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Measure Preserving Isomorphisms

M. Gheytaran Marzrood

Department of Mathematics
University of Tabriz, 5166617766, Tabriz, Iran
E-mail: m_gheytaran91@ms.tabrizu.ac.ir

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Abstract

In this note we study the relationship between the isomorphic and unitarily isomorphic measure preserving mappings. Also, we show that the concept of zero-product preserving mappings and unitarily isomorphic mappings are equivalent.

Keywords: *Measure preserving transformation, unitarily equivalent, isomorphic, unitarily isomorphic, zero-product.*

1 Introduction

Let (X, Σ, μ) be a probability measure space and let $\mathcal{A}$ be a sub-sigma algebra of Σ. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. Let $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ, that is, φ is non-singular. It is assumed that the Radon-Nikodym derivative $h_\varphi = d\mu \circ \varphi^{-1}/d\mu$ is finite-valued. In the setting of L^p-spaces the so called conditional expectation operator $E^{\varphi^{-1}(\Sigma)}$ with respect to $\varphi^{-1}(\Sigma)$ plays an important role. If there is no possibility of confusion, for each $0 \leq f \in L^0(\Sigma)$ or $f \in L^p(\Sigma)$, we write $E_\varphi f$ in place of $E^{\varphi^{-1}(\Sigma)} f$. For a deep study of conditional expectation operator we refer the reader to the monograph [7]. For a finite valued function $u \in L^0(\Sigma)$, the weighted composition operator W on $L^2(\Sigma)$ induced by u and non-singular measurable function φ is given by $W = M_u \circ C_\varphi$ where M_u is a multiplication operator and C_φ is a composition operator on

$L^2(\Sigma)$ defined by $M_u f = uf$ and $C_\varphi f = f\circ\varphi$, respectively. It is a classical fact that $W \in B(L^2(\Sigma))$, the algebra of all bounded linear operators on $L^2(\Sigma)$, if and only if $J := hE(|u|^2)\circ\varphi^{-1} \in L^\infty(\Sigma)$ and $W \in B(L^\infty(\Sigma))$ if and only if $u \in L^\infty(\Sigma)$ (see [3]).

We recall that the measure preserving transformations $\varphi_1, \varphi_2 : X \to X$ are said to be isomorphic if there is a bi-measurable, measure preserving bijection $\phi : X \to X$ such that $\varphi_1 \circ \phi = \phi \circ \varphi_2$ (see [5]). If ϕ is not necessarily measure preserving, we say that φ_1 and φ_2 are pseudo-isomorphic (see [8]). Also, the bounded linear operators C_{φ_1} and C_{φ_2} are said to be unitarily equivalent if there is a unitary transformation U such that $UC_{\varphi_1} = C_{\varphi_2}U$(in this case φ_1 and φ_2 are not necessarily measure preserving). Note that, if φ_1 and φ_2 are isomorphic then $\|C_{\varphi_1}\| = \|C_{\varphi_2}\| = 1$ and $\varphi_1 \circ \phi = \phi \circ \varphi_2$. Hence for each $f \in L^2(\Sigma)$, $C_\phi C_{\varphi_1} f = f \circ \varphi_1 \circ \phi = f \circ \phi \circ \varphi_2 = C_{\varphi_2} C_\phi f$. Also, since $h_\phi = 1$ then $C_\phi^* f = f \circ \phi^{-1} = C_\phi^{-1} f$, and so φ_1 and φ_2 are unitarily equivalent. Hence, isomorphic transformations are unitarily equivalent. For a fix measure preserving mapping $\varphi : X \to X$, define

$$\mathcal{W}_\varphi = \{uC_\varphi : E_\varphi(|u|^2)\circ\varphi^{-1} \in L^\infty(X)\},$$

$$\mathcal{K}_\varphi = \{u \in L^0(\Sigma) : uC_\varphi \in \mathcal{W}_\varphi\}.$$

