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# ON A TRACE INEQUALITY FOR ONE-SIDED POTENTIALS AND APPLICATIONS TO THE SOLVABILITY OF NONLINEAR INTEGRAL EQUATIONS

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*Dedicated to the memory of N. Muskhelishvili*

**Abstract**. Necessary and sufficient conditions, which govern a trace inequality for one-sided potentials in the "diagonal" case, are established. An application to the existence of positive solutions of a certain nonlinear integral equation is presented.

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## Introduction

It is well-known that the trace inequality for the Riesz potential

$$\int\limits_{R^n} |I_\alpha f(x)|^p v(x)\,dx \le c \int\limits_{R^n} |f(x)|^p\,dx, \quad 0<\alpha<n, \quad 1<p<\infty,$$

is of great importance for the spectral properties of the Schrödinger operator and has numerous applications to partial differential equations, Sobolev spaces, complex analysis, etc. (see [1]–[7]). The pointwise conditions derived in [3], [6], [7] turned out to be of particular interest for existence theorems and estimates of solutions of certain semilinear elliptic equations.

The trace inequality for the Riemann–Liouville transform $R_\alpha$ in the case where $p=2$ and $\alpha>1/2$ was derived in [8]. For the extention of this result when $1<p<\infty$ and $\alpha>1/p$ we refer to [9] (for more general transforms see [10]).

In the present paper criteria for the trace inequality for the Riemann–Liouville and Weyl operators in a more complicated case $0<\alpha<1/p$, $1<p<\infty$, are established. Some applications solvability problems of certain nonlinear integral equation are presented.

For the basic definitions and auxilliary results concerning fractional integrals on the line we refer to the monorgaph [11].

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## 1. Trace Inequalities

In this section we establish necessary and sufficient conditions for the validity of the trace inequality for the Riemann–Liouville and Weyl operators. Two-weighted inequalities are also derived.

Let $\nu$ be a locally finite Borel measure on $\Omega \subset R$. Denote by $L^p_\nu(\Omega)$ $(1 < p < \infty)$ a Lebesgue space with respect to the measure $\nu$ consisting of all $\nu$ -measurable functions $f$ for which

$$\|f\|_{L^p_\nu(\Omega)} = \Big(\int\limits_\Omega |f(x)|^p \, d\nu(x)\Big)^{1/p} < \infty.$$

If $d\nu(x) = v(x)\,dx$, with a locally integrable a.e. positive function $v$ on $\Omega$, then we use the notation $L^p_\nu(\Omega) \equiv L^p_v(\Omega)$. If $d\nu(x) = dx$ is a Lebesgue measure, then we assume that $L^p_\nu(\Omega) \equiv L^p(\Omega)$.

Let

$$R_\alpha f(x) = \int\limits_0^x f(y)(x-y)^{\alpha-1}\,dy, \quad x>0, \quad \alpha>0,$$

$$W_\alpha f(x) = \int\limits_x^\infty f(y)(y-x)^{\alpha-1}\,dy, \quad x>0, \quad \alpha>0$$

for measurable $f : R_+ \to R^1$.

**Theorem 1.1.** *Let $1<p<\infty$ and let $0<\alpha<\frac{1}{p}$. Then the inequality*

$$\int\limits_0^\infty |R_\alpha f(x)|^p v(x)\,dx \le c_0 \int\limits_0^\infty |f(x)|^p\,dx \tag{1.1}$$

*holds if and only if $W_\alpha v \in L^{p'}_{\mathrm{loc}}(R_+)$ and*

$$W_\alpha[W_\alpha v]^{p'}(x) \le c W_\alpha v(x) \ \ \textit{a.e.} \tag{1.2}$$

To prove this theorem we need

**Proposition 1.1.** *Let $1<p<\infty$, and let $0<\alpha<\frac{1}{p}$. If (1.1) is fulfilled, then*

$$\int\limits_x^{x+h} v(y)\,dy \le c\, h^{1-\alpha p} \tag{1.3}$$

*for all positive $x$ and $h$.*

*Proof.* By the duality argument, (1.1) is equivalent to the inequality

$$\|W_\alpha f\|_{L^{p'}(R_+)} \le c_0^{1/p} \|f\|_{L^{p'}_{v^{1-p'}}(R_+)}. \tag{1.4}$$

Replaing here $f(y)=\chi_{(x,x+h)}(y)v(y)$ for $0<h\le x$, we get

$$\int\limits_{x-h}^{x}\Big(\int\limits_{x}^{x+h}v(z)(z-y)^{\alpha-1}\,dz\Big)^{p'}dy\le c_0^{p'-1}\int\limits_{x}^{x+h}v(y)\,dy.$$

Hence (1.3) holds for all $x$ and $h$ with the condition $0<h\le x$. Now let $0<x<h<\infty$. Then taking into account the condition $0<\alpha<\frac{1}{p}$ we obtain

$$\begin{aligned}\int\limits_{x}^{x+h}v(y)\,dy&=\sum_{k=0}^{\infty}\int\limits_{x+\frac{h}{2^{k+1}}}^{x+\frac{h}{2^k}}v(y)\,dy\\&=\sum_{k=0}^{\infty}\Big(\int\limits_{x+\frac{h}{2^{k+1}}}^{x+\frac{h}{2^k}}v(y)\,dy\Big)\Big(\frac{h}{2^{k+1}}\Big)^{\alpha p-1}\Big(\frac{h}{2^{k+1}}\Big)^{1-\alpha p}\\&\le\sup_{\substack{t,a\\ t\le a}}\bigg(\Big(\int\limits_{a}^{a+t}v(y)\,dy\Big)t^{\alpha p-1}\bigg)h^{1-\alpha p}\sum_{k=0}^{\infty}2^{(k+1)(\alpha p-1)}\le c\,h^{1-\alpha p}.\end{aligned}$$

