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Two-Stage LAO Detection of Distribution for Arbitrarily Varying Object with the Pair of Families of Distributions

Farshin Hormozi-nejad

Islamic Azad University, Ahvaz Branch, Iran
E-mail: hormozi-nejad@iauahvaz.ac.ir

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Abstract

Multiple statistical hypotheses two-stage testing to make choice between hypotheses concerning distribution of discrete arbitrarily varying object is investigated. In the first stage one family of distributions is detected and then in the second stage, one distribution is denoted in mentioned family. The matrix of optimal asymptotic interdependencies of all pairs of the error probability exponents (reliabilities) are studied for arbitrarily varying object with the current states sequence known to the statistician. The goal of research is to express the optimal functional relations for all parts of reliabilities of LAO testing by two stages.

Keywords: *Logarithmically asymptotically optimal test, multiple hypotheses testing, multistage tests, error probability exponent.*

1 Introduction

In some works results of probability theory and statistics were obtained with application of information-theoretical methods and there are studies where statistical results provide ground for new findings in information theory [1, 9, 10, 15]. Hoeffding [10] and Tusnady [15] dealt with the error exponents for testing two simple statistical hypotheses. The exponent of error probability is called the reliability. In case of two hypotheses both reliabilities corresponding to two possible error probabilities could not be increased simultaneously, it is an accepted way to fix the value of one of the reliabilities and try to make the tests sequence get the greatest value of the remaining reliability. Such a test is called logarithmically asymptotically optimal (LAO). The need of testing of more than two hypotheses in many scientific and applied fields has essentially increased recently. Ahlswede *et al.* [1] and Haroutunian [6] formulated some problems of multiple hypotheses testing and identification. Haroutunian *et al.* [9] investigated the problem of LAO testing of multiple statistical hypotheses. Fu and Shen [5] and Haroutunian *et al.* [8] declared hypothesis testing for arbitrarily varying source. The model of the two-stage LAO testing in multiple hypotheses for a pair of families of distributions is investigated in [7, 12].

In some researches the problem of detection are investigated such as works of Chen and Papamarcou [2], Shalaby and Papamarcou [13], Tsitsiklis and Athans [14] and Willett and Warren [16]. The two-stage multiple hypotheses LAO test of distributed detection system for many families of distributions is investigated in [11]. This paper is dedicated to the two-stage detection of distribution concerning distributions of arbitrarily varying object.

2 Preliminaries

Random variable (RV) X characterizing the studied object takes values in the discrete finite set $\mathcal{X}$ and $\mathcal{P}(\mathcal{X})$ is the space of all distributions on $\mathcal{X}$. Suppose $\mathcal{G}$ be the alphabet of states of the object. The state $g \in \mathcal{G}$ of the object changes independently each moment of time n. S possible conditional probability distributions (PD) of X are given. Suppose $\mathcal{P}(g)$ be a set of conditional PDs as

$$\mathcal{P}(g) = \{P_1(x|g), P_2(x|g), ..., P_S(x|g)\}, \quad x \in \mathcal{X}, \quad g \in \mathcal{G}.$$

consists of S PDs of X which are divided to two disjoint families of PDs. The first family

$$\mathcal{P}_1(g) = \{P_1(x|g), P_2(x|g), ..., P_R(x|g)\}, \quad x \in \mathcal{X}, \quad g \in \mathcal{G},$$

includes R hypotheses and the second family

$$\mathcal{P}_2(g) = \{P_{R+1}(x|g), P_{R+2}(x|g), ..., P_S(x|g)\}, \ x \in \mathcal{X}, \ g \in \mathcal{G},$$

consists of $S - R$ hypotheses. It is not known which of these alternative hypotheses $H_s : P(x|g) = P_s(x|g), \ s = \overline{1, S}$, is in reality and it must be detected.

