

$$h^{(q)}=\Big(1,2,\dots,\frac{q-1}{2},\frac{q+3}{2},\frac{q+5}{2},\dots,q-1,\frac{q(1-q)}{2}\Big),$$

$$g^{(q,m)}=\Big(g_1,g_2,\dots,g_{(q-1)m}\Big)=(h^{(q)},h^{(q)},\dots,h^{(q)});$$

$y\equiv g^{(q,m)} \pmod q$ is to be understood elementwise;

$$\sigma_t^*(n)=\begin{cases}\Big(q^{(t+1)/2}+q^{(t+1)/4}+1\Big)\sigma_t(n), & \text{if } q\nmid n,\\ \Big(q^{(t+1)/2}+q^{(t+1)/4}+1\Big)\sigma_t(n)+q^{3(t+1)/4}\sigma_t\Big(\dfrac{n}{q}\Big), & \text{if } q|n,\end{cases}$$

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where

$$\sigma_t(n) = \sum_{d|n} d^t;$$

$$\rho_t^*(n) = q^{(3t+1)/4} \sum_{\delta d = n} \Big(\frac{\delta}{q}\Big) d^t + (-1)^{(t+1)/2} \sum_{d|n} \Big(\frac{d}{q}\Big) d^t,$$

where $(\frac{\cdot}{q})$ is the Jacobi symbol.

1. Let

$$F_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_2 + x_1x_3 + x_2x_4,$$
$$\Phi_2 = x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_4 + 2x_3x_4.$$

F_2 and Φ_2 are the only reduced positive quaternary quadratic forms with determinants 5 and 25 respectively [2, pp. 146, 147]. In [3] it is shown, that F_{2m} is a quadratic form of type $(2m, 5, \chi^m)$ $(\chi = \chi(d) = (\frac{d}{5}))$.

To construct a spherical function of order ν, we find a linear transformation which reduces the quadratic form to a sum of squares. Further, all possible monomials of order ν are constructed of the variables of the quadratic form. The found linear transformation are next applied to these monomials. The obtained polynomial is subjected to the action of the Laplace operator and for the resulting polynomials we choose identically vanishing linear combinations. A linear combination of the corresponding monomials of order ν is a spherical function of the same order with respect to the initial quadratic form.

Applying the Jacobi method it is easy to show that the linear transformation

$$\begin{gathered} x_1 = y_1 - \frac{1}{\sqrt{3}}\, y_2 - \frac{2}{\sqrt{15}}\, y_3, \quad x_2 = \frac{2}{\sqrt{3}}\, y_2 + \frac{1}{\sqrt{15}}\, y_3 - \frac{1}{\sqrt{5}}\, y_4, \\ x_3 = \sqrt{\frac{3}{5}}\, y_3 - \frac{1}{\sqrt{5}}\, y_4, \quad x_4 = \frac{2}{\sqrt{5}}\, y_4 \end{gathered} \tag{1.1}$$

takes Φ_2 into a sum of squares.

Lemma 1. (a) $\varphi = \varphi(x_1, \dots, x_4) = x_1^2 - 2x_4^2$ *is a spherical function of order* 2 *with respect to* Φ_2*;*

(b) $\vartheta(\tau; \Phi_2, \varphi) = 4z - 16z^2 + 8z^3 + 32z^4 - 20z^5 + \cdots \in S_4(\Gamma_0(5), 1)$*;*

(c) $\operatorname{ord}(\vartheta(\tau; \Phi_2, \varphi), i\infty, \Gamma_0(5)) = 1$.

