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Integrals Involving I-Function

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Abstract

In this paper, we have presented certain integrals involving product of the I-function with exponential function, Gauss's hypergeometric function and Fox's H-function. The results derived here are basic in nature and may include a number of known and new results as particular cases.

Keywords: *Exponential function, hypergeometric function, H-function, I-function, Mellin-Barnes type contour integral.*

1 Introduction

The Gaussian hypergeometric function is of fundamental importance in the theory of special functions. The importance of this function lies in the fact that almost all of the commonly used functions of applicable mathematics, mathematical physics, engineering and mathematical biology are expressible as its special cases.

The series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{1} $$

where $(a)_n$ is the Pochhammer symbol defined by

$$ (a)_n = \begin{cases} a(a+1)\dots(a+n-1), & n \in N \\ 1, & n = 0 \end{cases} \tag{2} $$

is called the Gauss's hypergeometric series after the famous German mathematician Carl Friedrich Gauss (1777-1855) who in the year 1812 introduced this series. It is represented by the symbol ${}_2F_1(a,b;c;z)$ and is called the Gauss's hypergeometric function also.

In 1961, Charles Fox [2] introduced a function which is more general than the Meijer's G-function and this function is well known in the literature of special functions as Fox's H-function or simply the H-function. This function is defined and represented by means of the following Mellin-Barnes type contour integral:

$$ H[z] = H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right] = \frac{1}{2\pi i}\int_L \theta(s) z^s ds, \tag{3} $$

where, for convenience,

$$ \theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}, \tag{4} $$

and L is a suitable contour of the Mellin-Barnes type which runs from $-i\infty$ to $+i\infty$, separating the poles of $\Gamma(b_j-\beta_j s)$, $j=1,\dots,m$ from those of $\Gamma(1-a_j+\alpha_j s)$, $j=1,\dots,n$. An empty product is interpreted as unity. The integers m,n,p,q satisfy the inequalities $0 \le n \le p$, $0 \le m \le q$; the coefficients α_j $(j=1,\dots,p)$, β_j $(j=1,\dots,q)$ are positive real numbers, and the complex parameters a_j $(j=1,\dots,p)$, b_j $(j=1,\dots,q)$ are so constrained that no poles of the integrand coincide. Owing to the popularity of the special functions, those are defined in (1) and (3) (c.f. [4], [3] and [6]), details regarding these are avoided.

The I-function, which is more general than the Fox's H-function, defined by V.P. Saxena [5], by means of the following Mellin-Barnes type contour integral:

$$ I[z] = I_{p_i,q_i:r}^{m,n}[z] = I_{p_i,q_i:r}^{m,n}\left[z \middle| \begin{matrix} (a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right] = \frac{1}{2\pi i}\int_L \phi(\xi) z^{\xi} d\xi, \tag{5} $$

where,

$$\phi(\xi) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j \xi)}{\Sigma_{i=1}^{r} \Big\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} \xi) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} \xi) \Big\}}, \tag{6}$$

$p_i, q_i (i = 1, \ldots, r), m, n$ are integers satisfying $0 \leq n \leq p_i$, $0 \leq m \leq q_i$; $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive and a_j, b_j, a_{ji}, b_{ji} are complex numbers. L is a suitable contour of the Mellin-Barnes type running from $\gamma - i\alpha$ to $\gamma + i\alpha$ (γ is real) in the complex ξ-plane. Details regarding existence conditions and various parametric restrictions of I-function, we may refer [5].