Let N-sample $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, be a vector of results of N independent observations of the RV X. The source of states of the object produces vector

$$\mathbf{g} = (g_1, g_2, ..., g_N), \quad g_n \in \mathcal{G}, \quad n = \overline{1, N}.$$

The purpose of the procedure is using sample $\mathbf{x}$ dependent on $\mathbf{g}$ for detecting the actual PD from given family of PDs. The probability of vector $\mathbf{x}$ for given states vector $\mathbf{g}$ is

$$P_s^N(\mathbf{x}|\mathbf{g}) = \prod_{n=1}^{N} P_s(x_n|g_n), \quad s = \overline{1,S}.$$

Let us introduce two sets of indices $\mathcal{D}_1 = \{\overline{1,R}\}$ and $\mathcal{D}_2 = \{\overline{R+1,S}\}$ and a pair of disjoint families of PDs

$$\mathcal{P}_1(g) = \{P_s(x|g), \quad s \in \mathcal{D}_1\}, \qquad \mathcal{P}_2(g) = \{P_s(x|g), \quad s \in \mathcal{D}_2\}.$$

The entropy of RV X with PD Q and the divergence (Kullback-Leibler distance) of PDs Q and P, are defined [3, 4] as follows:

$$\mathrm{H}_Q(X) \triangleq -\sum_{x\in\mathcal{X}} Q(x)\log Q(x),$$

$$\mathrm{D}\left(Q \parallel P\right) \triangleq \sum_{x\in\mathcal{X}} Q(x)\log\frac{Q(x)}{P(x)}.$$

The method of types is a base of our proofs, so here some definitions and estimates are reminded [3, 4, 9]. Let $N(g|\mathbf{g})$ be the number of repetitions of the element $g \in \mathcal{G}$ in the vector $\mathbf{g} \in \mathcal{G}^N$, and

$$\pi_{\mathbf{g}}\left(g\right) \triangleq N\left(g|\mathbf{g}\right)/N \ , \quad g \in \mathcal{G},$$

is the PD, called in information theory the type of vector $\mathbf{g}$. For a pair of vectors $\mathbf{x} \in \mathcal{X}^N$ and $\mathbf{g} \in \mathcal{G}^N$, let $N(x,g|\mathbf{x},\mathbf{g})$ be the number of occurrences of pair $(x,g) \in \mathcal{X} \times \mathcal{G}$ in the pair of vectors $(\mathbf{x},\mathbf{g})$. The joint type of the pair of vectors $(\mathbf{x},\mathbf{g})$ is defined by

$$Q_{\mathbf{x},\mathbf{g}}\left(x,g\right) \triangleq N\left(x,g|\mathbf{x},\mathbf{g}\right)/N \ , \quad x \in \mathcal{X}, \quad g \in \mathcal{G}.$$

The conditional type of $\mathbf{x}$ for given $\mathbf{g}$ is the conditional distribution defined by

$$Q_{\mathbf{x}|\mathbf{g}}\left(x|g\right) = \frac{Q_{\mathbf{x},\mathbf{g}}\left(x,g\right)}{\pi_{\mathbf{g}}\left(g\right)} = \frac{N\left(x,g|\mathbf{x},\mathbf{g}\right)}{N\left(g|\mathbf{g}\right)} \ , \quad x \in \mathcal{X}, \quad g \in \mathcal{G}.$$

Let X and G be RVs defined by probability distributions $Q(x|g)$ and $\pi(g)$. The conditional entropy of X respective to G is:

$$\mathrm{H}_{\pi,Q}(X|G) \triangleq -\sum_{x\in\mathcal{X},g\in\mathcal{G}} \pi(g)Q(x|g)\log Q(x|g).$$

The conditional divergence of the distribution $\pi \circ Q = \{\pi(g)Q(x|g),\ x \in \mathcal{X},\ g \in \mathcal{G}\}$ with respect to the distribution $\pi \circ P_s = \{\pi(g)P_s(x|g),\ x \in \mathcal{X},\ g \in \mathcal{G}\}$ is

$$\mathrm{D}\left(\pi \circ Q \parallel \pi \circ P_s\right) \triangleq \mathrm{D}\left(Q \parallel P_s|\pi\right) \triangleq \sum_{x\in\mathcal{X},g\in\mathcal{G}} \pi(g)Q(x|g)\log\frac{Q(x|g)}{P_s(x|g)}.$$