Therefore that (1.3) holds. Note that $c{=}c_0 2^{(1-\alpha)p}\max\Big\{1,\frac{2^{\alpha p-1}}{1-2^{\alpha p-1}}\Big\}$ in (1.3). □

*Proof of Theorem* 1.1. *Necessity.* Let us first show that, from (1.1) it follows that $W_\alpha v\in L^{p'}_{\rm loc}(R_+)$. For $f(y)=v(y)\chi_{(x,x+h)}(y)$ ($x\in R_+$ and $h>0$) from (1.1) we have

$$\int\limits_{x}^{x+h}\Big(W_\alpha(v\chi_{(x,x+h)})\Big)^{p'}(y)\,dy\le c\int\limits_{x}^{x+h}v(y)\,dy. \tag{1.5}$$

Let $v_1(y)=\chi_{(x,x+2h)}v(y)$ and $v_2(y)=\chi_{R_+\setminus(x,x+2h)}(y)v(y)$, where $x\in R_+$ and $h>0$. We have

$$\int\limits_{x}^{x+h}(W_\alpha v)^{p'}(y)\,dy\le c(I_1(x)+I_2(x)),$$

where

$$I_1(x)=\int\limits_{x}^{x+h}(W_\alpha v_1)^{p'}(y)\,dy\ \text{ and }\ I_2(x)=\int\limits_{x}^{x+h}(W_\alpha v_2)^{p'}(y)\,dy.$$

From (1.5) we get

$$I_1(x)\le c\int\limits_{x}^{x+2h}v(y)\,dy<\infty.$$

Thus $W_\alpha v_1\in L^{p'}_{\rm loc}(R_+)$.

Note that for $z > x+2h$ and $x<y<x+h$ we have $z-x \le 2(z-y)$. Now using (1.3), we come to the estimate

$$(W_\alpha v_2)(y) = \int\limits_{x+2h}^{\infty} (z-y)^{\alpha-1} v(z)\,dz \le 2^{1-\alpha} \int\limits_{x+h}^{\infty} (z-x)^{\alpha-1} v(z)\,dz$$
$$= 2^{1-\alpha} \int\limits_{h}^{\infty} t^{\alpha-2} \Big( \int\limits_{x}^{x+t} v(z)\,dz \Big)\,dt \le c\,2^{1-\alpha} \int\limits_{h}^{\infty} t^{\alpha-1-\alpha p}\,dt < \infty.$$

Therefore $W_\alpha v_2 \in L^{p'}_{\mathrm{loc}}(R_+)$. Thus $W_\alpha v \in L^{p'}_{\mathrm{loc}}(R_+)$.

Now we prove that (1.1) yields (1.2).

In the sequel the following equality

$$W_\alpha v(x) = (1-\alpha) \int\limits_0^\infty \tau^{\alpha-1} \Big( \int\limits_x^{x+\tau} v(y)\,dy \Big) \frac{d\tau}{\tau} \tag{1.6}$$

will be used.

Thus

$$W_\alpha \big[(W_\alpha v)^{p'}\big](x) = (1-\alpha) \int\limits_0^\infty \tau^{\alpha-1} \Big( \int\limits_x^{x+\tau} (W_\alpha v)^{p'}(y)\,dy \Big) \frac{d\tau}{\tau}\,. \tag{1.7}$$

Let $v_1$ and $v_2$ be defined as before. By (1.5) we have

$$\int\limits_x^{x+h} (W_\alpha v_1)^{p'}(y)\,dy \le c \int\limits_x^{x+2h} v(y)\,dy. \tag{1.8}$$

Then from (1.7) and (1.8) we derive the estimate

$$W_\alpha \big[(W_\alpha v_1)^{p'}\big](x) \le c \int\limits_0^\infty \tau^{\alpha-1} \Big( \int\limits_x^{x+2\tau} v(t)\,dt \Big) \frac{d\tau}{\tau} = c\,W_\alpha v(x). \tag{1.9}$$

It is easy to see that for $t \in (x, x+h)$

$$(W_\alpha v_2)(t) \le c \int\limits_h^\infty r^{\alpha-1} \Big( \int\limits_x^{x+r} v(y)\,dy \Big) \frac{dr}{r}\,.$$

Therefore (1.7) yields

$$W_\alpha \big[(W_\alpha v_2)^{p'}\big](x) \le c \int\limits_0^\infty t^{\alpha} \Big( \int\limits_t^\infty r^{\alpha-1} \Big( \int\limits_x^{x+r} v(y)\,dy \Big) \frac{dr}{r} \Big)^{p'} \frac{dt}{t}\,.$$

Integration by parts on the right-hand side of the last inequality leads to the estimate

$$\begin{aligned} & W_\alpha\Big[(W_\alpha v_2)^{p'}\Big](x) \\ & \le c\int_0^\infty r^\alpha\bigg(\int_r^\infty \tau^{\alpha-1}\Big(\int_x^{x+\tau} v(y)\,dy\Big)\frac{d\tau}{\tau}\bigg)^{p'-1}\Big(r^\alpha\int_x^{x+r} v(y)\,dy\Big)\frac{dr}{r}\,. \end{aligned} \tag{1.10}$$

Now recall that estimate (1.3) holds by Proposition 1.1. From inequality (1.3), by a simple computation, we obtain

$$W_\alpha\Big[(W_\alpha v_2)^{p'}\Big](x) \le c\int_0^\infty h^{\alpha-1}\Big(\int_x^{x+h} v(y)\,dy\Big)\frac{dh}{h}\,.$$

Thus

$$W_\alpha\Big[(W_\alpha v_2)^{p'}\Big](x) \le c(W_\alpha v)(x) \ \text{ a.e.} \tag{1.11}$$

Finally (1.9) and (1.11) imply (1.2).