Proof. Using transformation (1.1) we get

$$\begin{aligned} \varphi(x_1, \dots, x_4) = y_1^2 + \frac{1}{3}\, y_2^2 + \frac{4}{15}\, y_3^2 - \frac{8}{5}\, y_4^2 - \frac{2}{\sqrt{3}}\, y_1y_2 - \frac{4}{\sqrt{15}}\, y_1y_3 \\ + \frac{4}{3\sqrt{5}}\, y_2y_3 = \varphi(y_1, \dots, y_4), \end{aligned}$$

$$\sum_{i=1}^{4} \frac{\partial^2\varphi}{\partial y_i^2} = 2\Big(1 + \frac{1}{3} + \frac{4}{15} - \frac{8}{5}\Big) = 0.$$

Consequently, by definition (see [4], pp. 849, 853), φ is a spherical function of order 2 with respect to Φ_2 and, by Lemma 3 from [1], $\vartheta(\tau;\Phi_2,\varphi)\in S_4(\Gamma_0(5),1)$.

Having performed suitable calculations we get

$$\begin{aligned}\vartheta(\tau;\Phi_2,\varphi)&=\sum_{n=1}^{\infty}\Big(\sum_{\Phi_2=n}x_1^2-2x_4^2\Big)z^n\\&=4z-16z^2+8z^3+32z^4-20z^5+\cdots.\end{aligned}\tag{1.2}$$

It follows from (1.2) and formula (1.1) of [1] that

$$\operatorname{ord}\Big(\vartheta(\tau;\Phi_2,\varphi),i\infty,\Gamma_0(5)\Big)=1.\qquad\square$$

Theorem 2. *Let $C=(c_{sr})$ $(s=1,2,3;\ r=1,2,\dots,m-1)$ be the matrix whose elements are non-negative integers satisfying the conditions*

$$\left.\begin{aligned}4c_{1r}+2c_{2r}+2c_{3r}&=2m\\c_{1r}+c_{2r}=r,\quad c_{1r}&>0\end{aligned}\right\}\quad(r=1,2,\dots,m-1).\qquad\begin{aligned}(1.3)\\(1.4)\end{aligned}$$

Then for $m\geq 2$ the system of functions

$$\vartheta^{c_{1r}}(\tau;\Phi_2,\varphi)\vartheta^{c_{2r}}(\tau;Q_2^{(5)},1,h^{(5)})\vartheta^{c_{3r}}(\tau,F_2)\ \ (r=1,2,\dots,m-1)\tag{1.5}$$

is the basis of the space $S_{2m}(\Gamma_0(5),\chi^m)$.

Proof. By Lemma 1, (1.3) and Lemmas 3, 8, 9 from [1], functions (1.5) are cusp forms of type $(2m,\Gamma_0(5),\chi^m)$. By Lemma 1, (1.4) and Lemma 8 from [1], we have

$$\operatorname{ord}\Big(\vartheta^{c_{1r}}(\tau;\Phi_2,\varphi)\vartheta^{c_{2r}}(\tau;Q_2^{(5)},1,h^{(5)})\vartheta^{c_{3r}}(\tau,F_2),i\infty,\Gamma_0(5)\Big)=r$$
$$(r=1,2,\dots,m-1).$$

Functions (1.5) are linearly independent because their orders at the cusp $i\infty$ are different. Hence the theorem is proved as it is known (see [4], pp. 815, 816) that $\dim S_{2m}(\Gamma_0(5),\chi^m)=m-1$. $\square$

Corollary 3. *For $m=2$ the function $\vartheta(\tau;\Phi_2,\varphi)$ and for $m\geq 3$ the system of functions*

$$\vartheta^{r}(\tau;\Phi_2,\varphi)\vartheta^{m-2r}(\tau,F_2)\ \ (r=1,2,\dots,[m/2]),$$
$$\vartheta^{m-r}(\tau;\Phi_2,\varphi)\vartheta^{2r-m}(\tau;Q_2^{(5)},1,h^{(5)})\ \ (r=[m/2]+1,\dots,m-1)$$

is the basis of the space $S_{2m}(\Gamma_0(5),\chi^m)$.