Let $\mathcal{P}^N(\mathcal{X})$ be the set of all possible types on $\mathcal{X}^N$ for N observations, $\mathcal{T}_Q^N$ be the set of all vectors $\mathbf{x}$ of the type $Q \in \mathcal{P}^N(\mathcal{X})$, $\mathcal{P}^N(\mathcal{G})$ be the set of all types on $\mathcal{G}$ for given N, $\mathcal{P}(\mathcal{G})$ be the set of all possible probability distributions π on $\mathcal{G}$ and $\mathcal{Q}^N(X|\mathbf{g})$ be the set of all possible conditional types on $\mathcal{X}$ for given $\mathbf{g}$. Also suppose that $\mathcal{T}^N_{\pi_\mathbf{g},Q_{\mathbf{x}|\mathbf{g}}}(X|\mathbf{g})$ be the family of vectors $\mathbf{x}$ of the conditional type Q for given $\mathbf{g}$ of the type $\pi_\mathbf{g}$. The following well known properties of types will be used [3, 9]:

$$|\mathcal{Q}^N(X|\mathbf{g})| \le (N+1)^{|\mathcal{X}||\mathcal{G}|},$$

$$(N+1)^{-|\mathcal{X}||\mathcal{G}|}.\exp\left\{N\mathrm{H}_{\pi_\mathbf{g},Q_{\mathbf{x}|\mathbf{g}}}(X|G)\right\} \le |\mathcal{T}^N_{\pi_\mathbf{g},Q_{\mathbf{x}|\mathbf{g}}}(X|\mathbf{g})| \le \exp\left\{N\mathrm{H}_{\pi_\mathbf{g},Q_{\mathbf{x}|\mathbf{g}}}(X|G)\right\},$$

for $\mathbf{x} \in \mathcal{T}^N_{\pi_\mathbf{g},Q_{\mathbf{x}|\mathbf{g}}}(X|\mathbf{g})$:

$$P_s^N(\mathbf{x}|\mathbf{g}) = \exp\left\{-N(\mathrm{H}_{\pi,Q}(X|G) + \mathrm{D}(Q \parallel P_s|\pi)\right\}.$$

3 The Two-Stage LAO Detection

Suppose $N = N_1 + N_2$ be such that:

$$N_1 = \lceil \psi N \rceil, \quad N_2 = [(1-\psi)N], \quad 0 < \psi < 1,$$
$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2), \quad \mathbf{x} \in \mathcal{X}^N, \quad \mathcal{X}^N = \mathcal{X}^{N_1} \times \mathcal{X}^{N_2},$$
$$\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2), \quad \mathbf{g} \in \mathcal{G}^N, \quad \mathcal{G}^N = \mathcal{G}^{N_1} \times \mathcal{G}^{N_2}.$$

The two-stage procedure on the base of N-sample is denoted by Φ^N. Such test may be realized by a pair of tests $\varphi_1^{N_1}$ and $\varphi_2^{N_2}$ for two consecutive stages and it is written by $\Phi^N = (\varphi_1^{N_1}, \varphi_2^{N_2})$. The first stage is a non-randomized test $\varphi_1^{N_1}(\mathbf{x}_1, \mathbf{g}_1)$ based on the joint sample $(\mathbf{x}_1, \mathbf{g}_1)$. The next stage is a non-randomized test $\varphi_2^{N_2}(\mathbf{x}_2, \mathbf{g}_2, \varphi_1^{N_1})$ based on joint sample $(\mathbf{x}_2, \mathbf{g}_2)$ and the outcome of test $\varphi_1^{N_1}(\mathbf{x}_1, \mathbf{g}_1)$.