*Remark* 1.1. It follows from the proof of necessity of the Theorem 1.1 that if $c_0$ is the best constant in (1.2), then for $c$ from (1.3) we have

$$c = c_0^{p'-1}2^{p'-\alpha} + 2^{p'-1}(1-\alpha)^{p'}\,\frac{p'c_1}{\alpha(\alpha p-\alpha)^{p'-1}}\,,$$

where $c_1 = c_0 2^{(1-\alpha)p}\max\Big\{1, \frac{2^{\alpha p-1}}{1-2^{\alpha p-1}}\Big\}$.

*Sufficiency of Theorem* 1.1. In order to show the sufficiency, we shall need the following lemmas.

**Lemma 1.1.** *Let $1<p<\infty$ and $0<\alpha<1$. Then there exists a positive constant $c$ such that for all $f\in L^1_{\mathrm{loc}}(R_+)$, $f\ge 0$, and for arbitrary $x\in R_+$ the following inequality holds:*

$$(R_\alpha f(x))^p \le c\,R_\alpha\Big((R_\alpha f)^{p-1}f\Big)(x) \tag{1.12}$$

*(for $c$ we have $c=2^{\frac{1}{p-1}}$ if $p\le 2$ and $c=2^{p(p-1)}$ if $p>2$).*

*Proof.* First we assume that $R_\alpha f(x)<\infty$ and prove (1.12) for such $x$. We also assume that

$$V_\alpha f(x) \le (R_\alpha f(x))^p,$$

where $V_\alpha f(x) \equiv R_\alpha((R_\alpha f)^{p-1} f)(x)$, otherwise (1.12) is obvious for $c = 1$. Now let us assume that $1 < p \le 2$. Then we have

$$\begin{aligned}
(R_\alpha f(x))^p &= \int_0^x (x-y)^{\alpha-1} f(y) \Big( \int_0^x (x-z)^{\alpha-1} f(z)\, dz \Big)^{p-1} dy \\
&\le \int_0^x (x-y)^{\alpha-1} f(y) \Big( \int_0^y (x-z)^{\alpha-1} f(z)\, dz \Big)^{p-1} dy \\
&+ \int_0^x (x-y)^{\alpha-1} f(y) \Big( \int_y^x (x-z)^{\alpha-1} f(z)\, dz \Big)^{p-1} dy \\
&\equiv I_1(x) + I_2(x).
\end{aligned}$$

It is obvious that if $z < y < x$, then $y - z \le x - z$. Consequently,

$$I_1(x) \le \int_0^x (x-y)^{\alpha-1} f(y) \Big( \int_0^y (y-z)^{\alpha-1} f(z)\, dz \Big)^{p-1} dy = V_\alpha f(x).$$

Now we use Hölder's inequality with respect to the exponents $\frac{1}{p-1}$, $\frac{1}{2-p}$ and measure $d\sigma(y) = (x-y)^{\alpha-1} f(y)\, dy$. We have

$$\begin{aligned}
I_2(x) &\le \Big( \int_0^x (x-y)^{\alpha-1} f(y)\, dy \Big)^{2-p} \\
&\times \Big( \int_0^x \Big( \int_y^x (x-z)^{\alpha-1} f(z)\, dz \Big) (x-y)^{\alpha-1} f(y)\, dy \Big)^{p-1} \\
&= (R_\alpha f(x))^{2-p} (J(x))^{p-1},
\end{aligned}$$

where

$$J(x) \equiv \int_0^x \Big( \int_y^x (x-z)^{\alpha-1} f(z)\, dz \Big) (x-y)^{\alpha-1} f(y)\, dy.$$

Using Tonelli's theorem we have

$$J(x) = \int_0^x (x-z)^{\alpha-1} f(z) \Big( \int_0^z (x-y)^{\alpha-1} f(y)\, dy \Big) dz.$$

Further, it is obvious that the following simple inequality

$$\begin{aligned}
\int_0^z (x-y)^{\alpha-1} f(y)\, dy &\le \Big( \int_0^z (x-y)^{\alpha-1} f(y)\, dy \Big)^{p-1} (R_\alpha f(x))^{2-p} \\
&\le (R_\alpha f(z))^{p-1} (R_\alpha f(x))^{2-p}
\end{aligned}$$

holds, where $z < x$.

Taking into account the last estimate, we obtain

$$
\begin{aligned}
J(x) &\le \Big(\int_0^x (x-z)^{\alpha-1} f(z)(R_\alpha f(z))^{p-1}\,dz\Big)(R_\alpha f(x))^{2-p} \\
&= (V_\alpha f(x))(R_\alpha f(x))^{2-p}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
I_2(x) &\le (R_\alpha f(x))^{2-p}(R_\alpha f(x))^{(2-p)(p-1)}(V_\alpha f(x))^{p-1} \\
&= (R_\alpha f(x))^{p(2-p)}(V_\alpha f(x))^{p-1}.
\end{aligned}
$$

Combining the estimates for $I_1$ and $I_2$ we derive

$$
(R_\alpha f(x))^p \le V_\alpha f(x) + (R_\alpha f(x))^{p(2-p)}(V_\alpha f(x))^{p-1}.
$$

As we have assumed that $V_\alpha f(x) \le (R_\alpha f(x))^p$, we obtain

$$
V_\alpha f(x) = (V_\alpha f(x))^{2-p}(V_\alpha f(x))^{p-1} \le (V_\alpha f(x))^{p-1}(R_\alpha f(x))^{p(2-p)}.
$$