By Lemma 6 from [1], $\vartheta(\tau,F_{2m})-E(\tau,F_{2m})\in S_{2m}(\Gamma_0(5),\chi^m)$, where $E(\tau,F_{2m})$ is the Eisenstein series (see [4], pp. 875, 877). Hence, by Corollary 3, there are constants α and $\alpha_r^{(2m)}$ such that

$$\vartheta(\tau,F_4)=E(\tau,F_4)+\alpha\vartheta(\tau;\Phi_2,\varphi)\tag{1.6}$$

and for $m \geq 3$

$$\vartheta(\tau, F_{2m}) = E(\tau, F_{2m}) + \sum_{r=1}^{[m/2]} \alpha_r^{(2m)} \vartheta^r(\tau; \Phi_2, \varphi) \vartheta^{m-2r}(\tau, F_2)$$
$$+ \sum_{r=[m/2]+1}^{m-1} \alpha_r^{(2m)} \vartheta^{m-r}(\tau; \Phi_2, \varphi) \vartheta^{2r-m}(\tau; Q_2^{(5)}, 1, h^{(5)}). \qquad (1.7)$$

Using the expansions of $\vartheta(\tau, F_{2m})$, $E(\tau, F_{2m})$ from [3], that of $\vartheta(\tau; Q_2^{(5)}, 1, h^{(5)})$ from [1], Lemma 1, and equating, in both parts of (1.6) and (1.7), the coefficients of z and $z, z^2, \dots, z^{m-1}$ we get the systems of linear equations, whence we find the constants α and $\alpha_r^{(2m)}$. Equating, in both parts of (1.6) and (1.7), the coefficients of z^n, we obtain formulas for the arithmetical function $r(n, F_{2m})$.

The formulas for $r(n, F_{2m})$ when $m = 2, 3, \dots, 6$ have the form

$$r(n, F_4) = \frac{20}{13}\,\sigma_3^*(n) - \frac{25}{13} \sum_{\Phi_2 = n} x_1^2 - 2x_4^2,$$

$$r(n, F_6) = \frac{5}{67}\,\rho_5^*(n) + \frac{225}{67} \sum_{\Phi_2 \oplus F_2 = n} x_1^2 - 2x_4^2$$
$$- \frac{1250}{67} \sum_{\substack{\Phi_2 \oplus Q_2^{(5)} = n \\ y \equiv h^{(5)} \pmod 5}} x_1^2 - 2x_4^2,$$

$$r(n, F_8) = \frac{120}{13 \cdot 313}\,\sigma_7^*(n) + \frac{61850}{13 \cdot 313} \sum_{\Phi_2 \oplus F_4 = n} x_1^2 - 2x_4^2$$
$$- \frac{545625}{13 \cdot 313} \sum_{\Phi_4 = n} (x_1^2 - 2x_4^2)(x_5^2 - 2x_8^2)$$
$$- \frac{1550000}{13 \cdot 313} \sum_{\substack{\Phi_2 \oplus Q_4^{(5)} = n \\ y \equiv g^{(5,2)} \pmod 5}} x_1^2 - 2x_4^2,$$

$$r(n, F_{10}) = \frac{25}{191 \cdot 2161}\,\rho_9^*(n) + \frac{9830500}{191 \cdot 2161} \sum_{\Phi_2 \oplus F_6 = n} x_1^2 - 2x_4^2$$
$$- \frac{185896875}{2 \cdot 191 \cdot 2161} \sum_{\Phi_4 \oplus F_2 = n} (x_1^2 - 2x_4^2)(x_5^2 - 2x_8^2)$$
$$- \frac{596640625}{2 \cdot 191 \cdot 2161} \sum_{\substack{\Phi_4 \oplus Q_2^{(5)} = n \\ y \equiv h^{(5)} \pmod 5}} (x_1^2 - 2x_4^2)(x_5^2 - 2x_8^2)$$
$$- \frac{1228906250}{191 \cdot 2161} \sum_{\substack{\Phi_2 \oplus Q_6^{(5)} = n \\ y \equiv g^{(5,3)} \pmod 5}} x_1^2 - 2x_4^2,$$