3.1 First Stage of Two-stage Test

The first stage of decision making for detection of a family of PDs denoted by a test $\varphi_1^{N_1}(\mathbf{x}_1, \mathbf{g}_1)$, can be defined by division of the sample space $\mathcal{X}^{N_1}$ on two distinct subsets

$$\mathcal{A}_i^{N_1}(\mathbf{x}_1|\mathbf{g}_1) \triangleq \{\mathbf{x}_1 : \varphi_1^{N_1}(\mathbf{x}_1, \mathbf{g}_1) = i\}, \quad i = \overline{1,2}.$$

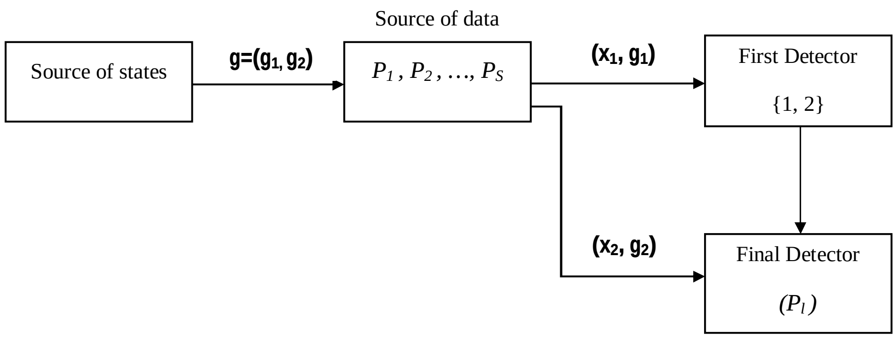

Figure 1: The two-stage detection with arbitrarily varying object

The set $\mathcal{A}_i^{N_1},\ i=1,2,$ consists all vectors $\mathbf{x}_1$ for which i-th family of PDs is adopted.

Let $\alpha'_{i|j}(\varphi_1^{N_1}),\ i\neq j,\ i,j=1,2,$ be the probability of the erroneous acceptance of the i-th family of PDs provided that the j-th family of PDs is true (that is the correct PD is in the j-th family):

$$\alpha'_{i|j}(\varphi_1^{N_1}) \triangleq \max_{\mathbf{g}_1\in\mathcal{G}^{N_1}} \max_{s\in\mathcal{D}_j} P_s^{N_1}(\mathcal{A}_i^{N_1}), \quad i\neq j,\ i,j=1,2. \tag{1}$$

The reliabilities of the infinite sequence of tests φ_1 are defined by

$$E'_{i|j}(\varphi_1) \triangleq \liminf_{N_1\to\infty}\left\{-\frac{1}{N_1}\log\alpha'_{i|j}(\varphi_1^{N_1})\right\}, \quad i,j=\overline{1,2}. \tag{2}$$

The matrix of reliabilities for the first stage of the test is $\mathbf{E}'(\varphi_1)$ and one can see from (1)-(2) that

$$E'_{j|j}=E'_{i|j}, \quad i,j=\overline{1,2},\ i\neq j.$$

For construction of the necessary LAO test φ_1^* for preliminarily given positive value $E'^{*}_{1|1}$, the following subsets of distributions are defined:

$$\mathcal{A}_1^{*N_1}(\mathbf{x}_1|\mathbf{g}_1)=\bigcup_{Q_{\mathbf{x}_1|\mathbf{g}_1}:\ \min\limits_{s\in\mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1})\leq E'^{*}_{1|1}} \mathcal{T}^{N_1}_{\pi_{\mathbf{g}_1},Q_{\mathbf{x}_1|\mathbf{g}_1}}(X|\mathbf{g}_1),$$

and $\quad \mathcal{A}_2^{*N_1}(\mathbf{x}_1|\mathbf{g}_1)=\mathcal{X}^{N_1}\setminus\mathcal{A}_1^{*N_1}(\mathbf{x}_1|\mathbf{g}_1).$