Hence

$$
\begin{aligned}
(R_\alpha f(x))^p &\le (V_\alpha f(x))^{p-1}(R_\alpha f(x))^{p(2-p)} + (V_\alpha f(x))^{p-1}(R_\alpha f(x))^{p(2-p)} \\
&= 2(V_\alpha f(x))^{p-1}(R_\alpha f(x))^{p(2-p)}.
\end{aligned}
$$

Using the fact $R_\alpha f(x) < \infty$ we deduce that

$$
(R_\alpha f(x))^{p-1} \le 2^{\frac{1}{p-1}}(V_\alpha f(x)).
$$

Now we shall deal with the case $p > 2$. Let us assume again that

$$
V_\alpha f(x) \le (R_\alpha f(x))^p,
$$

where

$$
V_\alpha f(x) \equiv R_\alpha\Big[(R_\alpha f)^{p-1} f\Big](x).
$$

As $p > 2$ we have

$$
\begin{aligned}
(R_\alpha f(x))^p &= \int_0^x f(y)(x-y)^{\alpha-1}\Big(\int_0^x (x-z)^{\alpha-1} f(z)\,dz\Big)^{p-1} dy \\
&\le 2^{p-1}\int_0^x f(y)(x-y)^{\alpha-1}\Big(\int_0^y (x-z)^{\alpha-1} f(z)\,dz\Big)^{p-1} dy \\
&+ 2^{p-1}\int_0^x f(y)(x-y)^{\alpha-1}\Big(\int_y^x (x-z)^{\alpha-1} f(z)\,dz\Big)^{p-1} dy \\
&\equiv 2^{p-1} I_1(x) + 2^{p-1} I_2(x).
\end{aligned}
$$

It is clear that if $z < y < x$, then $(x-z)^{\alpha-1} \le (y-z)^{\alpha-1}$. Therefore $I_1(x) \le V_\alpha f(x)$. Now we estimate $I_2(x)$. We obtain

$$\Big(\int_y^x (x-z)^{\alpha-1} f(z)\,dz\Big)^{p-1} = \Big(\int_y^x (x-z)^{\alpha-1} f(z)\,dz\Big)^{p-2}\Big(\int_y^x (x-z)^{\alpha-1} f(z)\,dz\Big)$$
$$\le (R_\alpha f(x))^{p-2}\int_y^x (x-z)^{\alpha-1} f(z)\,dz.$$

Using Tonelli's theorem and the last estimate we have

$$\begin{aligned} I_2(x) &\le (R_\alpha f(x))^{p-2}\int_0^x f(y)(x-y)^{\alpha-1}\Big(\int_y^x (x-z)^{\alpha-1} f(z)\,dz\Big)\,dy \\ &= (R_\alpha f(x))^{p-2}\int_0^x f(z)(x-z)^{\alpha-1}\Big(\int_0^z (x-y)^{\alpha-1} f(y)\,dy\Big)\,dz \\ &\le (R_\alpha f(x))^{p-2}\int_0^x f(z)(x-z)^{\alpha-1}\Big(\int_0^z (z-y)^{\alpha-1} f(y)\,dy\Big)\,dz. \end{aligned}$$

Using Hölder's inequality with respect to the exponents $p-1$ and $\frac{p-1}{p-2}$ we derive

$$\int_0^x (x-z)^{\alpha-1} f(z)\Big(\int_0^z (z-y)^{\alpha-1} f(y)\,dy\Big)\,dz \le \Big(\int_0^x (x-z)^{\alpha-1} f(z)\,dz\Big)^{\frac{p-2}{p-1}}$$
$$\times\Big(\int_0^x \Big(\int_0^z (z-y)^{\alpha-1} f(y)\,dy\Big)^{p-1}(x-z)^{\alpha-1} f(z)\,dz\Big)^{\frac{1}{p-1}}$$
$$= (R_\alpha f(x))^{\frac{p-2}{p-1}}(V_\alpha f(x))^{\frac{1}{p-1}}.$$

Combining these estimates we obtain

$$(R_\alpha f(x))^p \le 2^{p-1} V_\alpha f(x) + 2^{p-1}(R_\alpha f(x))^{\frac{p(p-2)}{p-1}}(V_\alpha f(x))^{\frac{1}{p-1}}.$$

From the inequality $V_\alpha f(x) \le (R_\alpha f(x))^p$ it follows that

$$V_\alpha f(x) = (V_\alpha f(x))^{\frac{1}{p-1}}(V_\alpha f(x))^{\frac{p-2}{p-1}} \le (V_\alpha f(x))^{\frac{1}{p-1}}(R_\alpha f(x))^{\frac{p(p-2)}{p-1}}.$$

Hence

$$\begin{aligned} (R_\alpha f(x))^p &\le 2^{p-1}\Big((V_\alpha f(x))^{\frac{1}{p-1}}(R_\alpha f(x))^{\frac{p(p-2)}{p-1}} + (V_\alpha f(x))^{\frac{1}{p-1}}(R_\alpha f(x))^{\frac{p(p-2)}{p-1}}\Big) \\ &= 2^p (V_\alpha f(x))^{\frac{1}{p-1}}(R_\alpha f(x))^{\frac{p(p-2)}{p-1}}. \end{aligned}$$

The last estimate yields

$$(R_\alpha f(x))^p \le 2^{p(p-1)}(V_\alpha f(x)),$$

where $2 < p < \infty$.

Next we shall show that (1.12) holds for $x$ satisfying $R_\alpha f(x) = \infty$.