For $r = 1$, (5) reduces to the Fox's H-function

$$I_{p_i,q_i:1}^{m,n} \left[z \middle| \begin{array}{l} (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i} \end{array} \right] = H_{p,q}^{m,n} \left[z \middle| \begin{array}{l} (a_j, \alpha_j)_{1,n}; (a_j, \alpha_j)_{n+1,p} \\ (b_j, \beta_j)_{1,m}; (b_j, \beta_j)_{m+1,q} \end{array} \right]$$

2 Required Results

We shall require the following results in the sequel:

The Mellin transform of the H-function follows from the definition (3) in view of the well-known Mellin inversion theorem. We have

$$\int_0^\infty x^{s-1} H_{p,q}^{m,n} \left[ax \middle| \begin{array}{l} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] dx = a^{-s} \theta(-s)$$

$$= a^{-s} \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j + \alpha_j s)}, \tag{7}$$

where,

$$A = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j > 0,$$

$$|\arg a| < \frac{1}{2} A\pi, \delta = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j > 0$$

and

$$-\min_{1 \leq j \leq m} [Re(b_j/\beta_j)] < Re(s) < \min_{1 \leq j \leq n} [Re\{(1 - a_j)/\alpha_j\}].$$

Lemma 2.1 *From Rainville [4], we have*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n - k) \tag{8}$$

3 Main Results

In this section, we have evaluated certain integrals involving product of the I-function with exponential function, Gauss's hypergeometric function and Fox's H-function.

First Integral

$$I_1 \equiv \int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xz}{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta}) \times I^{m,n}_{p_i,q_i:r}\left[yx^{\mu}(t-x)^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx$$
$$= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u} \times I^{m,n+2}_{p_i+2,q_i+1:r}\left[yt^{\mu+\nu}\left|\begin{matrix}(1-\rho-\zeta k,\mu),(1-\sigma-(\eta-1)k-u,\nu), & (a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}, & (1-\rho-\sigma-(\zeta+\eta-1)k-u,\mu+\nu)\end{matrix}\right.\right], \tag{9}$$

where,

$$f(k) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}, \tag{10}$$

provided

(i) $\mu \geq 0, \nu \geq 0$ (not both zero simultaneously)

(ii) ζ and η are non-negative integers such that $\zeta+\eta \geq 1$

(iii) $A_i > 0, B_i < 0; |\arg y| < \frac{1}{2}A_i\pi$, $\forall i \in 1,\dots,r$; where
$A_i = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ji} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ji}$,
$B_i = \frac{1}{2}(p_i-q_i) + \sum_{j=1}^{q_i}b_{ji} - \sum_{j=1}^{p_i}a_{ji}$

(iv) $Re(\rho) + \mu\min_{1\leq j\leq m}[Re(b_j/\beta_j)] > 0$,
$Re(\sigma) + \nu\min_{1\leq j\leq m}[Re(b_j/\beta_j)] > 0$.

Proof:

$$I_1 \equiv e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{(t-x)z}{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta}) \times I^{m,n}_{p_i,q_i:r}\left[yx^{\mu}(t-x)^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx$$

Now we replace $e^{(t-x)z}$ by $\sum_{u=0}^{\infty}\frac{(t-x)^u z^u}{u!}$ and express the hypergeometric function and the I-function with the help of (1) and (5) respectively, to get

$$\begin{aligned}I_1 &= e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\frac{(t-x)^u z^u}{u!}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k x^{\zeta k}(t-x)^{\eta k}}{k!}\\ &\times\frac{1}{2\pi i}\int_L \phi(\xi)y^{\xi}x^{\mu\xi}(t-x)^{\nu\xi}d\xi dx\\ &= e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k x^{\zeta k}(t-x)^{\eta k+u}}{k!}\frac{z^u}{u!}\\ &\times\frac{1}{2\pi i}\int_L \phi(\xi)y^{\xi}x^{\mu\xi}(t-x)^{\nu\xi}d\xi dx\end{aligned}$$

Now by the use of (8), the above result reduces to

$$\begin{aligned}I_1 &= e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k x^{\zeta k}(t-x)^{\eta k+u-k}}{k!}\frac{z^{u-k}}{(u-k)!}\\ &\times\frac{1}{2\pi i}\int_L \phi(\xi)y^{\xi}x^{\mu\xi}(t-x)^{\nu\xi}d\xi dx\end{aligned}$$