$$\begin{aligned} r(n, F_{12}) = {} & \frac{20}{601 \cdot 691}\, \sigma_{11}^{*}(n) + \frac{12379975}{601 \cdot 691} \sum_{\Phi_2 \oplus F_8 = n} x_1^2 - 2x_4^2 \\ & - \frac{115155625}{601 \cdot 691} \sum_{\Phi_4 \oplus F_4 = n} (x_1^2 - 2x_4^2)(x_5^2 - 2x_8^2) \\ & - \frac{497140625}{601 \cdot 691} \sum_{\Phi_6 = n} (x_1^2 - 2x_4^2)(x_5^2 - 2x_8^2)(x_9^2 - 2x_{12}^2) \\ & - \frac{3886875000}{601 \cdot 691} \sum_{\substack{\Phi_4 \oplus Q_4^{(5)} = n \\ y \equiv g^{(5,2)} \pmod 5}} (x_1^2 - 2x_4^2)(x_5^2 - 2x_8^2) \\ & - \frac{7737500000}{601 \cdot 691} \sum_{\substack{\Phi_2 \oplus Q_8^{(5)} = n \\ y \equiv g^{(5,4)} \pmod 5}} x_1^2 - 2x_4^2. \end{aligned}$$

2. Let

$$\begin{aligned} F_2 &= x_1^2 + x_2^2 + x_3^2 + 2x_4^2 + x_1x_2 + x_1x_3 + x_2x_4, \\ F_2^* &= 7x_1^2 + 6x_2^2 + 5x_3^2 + 2x_4^2 - 8x_1x_2 - 7x_1x_3 \\ &\quad + 2x_1x_4 + 4x_2x_3 - 3x_2x_4 - x_3x_4, \\ \Phi_2 &= x_1^2 + 2x_2^2 + 2x_3^2 + 4x_4^2 + x_1x_2 + x_1x_4 \\ &\quad + x_2x_3 + x_2x_4 + 2x_3x_4. \end{aligned}$$

F_2 is the only reduced positive quaternary quadratic form with determinant 13 ([2], p. 141) and F_2^* is adjoint to F_2. F_2 and F_2^* are quadratic forms of type $(2, 13, \chi)$ [5], and Φ_2 is a quadratic form of type $(2, 13, 1)$ [6], where $\chi = \chi(d) = (\frac{d}{13})$.

Using Lemma 5 from [1], it is easy to show that F_{2m} is a quadratic form of type $(2m, 13, \chi^m)$.

The following Lemmas are proved exactly as Lemma 1.

Lemma 4. (a) $\varphi_1 = \varphi_1(x_1, \dots, x_4) = x_1^2 - 2x_2^2$, $\varphi_2 = \varphi_2(x_1, \dots, x_4) = x_2^2 - x_3^2$, $\varphi_3 = \varphi_3(x_1, \dots, x_4) = 2x_1x_3 - x_4^2$ *are spherical functions of order* 2 *with respect to* Φ_2;

(b) $$\begin{aligned} \vartheta(\tau; \Phi_2, \varphi_1) &= 2z - 6z^2 - 2z^3 + 14z^4 - 10z^5 + 26z^6 \\ &\quad - 30z^7 - 38z^8 + 56z^9 + 42z^{10} + \cdots, \\ \vartheta(\tau; \Phi_2, \varphi_2) &= 2z^2 - 4z^3 + 2z^5 - 2z^6 + 18z^7 - 14z^8 - 24z^9 - 4z^{10} + \cdots, \\ \vartheta(\tau; \Phi_2, \varphi_3) &= 4z^3 - 4z^4 - 8z^6 - 8z^7 + 16z^8 + 12z^9 - 4z^{10} + \cdots; \end{aligned}$$

(c) $\vartheta(\tau; \Phi_2, \varphi_r) \in S_4(\Gamma_0(13), 1)$;

(d) $\operatorname{ord}(\vartheta(\tau; \Phi_2, \varphi_r), i\infty, \Gamma_0(13)) = r$ $(r = 1, 2, 3)$.