For $u \in \mathcal{K}_\varphi$, put $\|u\|_{\mathcal{K}_\varphi} = \|E_\varphi(|u|^2)\circ\varphi^{-1}\|^{1/2}$. It is easy to show that $(\mathcal{K}_\varphi, \|\cdot\|_{\mathcal{K}_\varphi})$ is a norm space ([4]). Let $\Lambda : \mathcal{A} \to \mathcal{B}$ be an additive surjective map between some operator algebras. The mapping Λ is said to be a zero-product preserving if $\Lambda(A)\Lambda(B) = 0$ whenever $AB = 0$ (see [9]). In this note we study the relationship between the isomorphic (pseudo-isomorphic), unitarily isomorphic measure preserving and zero-product preserving mappings.

2 Main Results

Proposition 2.1 *$\mathcal{W}_\varphi$ is a closed subspace of $\mathcal{B}(L^2(\Sigma))$.*

Proof. Clearly $\mathcal{W}_\varphi$ is a subspace of $\mathcal{B}(L^2(\Sigma))$. Let$\{u_nC_\varphi\} \subseteq \mathcal{W}_\varphi$ and $u_nC_\varphi \to T$ for some $T \in \mathcal{B}(L^2(\Sigma))$. We show that $T \in \mathcal{W}_\varphi$. Since $u_n = u_nC_\varphi(1) \to T(1) =: u$, then for every $f \in L^2(\Sigma)$ we have

$$\|u_nC_\varphi(f) - uC_\varphi(f)\| \le \|u_n - u\|\|C_\varphi\|\|f\| \le \|u_n - u\|\|f\|.$$

Thus $T = uC_\varphi \in \mathcal{W}_\varphi$, and so $\mathcal{W}_\varphi \subseteq \mathcal{B}(L^2(\Sigma))$ is close.

Proposition 2.2 $(\mathcal{K}_\varphi, \|\cdot\|_{\mathcal{K}_\varphi})$ *is a Banach space. In particular, $\mathcal{K}_\varphi$ is an order ideal.*

Proof. Define $\Lambda : \mathcal{K}_\varphi \longrightarrow \mathcal{W}_\varphi$ by $\Lambda(u) = uC_\varphi$. Then for each $u \in \mathcal{K}_\varphi$, $\|\Lambda(u)\|^2 = \|E_\varphi(|u|^2)\circ\varphi^{-1}\| = \|u\|^2_{\mathcal{K}_\varphi}$. Hence Λ is an isometry isomorphism and so, by Proposition 2.1, $\mathcal{K}_\varphi$ is also a Banach space. Now, if $u_2 \in \mathcal{K}_\varphi$ and $u_1 \leq u_2$, then $E_\varphi(|u_1|^2)\circ\varphi^{-1} \leq E_\varphi(|u_2|^2)\circ\varphi^{-1} < \infty$, and hence $u_1 \in \mathcal{K}_\varphi$.
The measure preserving transformations φ_1 and φ_2 are said to be unitarily isomorphic if there is a unitary transformation V on $L^2(\Sigma)$ such that $V\mathcal{W}_{\varphi_1} = \mathcal{W}_{\varphi_2}V$ (see [1, 2, 5]).

Theorem 2.3 *If φ_1 and φ_2 are isomorphic, then they are unitarily isomorphic.*

Proof. Let $uC_{\varphi_1} \in \mathcal{W}_{\varphi_1}$. Since $\varphi_1\circ\phi = \phi\circ\varphi_2$ and ϕ is a bijection, bi-measurable and measure preserving transformation, then C_ϕ is a unitary operator and for each $f \in L^2(\Sigma)$,

$$C_\phi(uC_{\varphi_1})(f) = (u\circ\phi)(f\circ\varphi_1\circ\phi) = (u\circ\phi)(f\circ\phi\circ\varphi_2) = ((u\circ\phi)C_{\varphi_2})C_\phi f.$$