Interchanging the order of integration and summation, we obtain

$$\begin{aligned}I_1 &= e^{-zt}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}\frac{1}{2\pi i}\int_L \phi(\xi)y^{\xi}\\ &\times\left\{\int_0^t x^{\rho+\zeta k+\mu\xi-1}(t-x)^{\sigma+(\eta-1)k+u+\nu\xi-1}dx\right\}d\xi,\end{aligned}$$

where $f(k)$ is given by (10).
On substituting $x = ts$ in the inner x-integral, the above expression reduces to

$$\begin{aligned}I_1 &= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u}\frac{1}{2\pi i}\int_L \phi(\xi)y^{\xi}t^{(\mu+\nu)\xi}\\ &\times\left\{\int_0^1 s^{\rho+\zeta k+\mu\xi-1}(1-s)^{\sigma+(\eta-1)k+u+\nu\xi-1}ds\right\}d\xi\\ &= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u}\frac{1}{2\pi i}\int_L \phi(\xi)\\ &\times\frac{\Gamma(\rho+\zeta k+\mu\xi)\Gamma(\sigma+(\eta-1)k+u+\nu\xi)}{\Gamma(\rho+\sigma+(\zeta+\eta-1)k+u+(\mu+\nu)\xi)}y^{\xi}t^{(\mu+\nu)\xi}d\xi\end{aligned}$$

Finally, interpreting the contour integral by virtue of (5), we obtain

$$I_1 = e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u}$$

$$\times I^{m,n+2}_{p_i+2,q_i+1:r}\left[yt^{\mu+\nu} \left| \begin{matrix} (1-\rho-\zeta k, \mu), (1-\sigma-(\eta-1)k-u, \nu), \\ (b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i}, \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (1-\rho-\sigma-(\zeta+\eta-1)k-u, \mu+\nu) \end{matrix} \right].$$

Second Integral

$$I_2 \equiv \int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-xz} {}_2F_1(\alpha, \beta; \gamma; ax^{\zeta}(t-x)^{\eta})$$

$$\times I^{m,n}_{p_i,q_i:r}\left[yx^{-\mu}(t-x)^{-\nu} \left| \begin{matrix} (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] dx$$

$$= e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u}$$

$$\times I^{m+2,n}_{p_i+1,q_i+2:r}\left[yt^{-\mu-\nu} \left| \begin{matrix} (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i}, \\ (\rho+\zeta k, \mu), (\sigma+(\eta-1)k+u, \nu), \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (\rho+\sigma+(\zeta+\eta-1)k+u, \mu+\nu) \\ (b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i} \end{matrix} \right], \qquad (11)$$

provided

$$Re(\rho) - \mu \max_{1\le j\le n} [Re\{(a_j-1)/\alpha_j\}] > 0,$$

$$Re(\sigma) - \nu \max_{1\le j\le n} [Re\{(a_j-1)/\alpha_j\}] > 0,$$

along with the sets of conditions (i) to (iii) given with I_1 and $f(k)$ is given by (10).

Third Integral

$$I_3 \equiv \int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-xz} {}_2F_1(\alpha, \beta; \gamma; ax^{\zeta}(t-x)^{\eta})$$

$$\times I^{m,n}_{p_i,q_i:r}\left[yx^{\mu}(t-x)^{-\nu} \left| \begin{matrix} (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] dx$$

$$= e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u}$$

$$\times I^{m+1,n+1}_{p_i+1,q_i+2:r}\left[yt^{\mu-\nu} \left| \begin{matrix} (1-\rho-\zeta k, \mu), \\ (\sigma+(\eta-1)k+u, \nu), (b_j, \beta_j)_{1,m}; \end{matrix} \right. \right.$$

$$\left.\begin{matrix} (a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_{ji},\beta_{ji})_{m+1,q_i},(1-\rho-\sigma-(\zeta+\eta-1)k-u,\mu-\nu) \end{matrix}\right], \tag{12}$$

provided $\mu > 0, \nu \geq 0$ such that $\mu - \nu \geq 0$,

$$Re(\rho) + \mu \min_{1\leq j\leq m}[Re(b_j/\beta_j)] > 0,$$

$$Re(\sigma) - \nu \max_{1\leq j\leq n}[Re\{(a_j-1)/\alpha_j\}] > 0,$$

along with the sets of conditions (i) to (iii) given with I_1 and $f(k)$ is given by (10).

