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Common Fixed Point for a Pair of Self-Maps Satisfying the Property E.A.

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Abstract

Using the notion of contractive modulus, we obtain a common fixed point for a pair of weakly compatible self-maps on a metric space, which satisfy the property E.A. Eventually our result is a modest generalization of an earlier result of the author. We also show that a common fixed point can be obtained from this result through the notions of compatibility and reciprocal continuity. In the linear setting of our second result, we get those of Rangamma et al as particular cases.

Keywords: *Complete metric space, compatible and weakly self-maps, reciprocal continuity and common fixed point*

1 Introduction

Let X denote a metric space endowed with metric d. If $x \in X$ and T is a self-map on X, we write Tx for the image of x under T, $T(X)$ for the range of T, and TS for the composition of self-maps S and T.

As a generalization for *commuting* maps, we have

Definition 1.1 *(Gerald Jungck, [2])***:** *Self-maps S and T on X are compatible if*

$$\lim_{n\to\infty} d\,(STx_n, TSx_n) = 0 \tag{1}$$

whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t \text{ for some } t \in X. \tag{2}$$

If there exists no sequence $\langle x_n \rangle_{n=1}^{\infty}$ *in X such that (2) holds good, the condition of compatibility is vacuously satisfied, and S and T are vacuously compatible.*
Various types of compatibility were derived by altering the condition (1).

To mention a few, we have

Definition 1.2 [4] *Self-maps S and T on X are compatible of type (A) if*

$$\lim_{n\to\infty} d\,(STx_n, TTx_n) = 0 \text{ and } \lim_{n\to\infty} d\,(TSx_n, SSx_n) = 0, \tag{3-a}$$

whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is with the choice (2).

Definition 1.3 [7] *Self-maps S and T on X are compatible of type (P) if*

$$\lim_{n\to\infty} d\,(SSx_n, TTx_n) = 0, \tag{3-b}$$

whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *satisfies the choice (2).*

Remark 1.4 Compatibility, compatibility of type (*A*), and compatibility of type (*P*) are equivalent, provided both *T* and *S* are continuous (Prop. 2.6, [7]).

Remark 1.5 Examples 3 and 4 of [6] and the Example of [9] respectively reveal that the notions of compatibility of type (*A*) and type (*P*) are independent of compatibility if *T* and *S* are not continuous.

In this paper, we assume that the compatibility and its types are *non-vacuous*.
Now taking $x_n = x$ for all *n*, all types of compatibility imply that
STx = *TSx* whenever $x \in X$ is such that *Tx* = *Sx*. (4)

This idea motivated Jungck [3] to define *weakly compatible* self-maps, which commute at coincidence points. Weakly compatible maps are also called *coincidentally commuting* or *partially commuting* [12]. A weakly compatible pair need not be compatible of either type ([3], [11] and [12]).

The following result was established in [8], for weakly compatible maps wherein $\phi:[0,\infty)\to[0,\infty)$ is a contractive modulus [14] such that $\phi(0)=0$ and $\phi(t)<t$ for $t>0$.

Theorem 1.6 *Let S and T be self-maps on X satisfying the inclusion*

$$T(X)\subset S(X) \tag{5}$$

$$d(Tx,Ty)\le\phi\left(\max\left\{d(Sx,Sy),d(Sx,Tx),d(Sy,Ty),\frac{1}{2}[d(Tx,Sy)+d(Ty,Sx]\right\}\right) \quad \textit{for all } x,y\in X, \tag{6}$$

where ϕ is non decreasing and upper semi continuous.
Suppose that one of the following conditions holds good:

(a) *S is onto and the space X is orbitally complete at some $x_0\in X$ in the sense that every Cauchy sub sequence of $\langle Sx_n\rangle_{n=1}^{\infty}$, where $Tx_{n-1}=Sx_n$ for all n, converges to some $z\in X$,*

(b) *S (X) or T (X) is orbitally complete at some $x_0\in X$. Then S and T have a coincidence point. Further if*

(c) *the pair (S, T) is weakly compatible, then S and T have a unique common fixed point.*

In this paper we obtain the conclusion of Theorem 1.6, by dropping the inclusion (5), weakening the contraction condition (6), and using the *property E. A.* (see below). Also we prove generalizations of theorems of [9] and [10] under certain conditions. These results were presented in National conference on Mathematics and its applications to Engineering, held at Vasavi College of Engineering, Hyderabad, Andhra Pradesh, India, in March 2008.

2 Notation and Main Results

We require the following notion due to Aamri and Moutawakil [1]:

Definition 2.1 *Self-maps S and T satisfy the property E. A. if there is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in X with the choice (2).*

Since non-compatibility implies the existence of the sequence $\langle x_n \rangle_{n=1}^{\infty}$ with the choice (2), the class of all pairs of self-maps with property E. A. is potentially wider than that of non compatible maps.