Let $k_n(x,y) = \chi_{(0,x)}(y)\min\{(x-y)^{\alpha-1}, n\}$, where $n \in N$. It is easy to verify that (1.12) holds if we replace $k(x,y) = \chi_{(0,x)}(y)(x-y)^{\alpha-1}$ by $k_n(x,y)$. Let $I = (a,b)$, where $0 < a < b < \infty$. Then

$$\int\limits_I k_n(x,y)f(y)dy < \infty \text{ and } \sup_{I,n}\int\limits_I k_n(x,y)f(y)\,dy = \infty.$$

Taking into account the above arguments we obtain

$$\Big(\int\limits_0^x \chi_I(y)k_n(x,y)f(y)\,dy\Big)^p$$
$$\le c\Big(\int\limits_0^x \chi_I(y)\Big(\int\limits_0^y \chi_I(z)k_n(y,z)f(z)\,dz\Big)^{p-1} f(z)k_n(x,z)\,dy\Big).$$

(In the last inequality we can assume that $f$ has a support in $I$. In this case $\int\limits_0^x k_n(x,y)f(y)\,dy < \infty$.) The constant $c$ is defined as follows: $c = 2^{\frac{1}{p-1}}$ if $1 < p \le 2$ and $c = 2^{p(p-1)}$ if $p > 2$. Taking the supremum with respect to all $I$ and passing $n$ to $+\infty$, we obtain (1.12) for all $x$. $\square$

*Remark* 1.2. Let $1 < p < \infty$, $0 < \alpha < 1$, $k_n(x,y) = \min\{n, (x-y)^{\alpha-1}\}$. Then for all $f \in L^1_{\text{loc}}(R_+)$ ($f \ge 0$) and for all $x \in R_+$ we have the inequality

$$\Big(\int\limits_0^x k_n(x,y)f(y)\,dy\Big)^p \le c\int\limits_0^x k_n(x,y)\Big(\int\limits_0^y k_n(x,y)f(z)\,dz\Big)^{p-1} f(y)\,dy,$$

where $c$ is the same as in inequlity (1.12).

**Lemma 1.2.** *Let $0 < \alpha < 1$, $v$ be a locally integrable a.e. positive function on $R_+$. Let there exist a constant $c > 0$ such that the inequality*

$$\|R_\alpha f\|_{L^p_{v_1}(R_+)} \le c_1\|f\|_{L^p(R_+)}, \quad v_1(x) = \Big[(W_\alpha v)(x)\Big]^{p'} \tag{1.13}$$

*holds for all $f \in L^p(R_+)$. Then*

$$\|R_\alpha f\|_{L^p_v(R_+)} \le c_2\|f\|_{L^p(R_+)}, \quad f \in L^p(R_+),$$

*where $c_2 = c_1^{1/p'}c^{1/p}$ and $c$ is the same as in* (1.12).

*Proof.* Let $f \ge 0$. Using Lemma 1.1, Tonelli's theorem and Hölder's inequality we have

$$\int_0^\infty (R_\alpha f(x))^p v(x)\,dx$$

$$\leq c\int_0^\infty R_\alpha\Big[f(R_\alpha f)^{p-1}\Big](x)v(x)\,dx = c\int_0^\infty (R_\alpha f)^{p-1}(y)f(y)(W_\alpha v)(y)\,dy$$

$$\leq c\Big(\int_0^\infty (f(y))^p\,dy\Big)^{\frac{1}{p}}\Big(\int_0^\infty (R_\alpha f(y))^p v_1(y)\,dy\Big)^{\frac{1}{p'}}$$

$$= c\|f\|_{L^p(R_+)}\|R_\alpha f\|^{p-1}_{L^p_{v_1}(R_+)} \leq c_1^{p-1}c\|f\|_{L^p(R_+)}\|f\|^{p-1}_{L^p(R_+)} = c_1^{p-1}c\|f\|^p_{L^p(R_+)}.$$

Hence

$$\|R_\alpha f\|_{L^p_v(R_+)} \leq c_1^{\frac{1}{p'}}c^{\frac{1}{p}}\|f\|_{L^p(R_+)}. \qquad \square$$

**Lemma 1.3.** *Let $1 < p < \infty$, $0 < \alpha < 1$, $W_\alpha v \in L^{p'}_{\rm loc}$, and let*

$$W_\alpha(W_\alpha v)^{p'}(x) \leq c_3(W_\alpha v)(x) \ \text{ a.e.}$$

*Then we have*

$$\|R_\alpha f\|_{L^p_{v_1}(R_+)} \leq c_4\|f\|_{L^p(R_+)}, \quad f \in L^p(R_+), \tag{1.14}$$

*where $v_1(x) = [(W_\alpha v)(x)]^{p'}$ and $c_4 = c\,c_3$ ($c$ is from (1.12)).*

*Proof.* Let $f \geq 0$ and let $I \subset R_+$ be a support of $f$, where $I$ has a form $I = (a,b)$, $0 < a < b < \infty$. Let $k_n(x,y) = \min\{(x-y)^{\alpha-1}, n\}$. Then using Lemma 1.1, Remark 1.1 and Tonelli's theorem we have

$$\int_0^\infty\Big(\int_0^x k_n(x,y)f(y)\,dy\Big)^p v_1(x)\,dx$$

$$\leq c\int_0^\infty\bigg(\int_0^x k_n(x,y)\Big(\int_0^y k_n(y,z)f(z)\,dz\Big)^{p-1}f(y)\,dy\bigg)v_1(x)\,dx$$

$$= c\int_0^\infty f(y)\Big(\int_0^y k_n(y,z)f(z)\,dz\Big)^{p-1}\Big(\int_y^\infty k_n(x,y)v_1(x)\,dx\Big)dy$$

$$\leq c\|f\|_{L^p(R_+)}\bigg(\int_I\Big(\int_0^y k_n(y,z)f(z)\,dz\Big)^p\Big(W_\alpha[(W_\alpha v)^{p'}](y)\Big)^{p'}dy\bigg)^{1/p'}$$

$$\leq c\,c_3\|f\|_{L^p(R_+)}\bigg(\int_I\Big(\int_0^y k_n(y,z)f(z)\,dz\Big)^p[(W_\alpha v)(y)]^{p'}dy\bigg)^{1/p'}.$$

