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A Common Fixed Point Theorem for Asymptotically Regular Multi-Valued Three Maps

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Abstract

In this paper, we prove a common fixed point theorem for asymptotically regular multi valued three maps. Our result generalizes and extends some recent results in the literature.

Keywords: *Asymptotically Regular Maps, Fixed Point, Multi-Valued Maps.*

1 Introduction

In 2006, P.D. Proinov [12] obtained two types of generalizations of Banach fixed point theorem. The first type involves Meir-Keeler [9] type conditions (see, for instance, Cho et al., [3], Lim [8], Park and Rhoades [11]) and the second type involves contractive guage functions (see, for instance, Boyd and Wong [1] and Kim et al., [7]). Proinov [12] obtained equivalence between these two types of contractive conditions and also obtained a new fixed point theorem generalizing some fixed point theorems of Jachymski [6] have extended Proinov [12] Theorem

4.1 into multi valued maps. In this paper we extend Theorem 2.2 of S.L. Singh et al. [16] for three maps.
Asymptotic regularity for single- valued map is due to Browder and Petryshyn [2].

Definition 1.1: *A self-map T on a metric space (X, d) is asymptotic regular if*

$$\lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0 \quad \text{for each } x \in X .$$

Rohades et al., [14] and Singh et al., [17] have extended this concept of asymptotic regularity to multi-valued maps as follows.

Definition 1.2: *Let (X, d) be a metric space and S: Y→CL(X). S is asymptotically regular at $x_o \in X$ if for any sequence $\{x_n\}$ in Y and each sequence $\{y_n\}$ in Y such that*
$y_n \in Sx_{n-1}$

$$\lim_{n \to \infty} (y_n , y_{n+1}) = 0.$$

Definition 1.3: *Let (X, d) be a metric space and S, T: Y→CL(X). A pair (S,T) is said to be asymptotically regular at $x_o \in X$, if for any sequence $\{x_n\}$ in X and each sequence* $\{y_n\}$ in X such that $y_n \in Sx_{n-1} \cup Tx_{n-1}$, $\lim_{n \to \infty} d(y_n , y_{n+1}) = 0$.

Definition 1.4: *Let f: Y→Y and S:Y→2^Y the collection of non-empty sub set of Y. Then the hybrid pair (S, f) are (IT)-Commuting on Y if $fSz \subseteq Sfz$ for all $z \in Y$.*

2 Common Fixed Point Theorem

The following theorem is extension and improves the Theorem of S.L. Singh et al., [16].

Theorem 2.1: *Let (X, d) be a metric space and f: Y→X and S, T: Y→CL(X) such that*
(C1). $SY \cup TY \subseteq fY$.

(C2). $H(Sx, Ty) \le \varphi(h(x,y))$ for all $x, y \in Y$,

where , $h(x,y) = d(fx,fy)+\gamma[d(Sx,fx)+d(Ty,fy)]$, $0 \le \gamma \le 1$ and $\varphi \in \Phi$ is continuous.

If the pair (S, T) is asymptotically regular at $x_o \in X$ and either S(Y) or T(Y) or f(Y) is a complete sub space of X. Then

(i). C(S, f) and
(ii). C(T, f) are non-empty. Further,

(iii). S and f have a common fixed point provided SSu=Su and S and f are (IT)-Commuting at a point $u \in C(S, f)$.
(iv). T and f have a common fixed point provided TTv=Tv and T and f are (IT)-Commuting at a point $v \in C(T, f)$.
(v). S, T and f have a common fixed point provided that (iii) and (iv) both are true.

Proof: We construct sequences $\{y_n\}$ and $\{x_n\}$ in Y in the following way.

Let y_1 be an element of Sx_0. Since Tx_1 is compact, we choose a point $y_2 \in Y$ such that $d(y_1, y_2) \le H(Sx_0, Tx_1)$. Again Tx_2 is compact we choose a point $y_3 \in Y$ such that

$d(y_2, y_3) \le H(Sx_1, Tx_2)$ continuing in the same manner we get

$d(y_n, y_{n+1}) \le H(Sx_{n-1}, Tx_n)$. Since $SY \cup TY \subseteq fY$, we may take

$y_n = fx_n \in Sx_{n-1} \cup Tx_{n-1}$ for n= 1,2,…….The asymptotic regularity of the pair (S,T) at x_0

implies that $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0$.

Fix $\varepsilon > 0$. Since $\varphi \in \Phi$ there exists $\delta > \varepsilon$ such that for any $t \in (0, \infty)$,

$$\in < t < \delta \Rightarrow \varphi(t) \le \varepsilon . \qquad (1)$$

Without loss of generality we may assume that $\delta \le 2\varepsilon$. By the asymptotic regularity of the pair (S,T) at x_0,

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = 0.$$

So, there exists an integer $N_1 \ge 1$ such that

$$d(y_n, y_{n+1}) \le H(Sx_{n-1}, Tx_n) < \frac{\delta - \varepsilon}{1 + 2\gamma}, \quad m \ge N_1 . \qquad (2)$$

By the induction we show that

$$d(y_n, y_m) \le H(Sx_{n-1}, Tx_{m-1}) < \frac{\delta + 2\gamma\varepsilon}{1 + 2\gamma}, \quad m \ge n \ge N_1 . \qquad (3)$$

Let $n > N_1$ be fixed .Then equation (3) holds for m = n+1.

