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A Class of Two-Step Newton's Methods with Accelerated Third Order Convergence

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Abstract

In this work we propose an improvement to the popular Newton's method based on the contra-harmonic mean while using quadrature rule derived from a Ostrowski-Gräuss type inequality developed in [19]. The order of convergence of this method for solving non-linear equations which have simple roots is shown to be three. Computer Algebra Systems (CAS), such as MAPLE 18 package, can be used successfully to take over the lengthy and tedious computations in deriving the asymptotic error of convergence. Furthermore, numerical experiments are made to show the efficiency and robustness of the suggested method.

Keywords: *Iterative methods, mean-based Newton's methods, order of convergence, efficiency index, CAS.*

1 Introduction

Many problems in mathematics and in many branches of science and engineering involve finding a simple root of a single non-linear equation $f(x) = 0$ where $f : I \subseteq R \to R$, for an open interval I, is a scalar function. Since it is not always possible to obtain its exact solution by the usual algebraic process, the numerical iterative methods can approximate the solution to a predetermined level of accuracy. This solution can be determined as a fixed point of some *iteration function* ϕ by means of the one-step iteration method

$$x_{n+1} = \phi(x_n), \quad n = 0, 1, 2, \ldots,$$

where x_0 is the initial guess. Among of these types of methods the celebrated Newton's Method (NM)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{1}$$

is probably the best known and most widely used, robust and general-purpose, iterative root-finding algorithm in computational mathematics. Essentially, it uses the fact that the tangent line to a curve is a good approximation to the curve near the point of tangency. It is well known that Newton's method is quadratically convergent to simple roots and linearly to multiple roots. Any one-step iterative method which depends explicitly on f and its first $r - 1$ derivatives cannot attain an order of convergence higher than r, according to ([13],Theorem 1.6). However, in order to overcome this theoretical limit many variants of Newton's method exist. In the last decade, several modified Newton-type methods have been proposed to improve the local order of convergence of Newton's method. All these modifications are in the direction of increasing the local order of convergence with the view of increasing their efficiency indices, often, at the expense of additional evaluations of functions, derivatives and changes in the points of iterations. Cubic and higher order of convergence without second and higher order derivative evaluation was first established in [18] using arithmetic mean Newton's method (AMN) derived using the trapezoidal rule for integration. This trend continued in [12] using harmonic mean (HMN) and the midpoint integration rule instead of the trapezoidal rule. Modifications in the Newton's method based on geometric mean (GMN) are suggested in [11] and based on (trapezoidal) p-power mean in [9]. A modification of Newton's method based on the Simpson 1/3-quadrature rule and arithmetic mean is suggested [6]. Replacing the arithmetic mean with the harmonic mean, in [7], the authors obtained a new variant with cubic convergence. We refer to the book [13] as a unifying presentation of the multi-step iterative methods constructed over fifty years and, also, to the classical book by Traub [15]. However, the derivation of all this variants involves an indefinite integral of the derivative of the function, and the relevant area is approximated by only classical quadrature rules e.g., trapezoid, midpoint, Simpson's, etc in order to reduce the error in the approximation. Ujevic in [16, 17] firstly adopted a quite different approach by using specially derived quadrature rule.

Aim of this paper is to present a new class of efficient Newton-like mean-based method with third-order convergence free from second or higher order derivative. Section 2 is devoted to the derivation of the method and the analysis of convergence is derived analytically, and then re-derived with the help of symbolic computation software package MAPLE 18, see [21]. Section 3 gives numerical results in order to compare the efficiency of the suggested method with the Newton's method and other relevant iterative methods.

2 New Variant of Newton's Method and Analysis of Convergence

Let β be a simple root of a sufficiently differentiable function $f(x)$. Consider the numerical solution of the equation $f(x) = 0$. Usually, derivation of Newtons method involves an indefinite integral of the derivative of the function, and the area is approximated by rectangles. In fact, from the Fundamental Theorem of Calculus, the equation can be written as

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt. \tag{2}$$

Interpolate f' in the interval $[x_n, x]$ by constant $f'(x_n)$ then, by taking $x = \beta$, we obtain

$$f(x_n) + (\beta - x_n)f'(x_n) \approx 0,$$

and finally, computing the iteration value for $x = x_{n+1}$, a new approximation x_{n+1} to β is given by (1). On the other hand, if we approximate the indefinite integral in (2) by the trapezoidal quadrature formula and taking $x = \beta$ we obtain

$$f(x_n) + \frac{1}{2}(\beta - x_n)(f'(x_n) - f'(\beta)) \approx 0,$$

we have a new approximation x_{n+1} to β proposed in [18] and given by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_{n+1})}, \quad n \geq 0.$$