Theorem 3.1. *If the positive value $E'^{*}_{1|1}$, is such that the following inequality hold*

$$E'^{*}_{1|1}<\min_{l\in\mathcal{D}_2,\ s\in\mathcal{D}_1} D(P_l||P_s), \tag{3}$$

then there exists a LAO sequence of procedures φ_1^* such that other reliability $E_{2|2}^{'*}$ is positive and is defined by

$$E_{2|2}^{'*} = \min_{\pi \in \mathcal{P}(\mathcal{G})} \min_{s \in \mathcal{D}_2} \inf_{Q: \min\limits_{l \in \mathcal{D}_1} \mathrm{D}(Q||P_l|\pi) \leq E_{1|1}^{'*}} D(Q||P_s|\pi).$$

Proof. By applying the properties of types and using the definition of the reliability, error probability is estimated as follows:

$$\alpha_{1|1}^{'}(\varphi_1^{*N_1}) = \max_{\mathbf{g}_1 \in \mathcal{G}^{N_1}} \max_{s \in \mathcal{D}_1} P_s^{N_1}\left(\overline{\mathcal{A}}_1^{*N_1}\right)$$

$$= \max_{\mathbf{g}_1 \in \mathcal{G}^{N_1}} \max_{s \in \mathcal{D}_1} P_s^{N_1}\Big(\bigcup_{Q_{\mathbf{x}_1|\mathbf{g}_1}: \min\limits_{s \in \mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1}) > E_{1|1}^{'*}} \mathcal{T}_{\pi_{\mathbf{g}_1}, Q_{\mathbf{x}_1|\mathbf{g}_1}}^{N_1} \Big)$$

$$\leq \max_{\mathbf{g}_1 \in \mathcal{G}^{N_1}} \max_{s \in \mathcal{D}_1} (N_1+1)^{|\mathcal{X}||\mathcal{G}|} \sup_{Q_{\mathbf{x}_1|\mathbf{g}_1}: \min\limits_{s \in \mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1}) > E_{1|1}^{'*}} P_s^{N_1}\left(\mathcal{T}_{\pi_{\mathbf{g}_1}, Q_{\mathbf{x}_1|\mathbf{g}_1}}^{N_1}\right)$$

$$\leq \max_{\mathbf{g}_1 \in \mathcal{G}^{N_1}} \max_{s \in \mathcal{D}_1} (N_1+1)^{|\mathcal{X}||\mathcal{G}|} \sup_{Q_{\mathbf{x}_1|\mathbf{g}_1}: \min\limits_{s \in \mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1}) > E_{1|1}^{'*}} \exp\left\{-N_1 \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1})\right\}$$

$$= \exp\left\{-N_1\Big[\min_{\pi_{\mathbf{g}_1} \in \mathcal{P}^{N_1}(\mathcal{G})} \min_{s \in \mathcal{D}_1} \inf_{Q_{\mathbf{x}_1|\mathbf{g}_1}: \min\limits_{s \in \mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1}) > E_{1|1}^{'*}} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1}) - o_{N_1}(1)\Big]\right\}$$

$$\leq \exp\left\{-N_1\{E_{1|1}^{'*} - o_{N_1}(1)\}\right\}.$$

where $o_{N_1}(1) \to 0$ is received by $N_1 \to \infty$. From here it follows that $E_{1|1}^{'}(\varphi_1^*) = E_{1|1}^{'*}$.

The another error probability is estimated as follows:

$$\alpha_{2|2}^{'}(\varphi_1^{*N_1}) = \max_{\mathbf{g}_1 \in \mathcal{G}^{N_1}} \max_{s \in \mathcal{D}_2} P_s^{N_1}\left(\mathcal{A}_1^{*N_1}\right)$$

$$= \max_{\mathbf{g}_1 \in \mathcal{G}^{N_1}} \max_{s \in \mathcal{D}_2} P_s^{N_1}\Big(\bigcup_{Q_{\mathbf{x}_1|\mathbf{g}_1}: \min\limits_{s \in \mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1}) \leq E_{1|1}^{'*}} \mathcal{T}_{\pi_{\mathbf{g}_1}, Q_{\mathbf{x}_1|\mathbf{g}_1}}^{N_1} \Big)$$