The last expression is finite as

$$\int_0^y k_n(y,z)f(z)\,dz \leq n\int_I f(z)\,dz \leq n|I|^{\frac{1}{p'}}\|f\|_{L^p(R_+)} < \infty$$

and

$$\int_I \Big((W_\alpha v)(y)\Big)^{p'} dy < \infty.$$

Consequently,

$$\left(\int_I \Big(\int_0^x k_n(x,y)f(y)\,dy\Big)^p v_1(x)\,dx\right)^{1/p} \le c_3\,c\|f\|_{L^p(R_+)},$$

where $c$ is from (1.12). Finally, we have (1.14). □

Combining the above-proved Lemmas we obtain sufficiency of Theorem 1.1. □

The next theorem concerns the boundedness of $W_\alpha$ and is proved just in the same way as the previous result.

**Theorem 1.2.** *Let $1 < p < \infty$ and $0 < \alpha < 1/p$. Then the inequality*

$$\|W_\alpha f\|_{L^p_v(R_+)} \le c\|f\|_{L^p(R_+)}, \quad f \in L^p(R_+),$$

*holds if and only if $R_\alpha v \in L^{p'}_{\mathrm{loc}}(R_+)$ and*

$$R_\alpha[R_\alpha v]^{p'}(x) \le c\,R_\alpha v(x)$$

*for a.a. $x \in R_+$.*

Let us consider the Riemann–Liouville and Weyl operators:

$$\mathcal{R}_\alpha f(x) = \int_{-\infty}^x (x-y)^{\alpha-1} f(y)\,dy, \quad x \in R,$$

$$\mathcal{W}_\alpha f(x) = \int_x^\infty (y-x)^{\alpha-1} f(y)\,dy, \quad x \in R.$$

The following statements follow analogously and their proofs are omitted.

**Theorem 1.3.** *Let $1 < p < \infty$ and $0 < \alpha < 1/p$. Assume that $v$ is a weight on $R$. Then $\mathcal{R}_\alpha$ is bounded from $L^p(R)$ to $L^p_v(R)$ if and only if $\mathcal{W}_\alpha v \in L^{p'}_{\mathrm{loc}}(R)$ and*

$$\mathcal{W}_\alpha[\mathcal{W}_\alpha v]^{p'}(x) \le c\,\mathcal{W}_\alpha v(x)$$

*for a.a. $x \in R$.*

**Theorem 1.4.** *Let $1 < p < \infty$ and $0 < \alpha < 1/p$. Then the inequality*

$$\|\mathcal{W}_\alpha f\|_{L^p_v(R)} \le c\|f\|_{L^p(R)}, \quad f \in L^p(R),$$

*holds if and only if $\mathcal{R}_\alpha v \in L^{p'}_{\mathrm{loc}}(R)$ and*

$$\mathcal{R}_\alpha[\mathcal{R}_\alpha v]^{p'}(x) \le c\,\mathcal{R}_\alpha v(x)$$

*for a.a. $x \in R$.*

Let us now consider the case of two weights.

Let $\mu$ and $\nu$ be locally finite Borel measures on $R_+$ and let

$$R_{\alpha,\mu}f(x)=\int\limits_{[0,x]} f(y)(x-y)^{\alpha-1}\,d\mu(y),$$

$$W_{\alpha,\nu}f(x)=\int\limits_{[x,\infty)} f(y)(y-x)^{\alpha-1}\,d\nu(y),$$

where $x\in R_+$ and $\alpha\in(0,1)$.

**Theorem 1.5.** *Let $1<p<\infty$ and $0<\alpha<1$. Assume that the measures $\mu$ and $\nu$ satisfy the conditions $W_{\alpha,\nu}(1)\in L^{p'}_{\mu,\mathrm{loc}}(R_+)$ and*

$$B\equiv\sup_{\substack{x,r\\ x>0,r>0}}\Big(\int\limits_r^\infty\frac{\nu(I_t(x))}{t^{1-\alpha}}\,\frac{dt}{t}\Big)^{1/p}\Big(\int\limits_0^r\frac{\mu(I_t(x))}{t^{\alpha-1}}\,\frac{dt}{t}\Big)^{1/p'}<\infty, \tag{1.15}$$

*where $I_r(x)$ is the interval of the type $[x,x+r)$. Then the inequality*

$$\int\limits_0^\infty|R_{\alpha,\mu}f(x)|^p\,d\nu(x)\le c_0\int\limits_0^\infty|f(x)|^p\,d\mu(x),\quad f\in L^p_\mu(R_+), \tag{1.16}$$

*holds if and only if*

$$W_{\alpha,\mu}\Big[W_{\alpha,\nu}(1)\Big]^{p'}(x)\le c\,W_{\alpha,\nu}(1)(x) \tag{1.17}$$

*for $\mu$-a.a. $x$.*

Sufficiency of this theorem follows using the following Lemmas:

**Lemma 1.4.** *Let $1<p<\infty$ and $0<\alpha<1$. Then there exists a positive constant $c$ such that for all $f\in L^1_{\mu,\mathrm{loc}}(R_+)$, $f\ge0$, and for arbitrary $x\in R_+$ the inequality*

$$\Big(R_{\alpha,\mu}f(x)\Big)^p\le c\,R_{\alpha,\mu}\Big((R_{\alpha,\mu}f)^{p-1}f\Big)(x) \tag{1.18}$$

*holds (for $c$ we have $c=2^{\frac{1}{p-1}}$ if $p\le2$ and $c=2^{p(p-1)}$ if $p>2$).*