To overcome the implicit problem in the right hand side of the above equation, it is straightforward to suggest the following two-step method: using the Newton's Method as a *predictor* and the new method as a *corrector*. Hence, the $(n + 1)$st value of Newton's method z_{n+1} is used instead of x_{n+1}, that is we get

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_{n+1})}, \; z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0. \tag{3}$$

Notice that (3) is obtained by using the arithmetic mean of $f'(x_n)$ and $f'(z_{n+1})$, for this reason this new method is called the arithmetic mean Newton's method

(AMN)(see [18]). Many authors have proposed to replace the arithmetic mean with other definitions of mean in the denominator of the equation (3) in order to obtain a new family of methods with third order of convergence which can be called *Based-Mean Newton's* methods (BMN). Again, instead of using trapezoidal quadrature rule it is possible to use also Simpson 1/3-quadrature rule to approximate the integral in (2) as in [6, 7, 8, 9]. Here, we propose, following ([19], Remark 1), to use a specially quadrature rules developed in the sense of a Ostrowski-Gräuss type inequality derived in [20, 14] and given by

$$\int_{x_n}^{x} f'(t)dt \approx (x - x_n)\left[h\left(\frac{f'(x_n) + f'(x)}{2}\right) + (1-h)f'\left(\frac{x_n + x}{2}\right)\right],$$

for $h \in [0, 1]$ a free parameter. The previous quadrature formula is an averaging of midpoint and trapezoid rule. Computing for $x = x_{n+1}$, we obtain a new approximation for $n \geq 0$

$$x_{n+1} = x_n - \frac{2f(x_n)}{h(f'(x_n) + f'(z_{n+1})) + 2(1-h)f'\left(\frac{x_n+z_{n+1}}{2}\right)}, \; z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{4}$$

We rearrange (4) in the following form

$$x_{n+1} = x_n - \frac{f(x_n)}{h\left(\frac{f'(x_n)+f'(z_{n+1})}{2}\right) + (1-h)f'\left(\frac{x_n+z_{n+1}}{2}\right)}, \; z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0. \tag{5}$$

In ([19], Theorem 3), the authors prove, for all $h \in [0, 1]$, the cubic convergence for the method (5). Along those lines, we propose to use the contra-harmonic mean instead of arithmetic mean obtaining a new variant of the Newton's method called *Contra-Harmonic Mean Newton's* method developed *in the sense of inequalities* (CHMN-I), that is

$$x_{n+1} = x_n - \frac{f(x_n)}{h\left(\frac{f'(x_n)^2+f'(z_{n+1})^2}{f'(x_n)+f'(z_{n+1})}\right) + (1-h)f'\left(\frac{x_n+z_{n+1}}{2}\right)}, \text{ for } h \in [0, 1], \tag{6}$$

where z_{n+1} is computed as follows

$$z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0. \tag{7}$$

Now, we compute the order of convergence of scheme (6)-(7) both analytically and by using symbolic computation. The symbols used here have usual meanings.

Theorem 2.1. *Let β be a simple root of a function $f : I \subseteq R \to R$, where I is an open interval. Assume f sufficiently differentiable in a neighborhood* *of β, $U_\beta = \{x : |x-\beta| \leq r\}$ for some $r \geq 0$. Furthermore, assume that the initial guess $x_0 \in U_\beta$ then the method defined by (6)-(7) is cubically convergent for all $h \in [0,1]$ with the following error equation*

$$e_{n+1} = \big(c_2^2(h+1) + \frac{1}{4}c_3(3h-1)\big)e_n^3 + O(e_n^4), \tag{8}$$

where $e_n = x_n - \beta$ and $c_k = \frac{f^{(k)}(\beta)}{k!f'(\beta)}$, $k = 1,2,3,\ldots$.

Proof. Let β be a simple zero of function $f(x) = 0$ (i.e. $f(\beta) = 0$ and $f'(\beta) \neq 0$). Since, by hypothesis, f is sufficiently smooth around β, expanding $f(x_n)$ and $f'(x_n)$ by Taylor series about β, we obtain

$$f(x_n) = f'(\beta)[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)], \tag{9}$$

and

$$f'(x_n) = f'(\beta)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)] \tag{10}$$

where $c_k = \frac{f^{(k)}(\beta)}{k!f'(\beta)}$. Direct division gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c3)e_n^3 + O(e_n^4). \tag{11}$$

From (7),(11) we have

$$z_{n+1} = \beta + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + O(e_n^4). \tag{12}$$

Again, expanding $f(z_{n+1})$ by Taylor series about β and using (12), we get

$$f'(z_{n+1}) = f'(\beta)[1 + 2c_2^2e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + O(e_n^4)]. \tag{13}$$