$$\leq \max_{\mathbf{g}_1 \in \mathcal{G}^{N_1}} \max_{s \in \mathcal{D}_2} (N_1+1)^{|\mathcal{X}||\mathcal{G}|} \sup_{Q_{\mathbf{x}_1|\mathbf{g}_1}: \min\limits_{s \in \mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1}) \leq E_{1|1}^{'*}} P_s^{N_1}\left(\mathcal{T}_{\pi_{\mathbf{g}_1}, Q_{\mathbf{x}_1|\mathbf{g}_1}}^{N_1}\right)$$

$$\leq \max_{\mathbf{g}_1 \in \mathcal{G}^{N_1}} \max_{s \in \mathcal{D}_2} (N_1+1)^{|\mathcal{X}||\mathcal{G}|} \sup_{Q_{\mathbf{x}_1|\mathbf{g}_1}: \min\limits_{s \in \mathcal{D}_2} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1}) \leq E_{1|1}^{'*}} \exp\left\{-N_1 \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1})\right\}$$

$$= \exp\Big\{ - N_1\Big[\min_{\pi_{\mathbf{g}_1} \in \mathcal{P}^{N_1}(\mathcal{G})} \min_{s \in \mathcal{D}_2} \inf_{Q_{\mathbf{x}_1|\mathbf{g}_1}: \min\limits_{s \in \mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1}) \leq E_{1|1}^{'*}} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1})$$

$$-o_{N_1}(1)\big]\Big\}. \tag{4}$$

Now let us prove the inverse inequality

$$\alpha'_{2|2}(\varphi_1^{*N_1}) = \max_{\mathbf{g}_1\in\mathcal{G}^{N_1}} \max_{s\in\mathcal{D}_2} P_s^{N_1}\left(\mathcal{A}_1^{*N_1}\right)$$

$$= \max_{\mathbf{g}_1\in\mathcal{G}^{N_1}} \max_{s\in\mathcal{D}_2} P_s^{N_1}\Bigg(\bigcup_{Q_{\mathbf{x}_1|\mathbf{g}_1}:\min\limits_{s\in\mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1})\leq E'^{*}_{1|1}} \mathcal{T}^{N_1}_{\pi_{\mathbf{g}_1},Q_{\mathbf{x}_1|\mathbf{g}_1}}\Bigg)$$

$$\geq \max_{\mathbf{g}_1\in\mathcal{G}^{N_1}} \max_{s\in\mathcal{D}_2} \sup_{Q_{\mathbf{x}_1|\mathbf{g}_1}:\min\limits_{s\in\mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1})\leq E'^{*}_{1|1}} P_s^{N_1}\left(\mathcal{T}^{N_1}_{\pi_{\mathbf{g}_1},Q_{\mathbf{x}_1|\mathbf{g}_1}}\right)$$

$$\geq \max_{\mathbf{g}_1\in\mathcal{G}^{N_1}} \max_{s\in\mathcal{D}_2} \sup_{Q_{\mathbf{x}_1|\mathbf{g}_1}:\min\limits_{s\in\mathcal{D}_2} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1})\leq E'^{*}_{1|1}} \exp\left\{-N_1 \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1})\right\}$$

$$= \exp\Big\{-N_1\big[\min_{\pi_{\mathbf{g}_1}\in\mathcal{P}^{N_1}(\mathcal{G})} \min_{s\in\mathcal{D}_2} \inf_{Q_{\mathbf{x}_1|\mathbf{g}_1}:\min\limits_{s\in\mathcal{D}_1} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1})\leq E'^{*}_{1|1}} \mathrm{D}(Q_{\mathbf{x}_1|\mathbf{g}_1}||P_s|\pi_{\mathbf{g}_1})$$

$$+o_{N_1}(1)\big]\Big\}. \tag{5}$$

According to the definition of the reliability (2), equations (4) and (5) will gain:

$$E'^{*}_{2|2} = \min_{\pi\in\mathcal{P}(\mathcal{G})} \min_{s\in\mathcal{D}_2} \inf_{Q:\min\limits_{l\in\mathcal{D}_1} \mathrm{D}(Q||P_l|\pi)\leq E'^{*}_{1|1}} D(Q||P_s|\pi).$$