Fourth Integral

$$\begin{aligned} I_4 \equiv & \int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xz}{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta}) \\ & \times I^{m,n}_{p_i,q_i:r}\left[yx^{\mu}(t-x)^{-\nu}\middle| \begin{matrix} (a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix}\right]dx \\ = & e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u} \\ & \times I^{m+1,n+1}_{p_i+2,q_i+1:r}\left[yt^{\mu-\nu}\middle| \begin{matrix} (1-\rho-\zeta k,\mu),(a_j,\alpha_j)_{1,n}; \\ (\sigma+(\eta-1)k+u,\nu), \end{matrix}\right. \\ & \left.\begin{matrix} (a_{ji},\alpha_{ji})_{n+1,p_i},(\rho+\sigma+(\zeta+\eta-1)k+u,\nu-\mu) \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix}\right], \end{aligned} \tag{13}$$

provided $\mu \geq 0, \nu > 0$ such that $\nu - \mu \geq 0$,

$$Re(\rho) - \mu \max_{1\leq j\leq n}[Re\{(a_j-1)/\alpha_j\}] > 0,$$

$$Re(\sigma) + \nu \min_{1\leq j\leq m}[Re(b_j/\beta_j)] > 0,$$

along with the sets of conditions (i) to (iii) given with I_1 and $f(k)$ is given by (10).

Fifth Integral

$$\begin{aligned} I_5 \equiv & \int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xz}{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta}) \\ & \times I^{m,n}_{p_i,q_i:r}\left[yx^{-\mu}(t-x)^{\nu}\middle| \begin{matrix} (a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix}\right]dx \\ = & e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u} \\ & \times I^{m+1,n+1}_{p_i+2,q_i+1:r}\left[yt^{-\mu+\nu}\middle| \begin{matrix} (1-\sigma-(\eta-1)k-u,\nu),(a_j,\alpha_j)_{1,n}; \\ (\rho+\zeta k,\mu), \end{matrix}\right. \end{aligned}$$

$$\left.\begin{matrix}(a_{ji},\alpha_{ji})_{n+1,p_i},(\rho+\sigma+(\zeta+\eta-1)k+u,\mu-\nu)\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right], \tag{14}$$

provided $\mu > 0, \nu \geq 0$ such that $\mu - \nu \geq 0$,

$$Re(\rho) + \mu \min_{1\leq j\leq m}[Re(b_j/\beta_j)] > 0,$$

$$Re(\sigma) - \nu \max_{1\leq j\leq n}[Re\{(a_j-1)/\alpha_j\}] > 0,$$

along with the sets of conditions (i) to (iii) given with I_1 and $f(k)$ is given by (10).

Sixth Integral

$$\begin{aligned}I_6 &\equiv \int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xz}{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta})\\&\quad\times I^{m,n}_{p_i,q_i:r}\left[yx^{-\mu}(t-x)^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx\\&= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u}\\&\quad\times I^{m+1,n+1}_{p_i+1,q_i+2:r}\left[yt^{-\mu+\nu}\left|\begin{matrix}(1-\sigma-(\eta-1)k-u,\nu),(a_j,\alpha_j)_{1,n};\\(\rho+\zeta k,\mu),(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},\end{matrix}\right.\right.\end{aligned}$$

$$\left.\begin{matrix}(a_{ji},\alpha_{ji})_{n+1,p_i}\\(1-\rho-\sigma-(\zeta+\eta-1)k-u,\nu-\mu)\end{matrix}\right], \tag{15}$$

provided $\mu \geq 0, \nu > 0$ such that $\nu - \mu \geq 0$,

$$Re(\rho) - \mu \max_{1\leq j\leq n}[Re\{(a_j-1)/\alpha_j\}] > 0,$$

$$Re(\sigma) + \nu \min_{1\leq j\leq m}[Re(b_j/\beta_j)] > 0,$$

along with the sets of conditions (i) to (iii) given with I_1 and $f(k)$ is given by (10).