Now, let $uC_{\varphi_1} \in \mathcal{W}_{\varphi_1}$. Then $\|E_{\varphi_1}(|u|^2)\circ\varphi_1^{-1}\| < \infty$. Since $E_\phi = I$ and $\|C_{\phi^{-1}}\| = h_{\phi^{-1}} = 1$, then for each $f \in L^2(\Sigma)$ we get that

$$\|(u\circ\phi)C_{\varphi_2}(f)\|^2 = \int |u|^2|f|^2\circ\varphi_2\circ\phi^{-1}d\mu = \int |u|^2|f|^2\circ\phi^{-1}\circ\varphi_1 d\mu$$

$$= \int E_{\varphi_1}(|u|^2)\circ\varphi_1^{-1}|f|^2\circ\phi^{-1}d\mu \leq \|E_{\varphi_1}(|u|^2\circ\varphi_1^{-1})\|_\infty\|C_{\phi^{-1}}\|^2\|f\|^2 < \infty.$$

Hence $(u\circ\phi)C_{\varphi_2}$ is in $\mathcal{W}_{\varphi_2}$ for each u in $\mathcal{K}_{\varphi_1}$, and consequently $C_\phi\mathcal{W}_{\varphi_1} \subseteq \mathcal{W}_{\varphi_2}C_\phi$. Now, if v is in $\mathcal{K}_{\varphi_2}$ then $v\circ\phi^{-1}$ is in $\mathcal{K}_{\varphi_1}$, thus $v = (v\circ\phi^{-1})\circ\phi$ is in $\mathcal{K}_{\varphi_2}$. It follows that each element of $\mathcal{W}_{\varphi_2}$ can be written as $(u\circ\phi)C_{\varphi_2}$ for some u in $\mathcal{K}_{\varphi_1}$. Thus $\mathcal{W}_{\varphi_2}C_\phi \subseteq C_\phi\mathcal{W}_{\varphi_1}$, and so φ_1 and φ_2 are unitarily isomorphic.

We recall that the measure preserving transformations φ_1, φ_2 are said to be pseudo-isomorphic if there is a bi-measurable bijection ϕ such that $\varphi_1\circ\phi = \phi\circ\varphi_2$. Note that ϕ is not necessarily measure preserving (see[8]). In [5] A. Lambert proved that unitarily isomorphic implies pseudo isomorphic. In the following theorem we give a simple proof for the converse of this fact.

Theorem 2.4 *If the measure preserving transformations φ_1 and φ_2 are pseudo-isomorphic, then they are unitarily isomorphic.*

Proof. Let $\varphi_1\circ\phi = \phi\circ\varphi_2$, where ϕ is a bi-measurable bijection. Put $h = \frac{d\mu\circ\phi^{-1}}{d\mu}$ and $w = \left(\frac{1}{\sqrt{h\circ\phi}}\right)$. Define $V : L^2(\Sigma) \to L^2(\Sigma)$ by $Vf = w(f\circ\phi)$. Then for each $f \in L^2(\Sigma)$ we have

$$\|Vf\|^2 = \int_X \frac{1}{h\circ\phi}|f|^2\circ\phi d\mu = \int_X \frac{1}{h}|f|^2\frac{d\mu\circ\phi^{-1}}{d\mu} = \int_X |f|^2 d\mu = \|f\|^2$$

Hence V is an isometry. Now, for each $g \in L^2(\Sigma)$, put $f = (w\circ\phi^{-1})^{-1}g\circ\phi^{-1} = \sqrt{h}g\circ\phi^{-1}$. Then $Vf = g$. Thus V is unitary. Now we show $V(uC_{\varphi_1}) = (u\circ\phi)C_{\varphi_2}V$, for any $u \in \mathcal{K}_{\varphi_1}$. Set $\upsilon = (\sqrt{\frac{h\circ\varphi_1}{h}}.u)\circ\phi$. Then $\upsilon \in \mathcal{K}_{\varphi_2}$, because