Lemma 5. (a) $\varphi_4 = \varphi_4(x_1, \dots, x_8) = x_4^2 - x_5^2$ *is a spherical function of order* 2 *with respect to* $\Phi_2 \oplus F_2^*$*;*

(b) $$\vartheta(\tau; \Phi_2 \oplus F_2^*, \varphi_4) = 8z^4 + 24z^6 - 24z^7 - 24z^8 - 48z^9 - 40z^{10} + \dots \in S_6(\Gamma_0(13), \chi);$$

(c) $\operatorname{ord}(\vartheta(\tau; \Phi_2 \oplus F_2^*, \varphi_4), i\infty, \Gamma_0(13)) = 4.$

Lemma 6. (a) $\varphi_5 = \varphi_5(x_1, \dots, x_4) = x_2^4 + x_3^4 - 6x_2^2x_3^2$ *and* $\varphi_6 = \varphi_6(x_1, \dots, x_4) = x_1x_4^3 + x_3x_4^3 - 6x_1^2x_3x_4$ *are spherical functions of order* 4 *with respect to* F_2^**;*

(b) $$\vartheta(\tau; F_2^*, \varphi_5) = 8z^5 + 8z^6 - 32z^7 + 8z^8 + 0 \cdot z^{10} + \cdots,$$
$$\vartheta(\tau; F_2^*, \varphi_6) = 12z^6 - 12z^7 - 12z^8 + 0 \cdot z^{10} + \cdots;$$

(c) $\vartheta(\tau; F_2^*, \varphi_r) \in S_6(\Gamma_0(13), \chi)$*;*

(d) $\operatorname{ord}(\vartheta(\tau; F_2^*, \varphi_r), i\infty, \Gamma_0(13)) = r$ $(r = 5, 6)$*.*

Lemma 7. (a) $\varphi_7 = \varphi_7(x_1, \dots, x_8) = x_7^4 - x_3^4 + 6x_2^2x_3^2 - 6x_2^2x_7^2$ *is a spherical function of order* 4 *with respect to* F_4^**;*

(b) $\vartheta(\tau; F_4^*, \varphi_7) = 48z^7 + 96z^9 - 96z^{10} + \dots \in S_8(\Gamma_0(13), 1)$*;*

(c) $\operatorname{ord}(\vartheta(\tau; F_4^*, \varphi_7), i\infty, \Gamma_0(13)) = 7.$

Theorem 8. *Let*

$$\lambda = \begin{cases} 2m + \left[\dfrac{m}{3}\right] - 1, & \text{if } m \equiv 0 \text{ or } 2 \pmod 3, \\ \dfrac{7}{3}(m-1), & \text{if } m \equiv 1 \pmod 3 \end{cases}$$

and let $C = (c_{sr})$ $(s = 1, 2, \dots, 9$*;* $r = 1, 2, \dots, \lambda)$ *be the matrix whose elements are non-negative integers satisfying the conditions*

$$\left.\begin{aligned} 4\sum_{s=1}^{8} c_{sr} + 2\sum_{s=4}^{9} c_{sr} + 2c_{7r} &= 2m \qquad (2.1)\\ \sum_{s=1}^{7} s \cdot c_{sr} + 7c_{8r} &= r \qquad (2.2)\\ c_{4r} = 0 \ \text{or} \ 1, \quad \sum_{s=1}^{7} c_{sr} &> 0 \end{aligned}\right\} \quad (r = 1, 2, \dots, \lambda).$$

Further, let $\varphi^{(r)} = \prod\limits_{s=1}^{7} \varphi_s^{c_{sr}}$ *denote the direct product of functions* φ_s*, defined by Lemmas* 4–7 $(\varphi_s^0 \equiv 1)$*. Then for each* $m \geq 2$*, the system of functions*

$$\vartheta\Big(\tau; \Phi_{2v_r} \oplus F_{2t_r}^*, \varphi^{(r)}\Big)\vartheta^{c_{8r}}(\tau; Q_6^{(13)}, 1, h^{(13)})\vartheta^{c_{9r}}(\tau, F_2^*) \qquad (2.3)$$
$$(r = 1, 2, \dots, \lambda),$$

where

$$v_r = \sum_{s=1}^{4} c_{sr}, \quad t_r = \sum_{s=4}^{7} c_{sr} + c_{7r} \qquad (2.4)$$

is the basis of the space $S_{2m}(\Gamma_0(13),\chi^m)$.