From (10) and (13), we easily get

$$f'(x_n)^2 = f'(\beta)^2[1 + 4c_2e_n + (4c_2^2 + 6c_3)e_n^2 + 4(2c_4 + 3c_2c_3)e_n^3 + O(e_n^4)] \tag{14}$$

and

$$f'(z_{n+1})^2 = f'(\beta)^2[1 + 4c_2^2e_n^2 + 8(c_2c_3 - c_2^3)e_n^3 + O(e_n^4)]. \tag{15}$$

Adding (10) and (13) we get

$$f'(x_n)+f'(z_{n+1}) = 2f'(\beta)\left[1+c_2e_n+\left(c_2^2+\frac{3}{2}c_3\right)e_n^2+2(c_2c_3-c_2^3+c_4)e_n^3+O(e_n^4)\right]. \tag{16}$$

In the same manner, adding (14) and (15) we get

$$f'(x_n)^2+f'(z_{n+1})^2 = 2f'(\beta)^2[1+2c_2e_n+(4c_2^2+3c_3)e_n^2+(10c_2c_3-4c_2^3+4c_4)e_n^3+O(e_n^4)]. \tag{17}$$

Thus, we obtain

$$h\left(\frac{f'(x_n)^2+f'(z_{n+1})^2}{f'(x_n)+f'(z_{n+1})}\right)=hf'(\beta)\Big[1+c_2e_n+\Big(2c_2^2+\frac{3}{2}c_3\Big)e_n^2 \\ +(-5c_2^3+2c_4+5c_2c_3)e_n^3+O(e_n^4)\Big]. \tag{18}$$

Furthermore, we have

$$\frac{z_{n+1}+x_n}{2}=\beta+\frac{1}{2}e_n+\frac{1}{2}e_n^2c_n+(c_3-c_2^2)e_n^3+O(e_n^4). \tag{19}$$

Expanding $f'(\frac{z_{n+1}+x_n}{2})$ by Taylor series about β and using (19), we obtain

$$(1-h)f'\left(\frac{z_{n+1}+x_n}{2}\right)=(1-h)f'(\beta)\Big[1+c_2e_n+\Big(c_2^2+\frac{3}{4}c_3\Big)e_n^2 \\ +\Big(-2c_2^3+\frac{7}{2}c_2c_3+\frac{1}{2}c_4\Big)e_n^3+O(e_n^4)\Big]. \tag{20}$$

Adding (18) and (20) we get

$$h\left(\frac{f'(x_n)^2+f'(z_{n+1})^2}{f'(x_n)+f'(z_{n+1})}\right)+(1-h)f'\left(\frac{z_{n+1}+x_n}{2}\right)= \\ f'(\beta)\Big[1+c_2e_n+(h-1)\Big(c_2^2+\frac{3}{4}c_3\Big)e_n^2+\frac{1}{2}\Big(-4c_2^3(1+2h) \\ +c_2c_3(7+3h)+c_4(1+3h)\Big)+O(e_n^4)\Big]. \tag{21}$$

Finally, dividing (9) by (21) we obtain

$$\frac{f(x_n)}{h\big(\frac{f'(x_n)^2+f'(z_{n+1})^2}{f'(x_n)+f'(z_{n+1})}\big)+(1-h)f'\big(\frac{z_{n+1}+x_n}{2}\big)}=e_n-\big(c_2^2(h+1)+\frac{1}{4}c_3(3h-1)\big)e_n^3+O(e_n^4). \tag{22}$$

Substituting (22) in (6) and, then, subtracting β from both sides we get the following error equation

$$e_{n+1}=\big(c_2^2(h+1)+\frac{1}{4}c_3(3h-1)\big)e_n^3+O(e_n^4), \tag{23}$$

which shows that the method (6)-(7) has third order convergence $\forall h\in[0,1]$. □

In this case, the computations are simple enough to do by hand but, in general, if for example higher order derivative are involved, this can be too time-consuming and, often, error-prone. In view of this issue, we would like to remark that all the above computations can be also derived employing the symbolic computation of the MAPLE 18 package to compute the Taylor series of f and its derivative f' about $x = \beta$, see [2, 3] for further details. As an example, consider method (6) restricted to the case $h = 1$ and run the following MAPLE 18 statements:

```
> N:=x->x-f(x)/D(f)(x):
> F:=x->x-f(x)*(D(f)(x)+(D(f)@N)(x))/(D(f)(x)^2+(D(f)@N)(x)^2):
> algsubs(f(beta) = 0, F(beta));
```

$$\beta$$

```
> algsubs(f(beta) = 0, D(F)(beta));
```

$$0$$

```
> algsubs(f(beta) = 0, (D@@2)(F)(beta));
```

$$0$$

```
> simplify(algsubs(f(beta) = 0, (D@@3)(F)(beta));
```

$$\frac{1}{2}\frac{D(f)(\beta)D^{(3)}(f)(\beta)+6D^{(2)}(f)(\beta)^2}{D(f)(\beta)^2}.$$