**Lemma 1.5.** *Let $0<\alpha<1$. Suppose that there exists a positive constant $c_1>0$ such that the inequality*

$$\|R_{\alpha,\mu}f\|_{L^p_{\nu_1}(R_+)}\le c_1\|f\|_{L^p_\mu(R_+)},\quad d\nu_1(x)=\Big[(W_{\alpha,\nu}(1))(x)\Big]^{p'}d\mu(x) \tag{1.19}$$

*holds for all $f\in L^p_\mu(R_+)$. Then*

$$\|R_{\alpha,\mu}f\|_{L^p_\nu(R_+)}\le c_2\|f\|_{L^p_\mu(R_+)},\quad f\in L^p_\mu(R_+).$$

*where $c_2=c_1^{1/p'}c^{1/p}$ and $c$ is the same as in (1.18).*

**Lemma 1.6.** *Let* $1 < p < \infty$, $0 < \alpha < 1$, $W_{\alpha,\nu}(1) \in L^{p'}_{\mu,\mathrm{loc}}(R_+)$, *and let*

$$W_{\alpha,\mu}(W_{\alpha,\nu}(1))^{p'}(x) \leq c_3(W_{\alpha,\nu}(1))(x) \ \ \mu\text{-a.e.}$$

*Then we have*

$$\|R_{\alpha,\mu}f\|_{L^p_{\nu_1}(R_+)} \leq c_4\|f\|_{L^p_\mu(R_+)}, \quad f \in L^p_\mu(R_+),$$

*where* $d\nu_1(x) = [(W_{\alpha,\nu}(1))(x)]^{p'}$ *and* $c_4 = cc_3$ ($c$ *is from* (1.18)).

These lemmas can be proved in the same way as Lemmas 1.1, 1.2 and 1.3 above.

Taking into account the proof of Theorem 1.1, we easily obtain necessity. Moreover, for the constant $c$ in condition (1.17) we have

$$c = 2^{p'-1}\Big(c_0^{p'-1}2^{1-\alpha} + (1-\alpha)^{p'}2^{(1-\alpha)p'}B^{p'}p'\Big),$$

where $c_0$ and $B$ are from (1.16) and (1.15), respectively.

Finally we note that the following proposition holds for the Volterra-type integral operator

$$K_\mu f(x) = \int\limits_{[0,x]} f(y)k(x,y)\,d\mu(y),$$

where $\mu$ is a locally finite Borel measure on $R_+$ and the kernel $k$ satisfies the condition: there exists a positive constant $b$ such that for all $x$, $y$, $z$, with $0 < y < z < x < \infty$, the inequality

$$k(x,y) \leq bk(z,y) \tag{1.20}$$

is fulfilled.

**Theorem 1.6.** *Let* $1 < p < \infty$, *the kernel* $k$ *satisfy condition* (1.20). *Let* $\nu$ *and* $\mu$ *be locally finite Borel measures on* $R_+$. *Suppose that* $K'_\nu(1) \in L^{p'}_\mu(R_+)$, *where*

$$K'_\nu g(x) = \int\limits_{[x,\infty)} g(y)k(y,x)\,d\nu(y).$$

*Then the condition*

$$K'_\mu\Big[K'_\nu(1)\Big]^{p'}(x) \leq cK'_\nu(1)(x), \ \ \mu\text{-a.e.}$$

*implies the boundedness of the operator* $K_\mu$ *from* $L^p_\mu(R_+)$ *to* $L^p_\nu(R_+)$.

The proof of this statement follows in the same way as sufficiency of Theorem 1.5 and therefore is omitted.

## 2. Application to the Existence of Positive Solutions of Nonlinear Integral Equations

The goal of this section is to give a characterization of the existence of positive solutions of some Volterra-type nonlinear integral equations.

First let us consider the integral equation

$$\varphi(x)=\int_x^\infty \frac{\varphi^p(t)}{(t-x)^{1-\alpha}}\,dt+\int_x^\infty \frac{v(t)}{(t-x)^{1-\alpha}}\,dt,\quad 0<\alpha<1 \tag{2.1}$$

with given non-negative $v\in L_{\rm loc}(R_+)$.

**Theorem 2.1.** *Let $1<p<\infty$, $0<\alpha<1$, $p'=\frac{p}{p-1}$, and $A_p=(p'-1)(p')^{-p}$.*

(i) *If $W_\alpha v\in L^p_{\rm loc}(R_+)$ and the inequality*

$$W_\alpha[W_\alpha v]^p(x)\le A_pW_\alpha v(x)\ \ a.e. \tag{2.2}$$

*holds, then (2.1) has a non-negative solution $\varphi\in L^p_{\rm loc}(R_+)$. Moreover, $(W_\alpha v)(x)\le\varphi(x)\le p'(W_\alpha v)(x)$.*

(ii) *If $0<\alpha<\frac{1}{p'}$ and (2.1) has a non-negative solution in $L^p_{\rm loc}(R_+)$, then $W_\alpha v\in L^p_{\rm loc}(R_+)$ and*

$$W_\alpha[(W_\alpha v)^p](x)\le c\,W_\alpha v(x)\ \ a.e. \tag{2.3}$$

*for some constant $c>0$.*

*Proof.* We shall use the following iteration procedure. Let $\varphi_0=0$, and let for $k=0,1,2,\dots$

$$\varphi_{k+1}(x)=W_\alpha(\varphi_k^p)(x)+W_\alpha v(x). \tag{2.4}$$

By induction it is easy to verify that

$$W_\alpha v(x)\le\varphi_k(x)\le\varphi_{k+1}(x),\quad k=0,1,2,\dots \tag{2.5}$$

From (2.4) we shall inductively derive an estimate of $\varphi_k(x)$.