3.2 Second Stage of the Two-Stage Test

The test $\varphi_2^{N_2}(\mathbf{x}_2, \mathbf{g}_2, \varphi_1^{N_1})$ can be defined by division of the sample space $\mathcal{X}^{N_2}$ to R (or $S-R$) distinct subsets. If the i-th family of PDs is accepted, then

$$\mathcal{B}_s^{N_2}(\mathbf{x}_2|\mathbf{g}_2, \varphi_1^{N_1}=i) \triangleq \{\mathbf{x}_2 : \varphi_2^{N_2}(\mathbf{x}_2, \mathbf{g}_2, \varphi_1^{N_1}) = s\}, \quad s\in\mathcal{D}_i, \quad i=\overline{1,2}.$$

The probability of the fallacious acceptance at the second stage of test of PD P_l, when P_s is correct, is

$$\alpha''_{l|s}(\varphi_2^{N_2}) \triangleq P_s^{N_2}(\mathcal{B}_l^{N_2}),\ l\neq s,\ l,s=\overline{1,S}.$$

The probability to reject P_s, when it is true and the first family of PDs is accepted, is

$$\alpha''_{s|s}(\varphi_2^{N_2}) \triangleq P_s^{N_2}(\overline{\mathcal{B}}_s^{N_2}) = \sum_{l\neq s} \alpha''_{l|s}(\varphi_2^{N_2}),\ l,s=\overline{1,S}. \tag{6}$$

Corresponding reliabilities for the second stage of test, are

$$E''_{l|s}(\varphi_2) \stackrel{\triangle}{=} \liminf_{N_2\to\infty} \left\{ -\frac{1}{N_2} \log \alpha''_{l|s}(\varphi_2^{N_2}) \right\}, \quad l,s=\overline{1,S}. \tag{7}$$

It follows from (6) and (7)

$$E''_{s|s}(\varphi_2) = \min_{l\neq s} E''_{l|s}(\varphi_2), \quad l,s=\overline{1,S}.$$

Theorem 3.2. *[6, 7] If at the first stage of test the first family of PDs is accepted, then for given positive values $E''_{s|s}$, $s=\overline{1,R-1}$ of the matrix of reliabilities $\mathbf{E}''(\varphi_2)$ let us consider the regions:*

$$\mathcal{R}''_s(\pi) = \left\{ Q : \min_{l\in\mathcal{D}_1} \mathrm{D}(Q||P_l|\pi) \leq E'^{*}_{1|1}, \quad \mathrm{D}(Q \parallel P_s|\pi) \leq E''_{s|s} \right\}, \quad s=\overline{1,R-1},$$

$$\mathcal{R}''_R(\pi) = \left\{ Q : \min_{l\in\mathcal{D}_1} \mathrm{D}(Q||P_l|\pi) \leq E'^{*}_{1|1}, \quad \mathrm{D}(Q \parallel P_s|\pi) > E''_{s|s}, \quad s=\overline{1,R-1} \right\},$$

and the following values of elements of the future matrix of reliabilities $\mathbf{E}''(\varphi_2^*)$ of the LAO test sequence:

$$E''^{*}_{s|s} = E''_{s|s}, \quad s=\overline{1,R-1},$$

$$E''^{*}_{l|s} = \min_{\pi\in\mathcal{P}(\mathcal{G})} \inf_{Q\in\mathcal{R}''_l} \mathrm{D}(Q \parallel P_s|\pi), \quad l=\overline{1,R},\ s=\overline{1,S},\ l\neq s.$$

If the following compatibility conditions are valid

$$E''_{1|1} < \min_{s=\overline{2,R}} \mathrm{D}(P_s \parallel P_1),$$

$$E''_{s|s} < \min[\min_{l=\overline{1,s-1}} E''^{*}_{l|s}, \min_{l=\overline{s+1,R}} \mathrm{D}(P_l \parallel P_s)], \quad 2\leq s\leq R-1,$$

then there exists a LAO sequence of tests φ_2^*, elements $E''^{*}_{l|s}$ of matrix of reliabilities $\mathbf{E}''(\varphi_2^*)$ of which are defined above and are positive.
If one compatibility condition is violated, then at least one element of the matrix $\mathbf{E}''(\varphi_2^*)$ is equal to zero.