$$\begin{aligned} V(uC_{\varphi_1})V^{-1}g &= V(uC_{\varphi_1})((w\circ\phi^{-1})^{-1}g\circ\phi^{-1}) \\ &= \frac{1}{\sqrt{h\circ\phi}}(u\circ\phi)((w\circ\phi^{-1})^{-1}\circ\varphi_1\circ\phi)(g\circ\phi^{-1}\circ\varphi_1\circ\phi) \\ &= \frac{1}{\sqrt{h\circ\phi}}(u\circ\phi)(w\circ\varphi_2)^{-1}(g\circ\varphi_2) \\ &= w(w\circ\varphi_2)^{-1}(u\circ\phi)(g\circ\varphi_2) = \upsilon(g\circ\varphi_2) = \upsilon C_{\varphi_2}g, \end{aligned}$$

and

$$\begin{aligned} \|\upsilon C_{\varphi_2}f\|^2 &= \int_X (\frac{h\circ\varphi_1}{h}|u|^2)\circ\phi(|f|^2\circ\varphi_2)d\mu \\ &= \int_X \frac{h\circ\varphi_1\circ\phi}{h\circ\phi}(|u|^2\circ\phi)(|f|^2\circ\phi^{-1}\circ\varphi_1\circ\phi)d\mu \\ &= \int_X \frac{h\circ\varphi_1}{h}|u|^2(|f|^2\circ\phi^{-1}\circ\varphi_1)d\mu\circ\phi^{-1} \\ &= \int_X (h\circ\varphi_1)E_{\varphi_1}(|u|^2)(|f|^2\circ\phi^{-1}\circ\varphi_1)d\mu \\ &= \int_X hE_{\varphi_1}(|u|^2)\circ\varphi_1^{-1}(|f|^2\circ\phi^{-1})d\mu \\ &\le \|E_{\varphi_1}(|u|^2)\circ\varphi_1^{-1}\|_\infty \int_X h|f|^2\circ\phi^{-1}d\mu \\ &\le \|E_{\varphi_1}(|u|^2)\circ\varphi_1^{-1}\|_\infty\|f\|^2 < \infty. \end{aligned}$$

Thus $\|\upsilon C_{\varphi_2}\| < \infty$, and so $V\mathcal{W}_{\varphi_1} = \mathcal{W}_{\varphi_2}V$.

Corollary 2.5 *Let $\Lambda : \mathcal{W}_{\varphi_1} \longrightarrow \mathcal{W}_{\varphi_2}$ be linear and surjection map.Then Λ zero-prouduct preserving if and only if φ_1 and φ_2 are pseudo-isomorphic.*

Proof. Let Λ be a zero-product preserving map. Then there exists an invertible bounded linear operator V such that $\Lambda(uC_{\varphi_1}) = V(uC_{\varphi_1})V^{-1}$, by [6]. Since Λ is surjection so $\mathcal{W}_{\varphi_2} = \Lambda(\mathcal{W}_{\varphi_1}) = V(\mathcal{W}_{\varphi_1})V^{-1}$. Consequently $V\mathcal{W}_{\varphi_1} = \mathcal{W}_{\varphi_2}V$. It follows that φ_1 and φ_2 are pseudo-isomorphic.
Conversely, assume that φ_1 and φ_2 are pseudo-isomorphic. So there is a unitary transformation V on $L^2(\Sigma)$ such that $V\mathcal{W}_{\varphi_1} = \mathcal{W}_{\varphi_2}V$. Now define $\Lambda : \mathcal{W}_{\varphi_1} \to \mathcal{W}_{\varphi_2}$ by $\Lambda(uC_{\varphi_1}) = V(uC_{\varphi_1})V^{-1}$. Thus, if $(u_1C_{\varphi_1})(u_2C_{\varphi_1}) = 0$, we get that

$$\Lambda(u_1C_{\varphi_1})\Lambda(u_1C_{\varphi_1}) = (V(u_1C_{\varphi_1})V^{-1})(V(u_2C_{\varphi_1})V^{-1}) = 0$$

and hence Λ is a zero-product preserving map.

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