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Group Analysis via Nonclassical Symmetries for Two-Dimensional Ricci Flow Equation

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Abstract

This paper is devoted to obtain the largest possible set of symmetries for the two-dimensional Ricci flow ((2D) Rf) equation. By using the classical symmetry method, the structure of Lie algebra of symmetries is obtained and the optimal system of subalgebras of the equation is constructed. Also some reduced equations and group invariant solutions are obtained. By applying the nonclassical symmetry method for the ((2D) Rf) equation we concluded that the analyzed model do not admit supplementary, nonclassical type, symmetries.

Keywords: *Lie symmetry group, Two-dimensional Ricci flow equation, Optimal system, Group invariant solution, Nonclassical symmetry.*

1 Introduction

The Ricci flow is an evolution equation that was introduced by Hamilton in his seminal paper, "Three-manifolds with positive Ricci curvature" in 1982 [8]. Ricci flow is a very useful tool for studying the special geometries which a manifold admits. If $(M, g(t))$ be a smooth Riemannian manifold, Ricci flow is defined by the equation

$$\frac{\partial}{\partial t}g(t) = -2Ric, \tag{1}$$

where Ric denotes the Ricci tensor of the metric g. By using the concept of Ricci flow, Grisha Perelman completely proved the Poincaré conjecture around

2003 [14]. The Ricci flow also is used as an approximation to the renormalization group flow for the two-dimensional nonlinear σ-model, in quantum field theory, see [7] and references therein. The ricci flow equation is related to one of the models used in obtaining the quantum theory of gravity. Because some difficulties appear when a quantum field theory is formulated, the studies focus on less dimensional models which are called mechanical models.

The symmetry group method plays a fundamental role in the analysis of differential equations.The theory of Lie symmetry groups of differential equations was first developed by Sophus Lie [10] at the end of the nineteenth century, which was called classical Lie method. Nowadays, application of Lie transformations group theory for constructing the solutions of nonlinear partial differential equations (PDEs) can be regarded as one of the most active fields of research in the theory of nonlinear PDEs and applications. The fact that symmetry reductions for many PDEs can not be obtained via the classical symmetry method, motivated the creation of several generalizations of the classical Lie group method for symmetry reductions. Consequently, several alternative reduction methods have been proposed, going beyond Lie's classical procedure and providing further solutions. The nonclassical symmetry method of reduction was devised originally by Bluman and Cole in 1969 [2], to find new exact solutions of the heat equation. The description of the method is presented in [5, 9]. Many authors have used the nonclassical method to solve PDEs. In [6] Clarkson and Mansfield have proposed an algorithm for calculating the determining equations associated to the nonclassical method. A new procedure for finding nonclassical symmetries has been proposed by Bila and Niesen in [1]. Classical and nonclassical symmetries of nonlinear PDEs may be applied to reduce the number of independent variables of the PDEs. Particularly, the PDEs can be reduced to ODES. The ODEs may also have symmetries which enable us to reduce the order of the equation and we can integrate to obtain exact solutions.

This paper is organized as follows: In section 2, by using the classical Lie symmetry method the most general Lie symmetry group of the (($2D$) Rf) equation is determined and the optimal system of one-dimensional subalgebras is constructed. Section 3, is devoted to obtain Lie invariants, similarity reduced equations corresponding to the infinitesimal and the most general group-invariant solutions of (($2D$) Rf) equation. In section 4, we focus on the nonclassical symmetries of the (($2D$) Rf) equation, symmetries generated when a supplementary condition, the invariance surface condition, is imposed. Some concluding remarks are presented at the end of the paper.

2 Lie Symmetries of ($(2D)$ Rf) Equation

As we know, transformations which map solutions of a differential equation to other solutions are called symmetries of the equation. The procedure of finding the Lie symmetry group of a PDE was described in many studies such as [13, 12]. Before performing the Lie symmetries of Ricci flow, let us restate the mechanical model of Ricci flow that introduced by Cimpoiaus and Constantinescu [4].

The metric tensor of the space, g_{ij}, can be written in the conformally flat frame

$$ds^2 = g_{ij}dx^i dx^j = 2e^{\phi(x,y,t)}dxdy = \frac{1}{2}e^{\phi(X,Y,t)}(dX^2 + dY^2) \tag{2}$$

using Cartesian coordinates X, Y or the complex variables $2x = Y + iX, 2y = Y - iX$. According to the equation (1), the function $\phi(X, Y, t)$ must satisfy the equation

$$\frac{\partial}{\partial t}e^{\phi} = \Delta\phi, \tag{3}$$

where Δ is Laplacian. By introducing the field

$$u(x, y, t) = e^{\phi},$$

the equation (3) takes the form $u_t = (\ln u)_{xy}$ or in the equivalent form:

$$u^2 u_t + u_y u_x - u u_{xy} = 0, \tag{4}$$

The infinitesimal generator of (4) is as follow:

$$X = \xi^1(x, y, t, u)\partial_x + \xi^2(x, y, t, u)\partial_y + \xi^3(x, y, t, u)\partial_t + \varphi(x, y, t, u)\partial_u \tag{5}$$