Proof. It follows from the definition of $\varphi^{(r)}$, (2.4) and ([1], Lemma 9) that

$$\begin{aligned}\vartheta\Big(\tau;\Phi_{2v_r}\oplus F^*_{2t_r},\varphi^{(r)}\Big) &= \vartheta^{c_{1r}}(\tau;\Phi_2,\varphi_1)\\ &\times\vartheta^{c_{2r}}(\tau;\Phi_2,\varphi_2)\vartheta^{c_{3r}}(\tau;\Phi_2,\varphi_3)\\ &\times\vartheta^{c_{4r}}\Big(\tau;\Phi_2\oplus F^*_2,\varphi_4\Big)\vartheta^{c_{5r}}(\tau;F^*_2,\varphi_5)\\ &\times\vartheta^{c_{6r}}(\tau;F^*_2,\varphi_6)\vartheta^{c_{7r}}(\tau;F^*_4,\varphi_7).\end{aligned}\tag{2.5}$$

Therefore (2.5), (2.1), above Lemmas 4–7 and Lemmas 3, 8, 9 from [1] imply that functions (2.3) are cusp forms of type $(2m,\Gamma_0(13),\chi^m)$. By Lemmas 4–7, (2.2) and Lemma 8 from [1], we have

$$\operatorname{ord}\Big(\vartheta(\tau;\Phi_{2v_r}\oplus F^*_{2t_r},\varphi^{(r)})\vartheta^{c_{8r}}(\tau;Q^{(13)}_6,1,h^{(13)})\vartheta^{c_{9r}}(\tau,F^*_2),i\infty,\Gamma_0(13)\Big)=r \quad (r=1,2,\dots,\lambda).$$

Functions (2.3) are linearly independent because their orders at the cusp $i\infty$ are different. Hence the theorem is proved as it is known that $\dim S_{2m}(\Gamma_0(13),\chi^m)=\lambda$ (see [4], pp. 815, 816)). □

By ([1], Lemma 6) $\vartheta(\tau,F_{2m})-E(\tau,F_{2m})\in S_{2m}(\Gamma_0(13),\chi^m)$. Hence by Theorem 8 there are constants $\alpha_r^{(2m)}$ such that

$$\vartheta(\tau,F_{2m})=E(\tau,F_{2m})$$
$$+\sum_{r=1}^{\lambda}\alpha_r^{(2m)}\vartheta\Big(\tau;\Phi_{2v_r}\oplus F^*_{2t_r},\varphi^{(r)}\Big)\vartheta^{c_{8r}}(\tau;Q^{(13)}_6,1,h^{(13)})\vartheta^{c_{9r}}(\tau,F^*_2).$$

Using the expansion of $\vartheta(\tau,F_2)$ from [5] we get the expansions of $\vartheta(\tau,F_{2m})=\vartheta^m(\tau,F_2)$. From formulas (1.5)–(1.9) of [1], when $q=13$, we get the expansions of $E(\tau,F_{2m})$. Then using the expansions of $\vartheta(\tau,F^*_2)$ from [5] and $\vartheta(\tau;Q^{(13)}_6,1,h^{(13)})$ from [1], Lemmas 4-7, exactly as above we get formulas for $r(n,F_{2m})$. When $m=2,\dots,5$ they have the form