First we prove

Theorem 2.2 *Let S and T be self-maps on X satisfying the contraction condition*
$d(Tx, Ty) \le \phi(\max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Tx, Sy), d(Ty, Sx\})$
for all $x, y \in X$, (7)
where ϕ is upper semi continuous contractive modulus. Suppose that

(d) *the pair (S, T) satisfies the property E. A.*
(e) *S is onto.*

Then S and T have a coincidence point. Further if the condition (c) *of Theorem* 1.6 *holds, then S and T will have a unique common fixed point.*

Proof. By virtue of the property E. A., there exists a sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that

$$\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_n = z \text{ for some } z \in X. \tag{8}$$

Suppose that S is onto. Then we can find some $u \in X$ such that $Su = z$.
Then by (7), we see that

$$d(Tx_n, Tu) \le \phi(\max\{d(Sx_n, Su), d(Sx_n, Tx_n), d(Su, Tu), d(Tx_n, Su), d(Tu, Sx_n\})$$

Applying the limit as $n \to \infty$ in this, and using $Su = z$ and upper semi-continuity of ϕ, we get

$$d(z, Tu) \le \phi(\max\{d(z, z), d(z, z), d(z, Tu), d(z, z), d(Tu, z\}) = \phi(d(z, Tu))$$

or $d(z, Tu) = 0$ or $Tu = z$. Thus u is a coincidence point of S and T so that $Tu = Su$ and (c) imply that $STu = TSu$ or $Sz = Tz$, proving the first part of the theorem.

Again, by (7),
$$d(Tx_n, Tz) \le \phi(\max\{d(Sx_n, Sz), d(Sx_n, Tx_n), d(Sz, Tz), d(Tx_n, Sz), d(Tz, Sx_n\})$$

in which proceeding the limit as $n \to \infty$ and using the coincidence at z, we get $d(z, Tz) \le \phi(d(z, Tz))$ or $d(z, Tz) = 0$ or $Tz = z$.
Thus $Sz = Tz = z$.

The uniqueness of the common fixed point follows directly from the choice of ϕ and (8).

Remark 2.3 Inequality (6) implies (7) whenever ϕ is non-decreasing, and for any $x_0 \in X$, the inclusion (5) generates a sequence of points $\langle x_n \rangle_{n=1}^{\infty}$ in X with the choice (2). From the proof of Theorem 1.6, it follows that the orbit $\langle Sx_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence. Since X is *orbitally complete*, we can find a point z in X such that (8) holds. Thus the pair (S,T) satisfies the property E. A. Hence by Theorem 2.2, S and T will have a unique common fixed point. Thus Theorem 1.6 is special case of Theorem 2.2 when ϕ is non-decreasing.
We can also obtain a common fixed point by replacing the condition (c) with the compatibility, (e) with reciprocal continuity (see below), and (9) with (6) in Theorem 2.2.

Definition 2.4 [5] *Self-maps S and T on X form a reciprocally continuous pair if for any* $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *with the choice (2), we have*

$$\lim_{n\to\infty} TSx_n = Tt \text{ and } \lim_{n\to\infty} STx_n = St. \tag{9}$$

Any pair of continuous maps will obviously be a reciprocally continuous one. Similarly, in the setting of common fixed point theorems for compatible maps satisfying contractive type conditions, continuity of one of the self-maps is sufficient to ensure the reciprocal continuity of the pair [5]. However a pair of maps S and T may be reciprocally continuous without both the maps being continuous ([5] and [13]). On the other hand, a vacuously compatible pair is necessarily reciprocally continuous. However the notions of (non-vacuous) compatibility and reciprocal continuity are independent of each other [13].
We now have

Theorem 2.5 *Let S and T be self-maps on the space* X *satisfying the contraction condition* (6) *with* ϕ *non-decreasing and upper semi-continuous and the condition* (*d*) *of Theorem* 2.2. *Suppose that*
(f) *the pair* (S,T) *is r. c. and compatible.*
Then S and T have a unique common fixed point.

Proof. By virtue of the property E. A., one can find a sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that (8) holds.
The convergence in (8) and *r. c.* imply that $\lim_{n\to\infty} STx_n = Sz$ and $\lim_{n\to\infty} TSx_n = Tz$.
While (8) together with compatibility will give

$$\lim_{n\to\infty} d(Sz,Tz) = \lim_{n\to\infty} d(STx_n, TSx_n) = 0 \text{ so that } Sz = Tz.$$

The rest of the proof follows from that of Theorem 2.2.
Writing $\phi(t) = ct$ for all $t \geq 0$ in Theorem 2.5, in view of Remark 2.3, we get

Corollary 2.6 [10] *Let S and T be self-maps satisfying the inclusion* (5) *and*

$$d(Tx,Ty) \leq c.\max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{1}{2}[d(Tx,Sy)+d(Ty,Sx]\right\}$$

for all $x, y \in X$, (10)

where $0 < c < 1$ *and* (*f*) *of Theorem* 2.5.

Suppose that

(g) *for any* $x_0 \in X$, the orbit $\langle Sx_n \rangle_{n=1}^{\infty}$ *converges to some z in X. Then S and T have a unique common fixed point.*

Finally, we replace the condition (*e*) that the map *S* is onto with its continuity and take the compatibility of type *P* in place of compatibility in Theorem 2.5 to get the following result, for which the proof is omitted:

Theorem 2.7 *Let S and T be self-maps on X satisfying the condition* (6) *where* ϕ *is upper semi continuous and* (*d*) *of Theorem* 2.2. *Suppose that*

(h) *the pair* (S,T) *is compatible of type* (*P*) *and S is continuous.*

Then S and T have a unique common fixed point.

Writing $\phi(t) = ct$ for all $t \geq 0$ in Theorem 2.7, we get

Corollary 2.8 [9] *Let S and T be self-maps on X satisfying the inclusion* (5) *and the inequality* (10), *and the conditions* (g) *and* (h). *Then S and T have a unique common fixed point.*

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