The integrals (11) to (15) can be proved on lines similar to those of integral (9).

Seventh Integral

$$I_7 \equiv \int_0^\infty x^{\eta-1} e^{ax} {}_2F_1(\alpha,\beta;\gamma;ax^\rho) I^{m_1,n_1}_{p_i,q_i:r}\left[zx^\sigma \left| \begin{matrix} (a_j,\alpha_j)_{1,n_1};(a_{ji},\alpha_{ji})_{n_1+1,p_i} \\ (b_j,\beta_j)_{1,m_1};(b_{ji},\beta_{ji})_{m_1+1,q_i} \end{matrix}\right.\right]$$
$$\times H^{m,n}_{p,q}\left[wx \left| \begin{matrix} (c_j,\gamma_j)_{1,n};(c_j,\gamma_j)_{n+1,p} \\ (d_j,\delta_j)_{1,m};(d_j,\delta_j)_{m+1,q} \end{matrix}\right.\right] dx$$
$$= w^{-\eta} \sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k) \frac{a^{u-k}}{(u-k)!} w^{-(\rho-1)k-u}$$
$$\times I^{m_1+n,n_1+m}_{p_i+q,q_i+p:r}\left[zw^{-\sigma} \left| \begin{matrix} (a_j,\alpha_j)_{1,n_1},(1-d_j-(\eta+(\rho-1)k+u)\delta_j,\sigma\delta_j)_{1,m}; \\ (b_j,\beta_j)_{1,m_1},(1-c_j-(\eta+(\rho-1)k+u)\gamma_j,\sigma\gamma_j)_{1,n}; \end{matrix}\right.\right.$$
$$\left.\begin{matrix} (a_{ji},\alpha_{ji})_{n_1+1,p_i},(1-d_j-(\eta+(\rho-1)k+u)\delta_j,\sigma\delta_j)_{m+1,q} \\ (b_{ji},\beta_{ji})_{m_1+1,q_i},(1-c_j-(\eta+(\rho-1)k+u)\gamma_j,\sigma\gamma_j)_{n+1,p} \end{matrix}\right], \tag{16}$$

where,

$$f(k) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}, \tag{17}$$

provided

(i) $\lambda > 0, |\arg z| < \frac{1}{2}\pi\lambda$

(ii) $\lambda \geq 0, |\arg z| \leq \frac{1}{2}\pi\lambda, Re(\mu+1) < 0$

(iii) $\lambda_1 > 0, |\arg w| < \frac{1}{2}\pi\lambda_1$

(iv) $\lambda_1 \geq 0, |\arg w| \leq \frac{1}{2}\pi\lambda_1, Re(\mu_1+1) < 0$

(v) $\sigma > 0, -\sigma \min_{1\leq j\leq m_1}[Re(b_j/\beta_j)] - \min_{1\leq j\leq m}[Re(d_j/\delta_j)] < Re(\eta) < \sigma < \min_{1\leq j\leq n_1}[Re\{(1-a_j)/\alpha_j\}] + \min_{1\leq j\leq n}[Re\{(1-c_j)/\gamma_j\}]$

and where,

$$\lambda = \sum_{j=1}^{n_1}\alpha_j + \sum_{j=1}^{m_1}\beta_j - \max_{1\leq i\leq r}\left[\sum_{j=n_1+1}^{p_i}\alpha_{ji} + \sum_{j=m_1+1}^{q_i}\beta_{ji}\right]$$
$$\mu = \sum_{j=1}^{m_1} b_j - \sum_{j=1}^{n_1} a_j - \min_{1\leq i\leq r}\left[\sum_{j=n_1+1}^{p_i} a_{ji} - \sum_{j=m_1+1}^{q_i} b_{ji} + \frac{p_i}{2} - \frac{q_i}{2}\right]$$
$$\lambda_1 = \sum_{j=1}^{m}\delta_j + \sum_{j=1}^{n}\gamma_j - \sum_{j=m+1}^{q}\delta_j - \sum_{j=n+1}^{p}\gamma_j$$
$$\mu_1 = \frac{1}{2}(p-q) + \sum_{j=1}^{q} d_j - \sum_{j=1}^{p} c_j$$