Cimpoiaus and Constantinescu, also obtained the Lie symmetry group of this equation [4]. They proved that this equation admits a 6-parameter Lie group, G, with the following infinitesimal generators for its Lie algebra, g.

$$\begin{array}{lll} X_1 = \partial_x, & X_2 = \partial_y, & X_3 = \partial_t, \\ X_4 = t\partial_t + u\partial_u, & X_5 = x\partial_x - u\partial_u, & X_6 = y\partial_y - u\partial_u. \end{array} \tag{6}$$

Since every linear combination of infinitesimal symmetries is an infinitesimal symmetry, there is an infinite number of one-dimensional subgroups for G. Therefore, it is important to determine which subgroups give different types of solutions. So, we must find invariant solutions which can not be transformed to each other by symmetry transformations in the full symmetry group. This led to the concept of an optimal system of subalgebra. For one-dimensional subalgebras, this classification problem is the same as the problem of classifying the orbits of the adjoint representation [12]. Optimal set of subalgebras is obtained by selecting only one representative from each class of equivalent subalgebras. The problem of classifying the orbits is solved by taking a general element in the Lie algebra and simplifying it as much as possible by imposing various adjoint transformation on it [13]. We have the following theorem:

A one-dimensional optimal system for Lie algebra of ((2D) Rf) equation is given by

$$\begin{array}{lll} 1) X_1 + aX_2 + bX_3, & 4) X_1 + cX_4 + dX_6, & 7) X_3 + cX_5 + dX_6, \\ 2) X_1 \pm X_2 + cX_4, & 5) X_2 \pm X_3 + cX_5, & 8) X_4 + cX_5 + dX_6, \\ 3) X_1 \pm X_3 + cX_6, & 6) X_2 + cX_4 + dX_5, & \end{array} \tag{7}$$

where a, b, c and d are real numbers and $a \neq 0, b \neq 0$ [11].

3 Similarity Reduction of ((2D) Rf) Equation

In this section, the two-dimensional Ricci flow equation will be reduced by expressing it in the new coordinates. The ((2D) Rf) equation is expressed in the coordinates (x, y, t, u), we must search for this equation's form in the suitable coordinates for reducing it. These new coordinates will be obtained by looking for independent invariants (z, w, f) corresponding to the generators of the symmetry group. Hence, by using the new coordinates and applying the chain rule, we obtain the reduced equation. We have listed the result for some cases in Table 1.

Table 1: Lie invariants, similarity solutions and reduced equations.

i	h_i	$\{z_i, w_i, v_i\}$	u_i	Similarity reduced equations
1	$X_1 + X_6$	$\{ye^{-x}, t, uy\}$	$\frac{f}{y}$	$f^2 f_w - z^2 f_z^2 + z f f_z + z^2 f f_{zz} = 0$
2	$X_2 + X_4$	$\{x, te^{-y}, ue^{-y}\}$	fe^y	$f^2 f_w - w f_z f_w + w f f_{zw} = 0$
3	$X_3 + X_5 + dX_6$	$\{\frac{y}{x^d}, \ln\frac{e^t}{x}, ux^{d+1}\}$	$\frac{f}{x^{d+1}}$	$f_w(f^2 - f_z) - dz f_z^2$ $+ f(df_z + dz f_{zz} + f_{zw}) = 0$
4	$X_2 + X_3 + X_5$	$\{\ln\frac{e^y}{x}, \ln\frac{e^t}{x}, ux\}$	$\frac{f}{x}$	$f^2 f_w - f_z^2 - f_w f_z + f f_{zz} + f f_{zw} = 0$
5	$X_2 + X_5$	$\{\ln\frac{e^y}{x}, t, ux\}$	$\frac{f}{x}$	$f^2 f_w - f_z^2 + f f_{zz} = 0$
6	$X_3 + X_6$	$\{x, t - \ln y, uy\}$	$\frac{f}{y}$	$f^2 f_w - f_z f_w + f f_{zw} = 0$
7	$X_1 + X_2$	$\{y - x, t, u\}$	f	$f^2 f_w - f_z^2 + f f_{zz} = 0$
8	$X_2 + X_3$	$\{x, t - y, u\}$	f	$f^2 f_w - f_z f_w + f f_{zw} = 0$

By reducing the equations obtained in Table 1 to ODEs we can solve them [11]. In Table 2, we obtain the invariant solutions of ((2D) Rf) equation corresponding to some of the similarity reduced equations.