Assuming (3) to hold for an integer $m \ge n$. We shall prove it for m+1.

By the triangle inequality, we get

$$d(y_n, y_{m+1}) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{m+1}) .$$

That is,

$d(y_n, y_{n+1}) \leq d(y_n, y_{n+1}) + H(Sx_n, Tx_m)$. (4)

We shall show that

$H(Sx_n, Tx_m) \leq \varepsilon$. (5)

If $H(Sx_n, Tx_m)$ not less than or equal ε , then

$\varepsilon < H(Sx_n, Tx_m) < \varphi(h(x_n, x_m)) \leq h(x_n, x_m) < \delta$.

$h(x_n, x_m) = d(x_n, x_m) + \gamma[d(x_n, Sx_n) + d(x_m, Tx_m)]$

$= d(x_n, x_m) + \gamma[d(x_n, x_{n+1}) + d(x_m, x_{m+1})]$.

Using (2) and (3) in this inequality yields,

$$h(x_n, x_m) < \frac{\delta + 2\gamma\varepsilon}{1+2\gamma} + \gamma\frac{\delta-\varepsilon}{1+2\gamma} + \gamma\frac{\delta-\varepsilon}{1+2\gamma} = \delta.$$

$$h(x_n, x_m) < \delta.$$

$\Rightarrow \varepsilon < h(x_n, x_m) \leq \varepsilon$, which is a contradiction .

Therefore, $H(Sx_n, Tx_m) \leq \varepsilon$. Hence (5).

(3) and (5) in (4) , we get

$$d(y_n, y_{m+1}) \leq d(y_n, y_{n+1}) + H(Sx_n, Tx_m)$$

$$< \frac{\delta-\varepsilon}{1+2\gamma} + \varepsilon.$$

$$= \frac{\delta-\varepsilon+\varepsilon+2\gamma\varepsilon}{1+2\gamma} = \frac{\delta+2\gamma\varepsilon}{1+2\gamma}.$$

$$d(y_n, y_{m+1}) < \frac{\delta+2\gamma\varepsilon}{1+2\gamma} .$$

This proves (3). Since $\delta \leq 2\varepsilon$, then (3) implies that $d(y_n, y_{m+1}) < 2\varepsilon$ for all integers m and n with $m \geq n \geq N_1$ and hence $\{y_n\}$ is a Cauchy sequence .

Suppose f(Y) is complete subspace of X, then there exists a point $u \in Y$ such that $fu = z$. To show that $z = fu \in Su$, We suppose otherwise and use the condition (ii) we have

$d(Su, Tx_n) \leq H(Su, Tx_n) \leq \varphi(h(u, x_n))$

$= \varphi(d(fu, fx_n)) + \gamma[d(Su,fu) + d(Tx_n, fx_n)])$.

Letting $n \to \infty$, we get

$d(Su, z) \leq \varphi(d(z, z)) + \gamma[d(Su, z) + d(z, z)])$

$= \varphi(0 + \gamma[d(Su, z)])$

$= \varphi(\gamma\, d(Su, z)) < d(Su, z)$, (Since, $\varphi(t) < t$)

Which is a contradiction.

Therefore, $z = fu \in Su$.
Consequently, C (S, f) is non-empty. This proves (i).

Since $SY \cup TY \subseteq fY$, there exists a point $v \in Y$ such that $z = fu = fv \in Tv$,
so by (ii)

$d(fv, Tv) = d(fu, Tv) \leq H(Su, Tv) \leq \varphi(h(u, v))$

$= \varphi(d(fu, fv)) + \gamma[d(Su,fu) + d(Tv, fv)])$

$= \varphi(d(z, z)) + \gamma[d(z,z) + d(Tv, fv)])$

$d(fv, Tv) \leq \varphi(d(Tv, fv)) < d(Tv, fv)$, which is a contradiction.

Therefore, $z = fu = fv \in Tv$.

Thus, C(T,f) is non-empty. This proves (ii).

Further, $Su = SSu$ and The (IT)-Commutative of S and f at $u \in$ C (S, f) implies that $Su \in Sfu \subseteq fSu$. So, Su is a common fixed point of S and f.

And $Tv = TTv$ and the (IT)-Commutative of T and f at $v \in$ C (S, f) implies that $Tv \in Tfv \subseteq fTv$. So, Tv is a common fixed point of T and f.

Since, $z = fu \in Su$ and $z = fu = fv \in Tv$.
Therefore, T, S and f have a common fixed point.

Analogous argument establishes the theorem when S(Y) or T(Y) is a complete sub space of X. This completes the proof.

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