Proof: We replace e^{ax} by $\sum_{u=0}^{\infty} \frac{a^u x^u}{u!}$ and express the hypergeometric function and the I-function with the help of (1) and (5) respectively, to obtain

$$I_7 = \int_0^\infty x^{\eta-1} \sum_{u=0}^{\infty} \frac{a^u x^u}{u!} \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k} \frac{a^k x^{\rho k}}{k!} \frac{1}{2\pi i} \int_L \phi(\xi) z^\xi x^{\sigma\xi}$$
$$\times H_{p,q}^{m,n}\left[wx \middle| \begin{matrix} (c_j,\gamma_j)_{1,n}; (c_j,\gamma_j)_{n+1,p} \\ (d_j,\delta_j)_{1,m}; (d_j,\delta_j)_{m+1,q} \end{matrix} \right] d\xi dx$$
$$= \int_0^\infty x^{\eta-1} \sum_{u=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k} \frac{a^k x^{\rho k+u}}{k!} \frac{a^u}{u!} \frac{1}{2\pi i} \int_L \phi(\xi) z^\xi x^{\sigma\xi}$$
$$\times H_{p,q}^{m,n}\left[wx \middle| \begin{matrix} (c_j,\gamma_j)_{1,n}; (c_j,\gamma_j)_{n+1,p} \\ (d_j,\delta_j)_{1,m}; (d_j,\delta_j)_{m+1,q} \end{matrix} \right] d\xi dx$$

Now by the use of (8), the above result reduces to

$$I_7 = \int_0^\infty x^{\eta-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k} \frac{a^k x^{\rho k+u-k}}{k!} \frac{a^{u-k}}{(u-k)!} \frac{1}{2\pi i} \int_L \phi(\xi) z^\xi x^{\sigma\xi}$$
$$\times H_{p,q}^{m,n}\left[wx \middle| \begin{matrix} (c_j,\gamma_j)_{1,n}; (c_j,\gamma_j)_{n+1,p} \\ (d_j,\delta_j)_{1,m}; (d_j,\delta_j)_{m+1,q} \end{matrix} \right] d\xi dx$$

Interchanging the order of integration and summation, we obtain

$$I_7 = \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{a^{u-k}}{(u-k)!} \frac{1}{2\pi i} \int_L \phi(\xi) z^\xi \left\{ \int_0^\infty x^{\eta+(\rho-1)k+u+\sigma\xi-1} \right.$$
$$\left. \times H_{p,q}^{m,n}\left[wx \middle| \begin{matrix} (c_j,\gamma_j)_{1,n}; (c_j,\gamma_j)_{n+1,p} \\ (d_j,\delta_j)_{1,m}; (d_j,\delta_j)_{m+1,q} \end{matrix} \right] dx \right\} d\xi,$$

where $f(k)$ is given by (17).
Now we use the Mellin transform of H-function by virtue of (7), so that

$$I_7 = \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{a^{u-k}}{(u-k)!} \frac{1}{2\pi i} \int_L \phi(\xi) z^\xi w^{-(\eta+(\rho-1)k+u+\sigma\xi)}$$
$$\times \frac{\prod_{j=1}^{m} \Gamma(d_j + \delta_j(\eta+(\rho-1)k+u+\sigma\xi))}{\prod_{j=m+1}^{q} \Gamma(1-d_j-\delta_j(\eta+(\rho-1)k+u+\sigma\xi))}$$
$$\times \frac{\prod_{j=1}^{n} \Gamma(1-c_j-\gamma_j(\eta+(\rho-1)k+u+\sigma\xi))}{\prod_{j=n+1}^{p} \Gamma(c_j+\gamma_j(\eta+(\rho-1)k+u+\sigma\xi))} d\xi$$
$$= w^{-\eta} \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{a^{u-k}}{(u-k)!} w^{-(\rho-1)k-u} \frac{1}{2\pi i} \int_L \phi(\xi)$$
$$\times \frac{\prod_{j=1}^{m} \Gamma(d_j + (\eta+(\rho-1)k+u)\delta_j + \sigma\delta_j\xi)}{\prod_{j=m+1}^{q} \Gamma(1-d_j-(\eta+(\rho-1)k+u)\delta_j - \sigma\delta_j\xi)}$$
$$\times \frac{\prod_{j=1}^{n} \Gamma(1-c_j-(\eta+(\rho-1)k+u)\gamma_j - \sigma\gamma_j\xi)}{\prod_{j=n+1}^{p} \Gamma(c_j+(\eta+(\rho-1)k+u)\gamma_j + \sigma\gamma_j\xi)} z^\xi w^{-\sigma\xi} d\xi$$