Table 2: Group invariant solutions of the ((2D) Rf) equation.

$\mathcal{A}_j^i$	Invariant solution	$\mathcal{A}_j^i$	Invariant solution
$\mathcal{A}_1^2$	$-2s+c_1$	$\mathcal{A}_4^3$	$\frac{2c_1e^{c_1(s+c_2)}}{-1+e^{c_1(s+c_2)}}$
$\mathcal{A}_1^3$	$\frac{1}{2c_1^2}(1-\tanh(\frac{\ln s-c_2}{2c_1})^2)$	$\mathcal{A}_5^2$	$\frac{1}{2c_1^2}(1-\tanh(\frac{s+c_2}{2c_1})^2)$
$\mathcal{A}_2^1$	c_1	$\mathcal{A}_7^1$	$c_2e^{c_1s}$
$\mathcal{A}_3^2$	$c_2s^{c_1}$	$\mathcal{A}_7^4$	$\frac{-c_1^2e^{\frac{c_1}{s}}}{e^{\frac{c_1}{s}}(c_1-s)-sc_1^2c_2}$
$\mathcal{A}_3^1$	$\frac{c_1(1+c_1)s^{c_1}}{-s^{c_1}(1+c_1-ds)+dc_1c_2(1+c_1)(ds-1)^{c_1+1}}$	$\mathcal{A}_8^1$	$\frac{c_1e^{c_1(s+c_2)}}{1-e^{c_1(s+c_2)}}$

4 Nonclassical Symmetries of ((2D) Rf) Equation

In this section, we will apply the so called nonclassical symmetry method [2]. Beside the classical symmetries, the nonclassical symmetry method can be used to find some other solutions for a system of PDEs and ODEs. The nonclassical symmetry method has become the focus of a lot of research and many applications to physically important partial differential equations as in [1, 6, 5, 9]. Here, we follow the method used by Cai Guoliang *et al*, for obtaining the non-classical symmetries of the Burgers-Fisher equation based on compatibility of evolution equations [3]. For the non-classical method, we must add the invariance surface condition to the given equation, and then apply the classical symmetry method. This can also be conveniently written as:

$$X^{(2)}\Delta_1 \mid_{\Delta_1=0,\Delta_2=0}= 0, \tag{8}$$

where X is defined in (5) and Δ_1 and Δ_2 are given as:

$$\Delta_1 := u^2u_t + u_yu_x - uu_{xy}, \qquad \Delta_2 := \varphi - \xi^1u_x - \xi^2u_y - \xi^3u_t \tag{9}$$

Without loss of generality we choose $\xi^3 = 1$. In this case using Δ_2 we have:

$$u_t = \varphi - \xi^1u_x - \xi^2u_y. \tag{10}$$

Total differentiation D_t of the equation gives

$$\begin{aligned} D_t(u^2u_t) &= D_t(uu_{xy} - u_xu_y) = u_tu_{xy} + uu_{xyt} - u_{xt}u_y - u_xu_{yt} \\ &= (\varphi - \xi^1u_x - \xi^2u_y)u_{xy} + u(\varphi - \xi^1u_x - \xi^2u_y)_{xy} \\ &\quad -(\varphi - \xi^1u_x - \xi^2u_y)_xu_y - u_x(\varphi - \xi^1u_x - \xi^2u_y)_y \\ &= \varphi u_{xy} - \xi^1u_xu_{xy} - \xi^2u_yu_{xy} + u(\varphi^{xy} - \xi^1u_{xxy} - \xi^2u_{xyy}) \\ &\quad -(\varphi^x - \xi^1u_{xx} - \xi^2u_{yx})u_y - u_x(\varphi^y - \xi^1u_{xy} - \xi^2u_{yy}) \end{aligned} \tag{11}$$

and

$$
\begin{aligned}
D_t(u^2u_t) &= D_t(u^2(\varphi - \xi^1 u_x - \xi^2 u_y)) = 2uu_t\varphi + u^2\varphi^t - 2uu_tu_x\xi^1 \\
&\quad - u^2u_{xt}\xi^1 - 2uu_tu_y\xi^2 - u^2u_{yt}\xi^2 \\
&= 2u(\varphi - \xi^1u_x - \xi^2u_y)\varphi + u^2\varphi^t - 2u(\varphi - \xi^1u_x - \xi^2u_y)u_x\xi^1 \\
&\quad - u^2(\varphi - \xi^1u_x - \xi^2u_y)_x\xi^1 - 2u(\varphi - \xi^1u_x - \xi^2u_y)u_y\xi^2 \\
&\quad - u^2(\varphi - \xi^1u_x - \xi^2u_y)_y\xi^2
\end{aligned} \tag{12}
$$