$$\begin{aligned}r(n,F_4)={}&\frac{12}{7\cdot17}\,\sigma_3^*(n)+\frac{330}{7\cdot17}\sum_{\Phi_2=n}x_1^2-2x_2^2+\frac{1342}{7\cdot17}\sum_{\Phi_2=n}x_2^2-x_3^2\\ &-\frac{1013}{7\cdot17}\sum_{\Phi_2=n}2x_1x_3-x_4^2,\\ r(n,F_6)={}&\frac{13}{109\cdot307}\,\rho_5^*(n)+\frac{416694}{109\cdot307}\sum_{\Phi_2\oplus F_2^*=n}x_1^2-2x_2^2\\ &+\frac{3425570}{109\cdot307}\sum_{\Phi_2\oplus F_2^*=n}x_2^2-x_3^2+\frac{4662503}{109\cdot307}\sum_{\Phi_2\oplus F_2^*=n}2x_1x_3-x_4^2\\ &-\frac{2905512}{109\cdot307}\sum_{\Phi_2\oplus F_2^*=n}x_4^2-x_5^2-\frac{1698555}{109\cdot307}\sum_{F_2^*=n}x_2^4+x_3^4-6x_2^2x_3^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{2615880}{109\cdot 307}\sum_{F_2^*=n} x_1x_4^3+x_3x_4^3-6x_1^2x_3x_4,\\
r(n,F_8)&=\frac{24}{17\cdot 14281}\,\sigma_7^*(n)+\frac{5481876}{17\cdot 14281}\sum_{\Phi_2\oplus F_4^*=n} x_1^2-2x_2^2\\
&+\frac{83647460}{17\cdot 14281}\sum_{\Phi_2\oplus F_4^*=n} x_2^2-x_3^2\\
&+\frac{251775638}{17\cdot 14281}\sum_{\Phi_2\oplus F_4^*=n} 2x_1x_3-x_4^2\\
&+\frac{239040860}{17\cdot 14281}\sum_{\Phi_4=n}(2x_1x_3-x_4^2)(x_5^2-2x_6^2)\\
&+\frac{591029498}{17\cdot 14281}\sum_{\Phi_4=n}(2x_1x_3-x_4^2)(x_6^2-x_7^2)\\
&+\frac{877045926}{17\cdot 14281}\sum_{\Phi_4=n}(2x_1x_3-x_4^2)(2x_5x_7-x_8^2)\\
&-\frac{315282279}{17\cdot 14281}\sum_{F_4^*=n} x_7^4-x_3^4+6x_2^2x_3^2-6x_2^2x_7^2,\\
r(n,F_{10})&=\frac{1}{144547171}\,\rho_9^*(n)+\frac{4305040872}{144547171}\sum_{\Phi_2\oplus F_6^*=n} x_1^2-2x_2^2\\
&+\frac{106015990372}{144547171}\sum_{\Phi_2\oplus F_6^*=n} x_2^2-x_3^2\\
&+\frac{541004173576}{144547171}\sum_{\Phi_2\oplus F_6^*=n} 2x_1x_3-x_4^2\\
&+\frac{1284103896202}{144547171}\sum_{\Phi_4\oplus F_2^*=n}(x_1^2-2x_2^2)(2x_5x_7-x_8^2)\\
&+\frac{7148202390955}{144547171}\sum_{\Phi_4\oplus F_2^*=n}(x_2^2-x_3^2)(2x_5x_7-x_8^2)\\
&+\frac{11816421944265}{144547171}\sum_{\Phi_4\oplus F_2^*=n}(2x_1x_3-x_4^2)(2x_5x_7-x_8^2)\\
&+\frac{8100018167088}{144547171}\sum_{\Phi_2\oplus F_2^*=n}(x_1^2-2x_2^2)(x_5x_8^3+x_7x_8^3-6x_5^2x_7x_8)\\
&+\frac{22777304972068}{144547171}\sum_{\Phi_2\oplus F_2^*=n}(x_2^2-x_3^2)(x_5x_8^3+x_7x_8^3-6x_5^2x_7x_8)\\
&+\frac{41871730633632}{144547171}\sum_{\Phi_2\oplus F_2^*=n}(2x_1x_3-x_4^2)(x_5x_8^3+x_7x_8^3-6x_5^2x_7x_8)
\end{aligned}
$$

$$-\frac{7694378154216}{144547171}\sum_{\substack{\Phi_2\oplus Q_6^{(13)}=n\\ y\equiv h^{(13)}\ (\mathrm{mod}\ 13)}} 2x_1x_3-x_4^2.$$

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