Finally, interpreting the contour integral by virtue of (5), we obtain

$$I_7 = w^{-\eta}\sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{a^{u-k}}{(u-k)!}w^{-(\rho-1)k-u}$$
$$\times I^{m_1+n,n_1+m}_{p_i+q,q_i+p:r}\left[zw^{-\sigma}\left|\begin{array}{l}(a_j,\alpha_j)_{1,n_1},(1-d_j-(\eta+(\rho-1)k+u)\delta_j,\sigma\delta_j)_{1,m};\\(b_j,\beta_j)_{1,m_1},(1-c_j-(\eta+(\rho-1)k+u)\gamma_j,\sigma\gamma_j)_{1,n};\end{array}\right.\right.$$
$$\left.\begin{array}{l}(a_{ji},\alpha_{ji})_{n_1+1,p_i},(1-d_j-(\eta+(\rho-1)k+u)\delta_j,\sigma\delta_j)_{m+1,q}\\(b_{ji},\beta_{ji})_{m_1+1,q_i},(1-c_j-(\eta+(\rho-1)k+u)\gamma_j,\sigma\gamma_j)_{n+1,p}\end{array}\right].$$

4 Particular Cases

Putting $r=1$, $t=1$ and $\eta=0$ in (9), (11), (12), (13), (14) and (15) the following known as well as new results may be realised:

(i) Integral (9) leads to the known result [1, p. 246, eq. (2.2)]:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-xz}{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta})H^{m,n}_{p,q}\left[yx^{\mu}(1-x)^{\nu}\left|\begin{array}{l}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\right.\right]dx$$
$$=e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!}H^{m,n+2}_{p+2,q+1}\left[y\left|\begin{array}{l}(1-\rho-\zeta k,\mu),\\(b_j,\beta_j)_{1,q},\end{array}\right.\right.$$
$$\left.\begin{array}{l}(1-\sigma+k-u,\nu),(a_j,\alpha_j)_{1,p}\\(1-\rho-\sigma-(\zeta-1)k-u,\mu+\nu)\end{array}\right], \qquad (18)$$

where,

$$f(k)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}.$$

(ii) Integral (11) leads to the another known result [1, p. 248, eq. (3.1)]:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-xz}{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta})H^{m,n}_{p,q}\left[yx^{-\mu}(1-x)^{-\nu}\left|\begin{array}{l}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\right.\right]dx$$
$$=e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!}H^{m+2,n}_{p+1,q+2}\left[y\left|\begin{array}{l}(a_j,\alpha_j)_{1,p},\\(\rho+\zeta k,\mu),\end{array}\right.\right.$$
$$\left.\begin{array}{l}(\rho+\sigma+(\zeta-1)k+u,\mu+\nu)\\(\sigma-k+u,\nu),(b_j,\beta_j)_{1,q}\end{array}\right], \qquad (19)$$

where,

$$f(k)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}.$$

(iii) Integral (12) reduces to the known result [1, p. 248, eq. (3.2)]:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-xz}{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta})H_{p,q}^{m,n}\left[yx^{\mu}(1-x)^{-\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]dx$$
$$=e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}H_{p+1,q+2}^{m+1,n+1}\left[y\left|\begin{matrix}(1-\rho-\zeta k,\mu),\\(\sigma-k+u,\nu),(b_j,\beta_j)_{1,q},\end{matrix}\right.\right.$$
$$\left.\begin{matrix}(a_j,\alpha_j)_{1,p}\\(1-\rho-\sigma-(\zeta-1)k-u,\mu-\nu)\end{matrix}\right], \tag{20}$$