When the second family of PDs is accepted, then Theorem 3.2 with replacing $s=\overline{R+1,S}$ will be used.

3.3 Reliabilities of Two-Stage Test

In the two-stage decision making, the test Φ^{*N} can be defined by partition of the sample space $\mathcal{X}^N$ to S separate subsets as follows

$$\mathcal{C}_s^N \triangleq \mathcal{A}_i^{*N_1} \times \mathcal{B}_s^{N_2}, \quad s \in \mathcal{D}_i, \quad i = 1, 2.$$

Definitions of error probabilities $\alpha'''_{l|s}(\Phi^{*N})$ and reliabilities $E'''_{l|s}(\Phi_2^*)$ can be used similar to Section 3. So error probabilities are considered as follows
a) if $l,\ s \in \mathcal{D}_i, \quad i = 1, 2$ then

$$\alpha'''_{l|s}(\Phi^{*N}) = P_s^{N_1}(\mathcal{A}_i^{*N_1}) \cdot P_s^{N_2}(\mathcal{B}_l^{N_2}) \tag{8}$$

b) if $s \in \mathcal{D}_i$ and $l \in \mathcal{D}_j, \quad i, j = 1, 2, \quad i \neq j$ then

$$\alpha'''_{l|s}(\Phi^{*N}) = P_s^{N_1}(\mathcal{A}_j^{*N_1}) \cdot P_s^{N_2}(\mathcal{B}_l^{N_2}) \tag{9}$$

By using properties of types the following equalities are created:

$$\lim_{N\to\infty} \left\{ -\frac{1}{N} \log P_s^N(\mathcal{A}_j^{*N}) \right\} = \min_{\pi \in \mathcal{P}(\mathcal{G})} \inf_{Q:Q\in\mathcal{A}_j^*} D(Q||P_s|\pi) \triangleq E_{j|s}^I, \quad s \notin \mathcal{D}_j. \tag{10}$$

According to equations (8)–(10) and definition of reliabilities are obtained:
a) if $l,\ s \in \mathcal{D}_i, \quad i = 1, 2$ then

$$E'''_{l|s}(\Phi^*) = (1 - \psi) E''^{*}_{l|s}, \tag{11}$$

b) if $s \in \mathcal{D}_i$ and $l \in \mathcal{D}_j, \quad i, j = 1, 2, \quad i \neq j$ then

$$E'''_{l|s}(\Phi^*) = \psi E_{j|s}^I + (1 - \psi) E''^{*}_{l|s}, \tag{12}$$

c) if $s \in \mathcal{D}_i,\ i = 1, 2$ then

$$E'''_{s|s}(\Phi^*) = \min_{l \neq s} E'''_{l|s}(\Phi_2^*), \tag{13}$$

Theorem 3.3. *If all compatibility conditions of Theorems 3.1 and 3.2 are satisfied, then elements of matrix of reliabilities* $\mathbf{E}'''(\Phi^*)$ *of the two-stage detection* Φ^* *are defined in equations (11)–(13).*
When one of compatibility condition is violated, then at least one element of $\mathbf{E}'''(\Phi^*)$ *is equal to zero.*

4 Discussion and Conclusion

Detection of distribution for multiple hypotheses two-stage test concerning distributions from arbitrarily varying object is considered and the optimal functional relations between the reliabilities of two-stage detection are investigated. It can be shown that the number of operations of the two-stage test is less than this of one-stage detection. It is shown for one invariant object in [7, 12]. Also the problem with arbitrarily varying object can be examined for hypotheses detection concerning two or more arbitrarily varying objects and at for many families of PDs.

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