Let

$$\varphi_k(x)\le c_kW_\alpha v(x) \tag{2.6}$$

for some $k=0,1,\dots$. It is obvious that $c_1=1$. Then (2.2), (2.3) and (2.6) yield

$$\varphi_{k+1}(x)\le(A_pc_k^p+1)(W_\alpha v)(x),$$

where $A_p$ is the constant from (2.2). Thus $c_{k+1}=A_pc_k^p+1$ for $k=1,2,\dots$. Now by induction and the definition of $A_p$ we deduce that the sequence $(c_k)_k$ is increasing. Indeed, it is obvious that $c_1<c_2$. Let $c_k<c_{k+1}$. Then

$$c_{k+1}(x)=A_pc_k^p+1<A_pc_{k+1}^p+1=c_{k+2}.$$

It is also clear that $(c_k)_k$ is bounded from the above by $p'$ and consequently it converges. As the equation $z=A_pz^p+1$ has only one solution, $x=p'$, it follows that $\lim_{k\to\infty} c_k = p'$. On the other hand, the sequence $(\varphi_k)_k$ is nondecreasing and by (2.6) we get

$$\varphi(x) = \lim_{k\to\infty} \varphi_k(x) \le p'(W_\alpha v)(x).$$

By our assumption $W_\alpha v \in L^p_{\mathrm{loc}}(R_+)$ and from the preceding estimate we conclude that $\varphi \in L^p_{\mathrm{loc}}(R_+)$. Moreover, $(W_\alpha v)(x) \le \varphi(x) \le p'(W_\alpha v)(x)$.

Now we prove the statement (ii). Suppose (2.1) has a solution $\varphi \in L^p_{\mathrm{loc}}(R^+)$. We have

$$W_\alpha(\varphi^p)(x) \le \varphi(x) < \infty \ \text{ a.e.} \tag{2.7}$$

Hence $W_\alpha(\varphi^p) \in L^p_{\mathrm{loc}}(R_+)$. Then from (2.7) we get

$$W_\alpha\Big[W_\alpha(\varphi^p)(x)\Big]^p(x) \le W_\alpha(\varphi^p)(x) \ \text{ a.e.}$$

Applying Theorem 1.1, we deduce that

$$\|R_\alpha f\|_{L^{p'}_\rho} \le \|f\|_{L^{p'}},$$

where $\rho(x) = \varphi^p(x)$. But $(W_\alpha v)(x) \le \varphi(x)$. Due to (2.7) we get

$$\|R_\alpha f\|_{L^{p'}_{\rho_1}} \le c\|f\|_{L^{p'}}$$

with $\rho_1(x) = (W_\alpha v)^p(x)$. Using Lemma 1.2 we arrive at the inequality

$$\|R_\alpha f\|_{L^{p'}_v} \le \|f\|_{L^{p'}}.$$

Applying Theorem 1.1 we come to condition (2.3). □

Analogously, we can prove

**Theorem 2.2.** *Let $1 < p < \infty$, $0 < \alpha < 1$, and let $A_p = (p'-1)(p')^{-p}$.*
(i) *If $R_\alpha v \in L^p_{\mathrm{loc}}(R_+)$ and the inequality*

$$R_\alpha[R_\alpha v]^p(x) \le A_p R_\alpha v(x) \ \text{ a.e.}$$

*holds, then the integral equation*

$$\varphi(x) = R_\alpha(\varphi^p)(x) + R_\alpha(v)(x) \tag{2.8}$$

*has a non-negative solution $\varphi \in L^p_{\mathrm{loc}}(R_+)$. Moreover, $R_\alpha v(x) \le \varphi(x) \le p'(R_\alpha v)(x)$.*

(ii) *If $0 < \alpha < \frac{1}{p'}$ and (2.8) has a non-negative solution in $L^p_{\mathrm{loc}}(R_+)$, then $R_\alpha v \in L^p_{\mathrm{loc}}(R_+)$ and*

$$R_\alpha[(R_\alpha v)^p](x) \le c\, R_\alpha v(x) \ \text{ a.e.}$$

*for some constant $c > 0$.*

## References


1. D. R. Adams and L. I. Hedberg, Function spaces and potential theory. *Springer-Verlag, Berlin,* 1996.
2. V. G. Mazya, Sobolev spaces. *Springer, Berlin,* 1985.
3. V. G. Mazya and I. E. Verbitsky, Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers. *Ark. Mat.* **33**(1995), 81–115.
4. E. T. Sawyer and R. L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Amer. J. Math.* **114**(1992), 813–874.
5. W. P. Ziemer, Weakly differentiable functions. *Springer-Verlag, New York*, 1989.
6. I. E. Verbitsky and R. L. Wheeden, Weighted estimates for fractional integrals and applications to semilinear equatuions. *J. Funct. Anal.* **129**(1995), 221–241.
7. I. E. Verbitsky, Superlinear equalities, potential theory and weighted norm inequalities. In: *Nonlinear Analysis, Function Spaces and Applications* 6, *Proceedings, Mathematical Institute Czech Academy of Sciences, Olimpia Press, Prague*, 1998.
8. J. Newman and M. Solomyak, Two-sided estimates on singular values for a class of integral operators on the semi-axis. *Integral Equations Operator Theory* **20**(1994), 335–349.
9. A. Meskhi, Solution of some weight problems for the Riemann–Liouville and Weyl operator. *Georgian Math. J.* **5**(1998), No. 6, 565–574.
10. V. Kokilashvili and A. Meskhi, Boundedness and compactness criteria for some classical integral operators. *In: Lecture Notes in Pure and Applied Mathematics*, 213, *"Function Spaces* V*", Proceedings of the Conference, Poznań, Poland* (*Ed. H. Hudzik and L. Skrzypczak*), 279–296, *New York, Bazel, Marcel Dekker,* 2000.
11. S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives. Theory and Applications. *Gordon and Breach Science Publishers, London–New York*, 1993.


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