where,

$$f(k)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}.$$

(iv) Integral (13) leads to the known result [1, p. 249, eq. (3.3)]:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-xz}{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta})H_{p,q}^{m,n}\left[yx^{\mu}(1-x)^{-\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]dx$$
$$=e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}H_{p+2,q+1}^{m+1,n+1}\left[y\left|\begin{matrix}(1-\rho-\zeta k,\mu),(a_j,\alpha_j)_{1,p},\\(\sigma-k+u,\nu),\end{matrix}\right.\right.$$
$$\left.\begin{matrix}(\rho+\sigma+(\zeta-1)k+u,\nu-\mu)\\(b_j,\beta_j)_{1,q}\end{matrix}\right], \tag{21}$$

where,

$$f(k)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}.$$

(v) Integral (14) reduces to the result:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-xz}{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta})H_{p,q}^{m,n}\left[yx^{-\mu}(1-x)^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]dx$$
$$=e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}H_{p+2,q+1}^{m+1,n+1}\left[y\left|\begin{matrix}(1-\sigma+k-u,\nu),(a_j,\alpha_j)_{1,p},\\(\rho+\zeta k,\mu),\end{matrix}\right.\right.$$
$$\left.\begin{matrix}(\rho+\sigma+(\zeta-1)k+u,\mu-\nu)\\(b_j,\beta_j)_{1,q}\end{matrix}\right], \tag{22}$$

where,

$$f(k)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}.$$

(vi) Integral (15) leads to the result:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-xz}{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta})H_{p,q}^{m,n}\left[yx^{-\mu}(1-x)^{\nu}\middle| \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]dx$$

$$=e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}H_{p+1,q+2}^{m+1,n+1}\left[y\middle| \begin{matrix}(1-\sigma+k-u,\nu),\\(\rho+\zeta k,\mu),(b_j,\beta_j)_{1,q},\end{matrix}\right.$$

$$\left.\begin{matrix}(a_j,\alpha_j)_{1,p}\\(1-\rho-\sigma-(\zeta-1)k-u,\nu-\mu)\end{matrix}\right], \tag{23}$$

where,

$$f(k)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}.$$

(vii) Putting $a=0$ in (16), the exponential function e^{ax} and the hypergeometric function reduces to unity and consequently it leads to a result by V.P. Saxena [5, p. 66, eq. (4.5.1)]:

$$\int_0^{\infty} x^{\eta-1}I_{p_i,q_i:r}^{m_1,n_1}\left[zx^{\sigma}\middle| \begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right]$$

$$\times H_{p,q}^{m,n}\left[wx\middle| \begin{matrix}(c_j,\gamma_j)_{1,n};(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m};(d_j,\delta_j)_{m+1,q}\end{matrix}\right]dx$$

$$=w^{-\eta}I_{p_i+q,q_i+p:r}^{m_1+n,n_1+m}\left[zw^{-\sigma}\middle| \begin{matrix}(a_j,\alpha_j)_{1,n_1},(1-d_j-\eta\delta_j,\sigma\delta_j)_{1,m};\\(b_j,\beta_j)_{1,m_1},(1-c_j-\eta\gamma_j,\sigma\gamma_j)_{1,n};\end{matrix}\right.$$

$$\left.\begin{matrix}(a_{ji},\alpha_{ji})_{n_1+1,p_i},(1-d_j-\eta\delta_j,\sigma\delta_j)_{m+1,q}\\(b_{ji},\beta_{ji})_{m_1+1,q_i},(1-c_j-\eta\gamma_j,\sigma\gamma_j)_{n+1,p}\end{matrix}\right]. \tag{24}$$

5 Conclusion

The I-function, presented in this paper, is quite basic in nature. Therefore, on specializing the parameters of this function, we may obtain various other special functions such as Fox's H-function, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modified Bessel function, Whittaker function, exponential function, binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

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