It follows that

$$F(\beta) = \beta, F'(\beta) = F''(\beta) = 0, F'''(\beta) = \frac{1}{2}\frac{f'(\beta)f'''(\beta) + 6f''(\beta)^2}{f'(\beta)^2} \tag{24}$$

and, thus, by the Schröder-Traub's Theorem ([13], Theorem 1.1), the method (6)-(7) is of order 3.
Expanding in Taylor series $F(x_n)$ around $x = \beta$ and by using (24) it is possible to rederived the error equation (8).

Remark 2.2.

1. *The family of two-step iterative method presented in this paper recaptures some previous quadrature based methods. For $h = 0$, it recaptures the Midpoint Newton's method, see [4, 12]. For $h = 1$, it recaptures the Contra-Harmonic Newton's method, see [1]. For different choices of h we obtain different variants of Newton's method with third order convergence.*

2. *At each iteration, this variant requires evaluations of one function f and three derivatives f'. We have, previously, proved that the current method converge cubically to a simple root and, thus, as a byproduct, the efficiency index is found to be $EI \sim 1.3161$ which is less than the efficiency index of Newton's method ($EI \sim 1.414$) except for the cases for which $h = 0$ ($EI \sim 1.4422$) and $h = 1$ ($EI \sim 1.4422$). However, our method takes lesser number of iterations than Newton's method.*

3. *The suggested method is not optimal in the sense of Kung and Traub, see [10, 15].*

4. *In [8], the authors introduced a new class of cubic convergent methods based on p-power means exploited in the trapezoidal formula for 3 knots and introducing also same parameters k_1, k_2 and k with the condition $k = k_1 + k_2$, i.e.*

$$x_{n+1} = x_n - \frac{kf(x_n)}{k_1 sign(f'(x_n))\Big(\frac{f'(x_n)^p + f'(z_{n+1})^p}{2}\Big)^{\frac{1}{p}} + k_2 f'\Big(\frac{x_n + z_{n+1}}{2}\Big)}.$$

The efficiency index of the method is $EI \sim 1.3161$. For different choices of p, it is possible to obtain different variants of Newton's method such as arithmetic mean, harmonic mean, quadratic mean, geometric mean and square-mean root. Our main result Theorem 2.1 can be also viewed as an extension of ([8], Theorem 4.1) to the not included contra-harmonic mean case.

3 Numerical Results and Conclusions

In this section, we present the results of numerical calculations to compare the efficiency of the proposed method (CHMN-I) with Newton's method (NM) and with the most classical variants defined in Section 1. We use the following stopping criterion for iterative process: $|x_{n+1} - x_n| < \varepsilon$ and $|f(x_{n+1})| < \varepsilon$ for a fixed ε, the precision of the computer. For example, here we fix $\varepsilon = 10^{-14}$. This means that if the stopping criterion is satisfied, x_{n+1} is taken as the exact root β computed. As convergence criterion it is required that the distance of two consecutive approximations δ for the root is less than 10^{-14}. Numerical computations are performed in MAPLE 18 environment with 64 digit floating point arithmetics (`Digits:=64`). Different test functions, the initial guess x_0, the number of iterations are given in Table 1. The following test functions are used in the numerical results:

1. $f_1(x) = x^3 + 4x^2 - 10$, $\beta = 1.365230013414097$,

2. $f_2(x) = \sin^2(x) - x^2 + 1$, $\beta = -1.404491648215341$,

Table 1: Number of iterations for the test functions

Function	x_0	NM	AMN	HMN	GMN	CHMN-I
$f_1(x)$	1	5	3	3	3	3
$f_2(x)$	1	6	4	3	4	4
$f_3(x)$	3	6	4	4	4	3
$f_4(x)$	3	6	5	4	4	4

3. $f_3(x) = x^2 - e^x - 3x + 2, \ \beta = 0.2575302854398608,$

4. $f_4(x) = (x - 1)^3 - 1, \ \beta = 2$

Finally, numerical results show that the proposed method is in accordance with the developed theory and it can compete with the classical Newton's method. Though our method needs more function evaluations at each iteration, when compared to the Newton's method it is evident by Table 1 that the number of the required iterations is less than that of Newton's method.

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