On the other hand we have:

$$
\begin{aligned}
&D_x(u^2u_t) = D_x(uu_{xy} - u_xu_y) \\
&\Rightarrow 2uu_xu_t + u^2u_{xt} = uu_{xxy} - u_{xx}u_y \\
&\Rightarrow 2uu_x(\varphi - \xi^1u_x - \xi^2u_y) + u^2(\varphi - \xi^1u_x - \xi^2u_y)_x = uu_{xxy} - u_{xx}u_y
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
&D_y(u^2u_t) = D_y(uu_{xy} - u_xu_y) \\
&\Rightarrow 2uu_yu_t + u^2u_{yt} = uu_{xyy} - u_{yy}u_x \\
&\Rightarrow 2uu_y(\varphi - \xi^1u_x - \xi^2u_y) + u^2(\varphi - \xi^1u_x - \xi^2u_y)_y = uu_{xyy} - u_{yy}u_x.
\end{aligned} \tag{14}
$$

By equality of (11) and (12) and substituting the (13) and (14) in them, the governing equation is obtained as follow

$$
\begin{aligned}
2u\varphi^2 + u^2\varphi^t - 2uu_x\xi^1\varphi - 2uu_y\xi^2\varphi - \varphi u_{xy} - u\varphi^{xy} + \varphi^xu_y + \varphi^yu_x = 0 \\
\Rightarrow 2uu_t\varphi + u^2\varphi^t - \varphi u_{xy} - u\varphi^{xy} + \varphi^xu_y + \varphi^yu_x = 0
\end{aligned} \tag{15}
$$

where φ^t, φ^x, φ^y and φ^{xy} are given by

$$
\begin{aligned}
&\varphi^x = D_x(\varphi - \xi^1u_x - \xi^2u_y - \xi^3u_t) + \xi^1u_{xx} + \xi^2u_{xy} + \xi^3u_{xt} \\
&\varphi^y = D_y(\varphi - \xi^1u_x - \xi^2u_y - \xi^3u_t) + \xi^1u_{xy} + \xi^2u_{yy} + \xi^3u_{yt}, \\
&\varphi^t = D_t(\varphi - \xi^1u_x - \xi^2u_y - \xi^3u_t) + \xi^1u_{xt} + \xi^2u_{yt} + \xi^3u_{tt} \\
&\varphi^{xy} = D_xD_y(\varphi - \xi^1u_x - \xi^2u_y - \xi^3u_t) + \xi^1u_{xxy} + \xi^2u_{xyy} + \xi^3u_{xyt}
\end{aligned} \tag{16}
$$

Substituting them into the (15), we can get the determining equations for the symmetries of the ((2D) Rf) equation. By substituting $\xi^3 = 1$ into the determining equations, we obtain the determining equations of the nonclassical symmetries of the original equation (4). Solving the system obtained by this procedure, the only solutions we found were exactly the solution obtained through the classical symmetry approach (6). This means that no supplementary symmetries, of non-classical type, are specific for ((2D) Rf) equation.

5 Conclusion

In this paper, by using the adjoint representation of the symmetry group on its Lie algebra, we have constructed an optimal system of one-dimensional subalgebras for a well-known partial differential equation in mathematical physics called: two-dimensional Ricci flow equation. Moreover, by applying the criterion of invariance of the equation under the prolonged infinitesimal generators, we find the most general Lie point symmetries group of the $((2D)$ Rf) equation. Also by applying the nonclassical symmetry method for the $((2D)$ Rf) equation we concluded that the analyzed model do not admit supplementary, nonclassical type symmetries.

References

[1] N. Bila and J. Niesen, On a new procedure for finding nonclassical symmetries, *Journal of Symbolic Computation*, 38(2004), 1523-1533.

[2] G.W. Bluman and J.D. Cole, The general similarity solutions of the heat equation, *Journal of Mathematics and Mechanics*, 18(1969), 1025-1042.

[3] G. Cai, Y. Wang and F. Zhang, Nonclassical symmetries and group invariant solutions of Burgers-Fisher equations, *World Journal of Modelling and Simulation*, 3(4) (2007), 305-309.

[4] R. Cimpoiasu and R. Constantinescu, Symmetries and invariants for the 2D-Ricci flow model, *Journal of Nonlinear Mathematical Physics*, 13(2) (2006), 285-292.

[5] P.A. Clarkson, Nonclassical symmetry reductions of the Boussinesq equation, *Chaos, Solitons, Fractals*, 5(1995), 2261-2301.

[6] P.A. Clarkson and E.L. Mansfield, Algorithms for the nonclassical method of symmetry reductions, *SIAM Journal on Applied Mathematics*, 55(1994), 1693-1719.

[7] K. Gawedzki, Lectures on conformal field theory, Quantum fields and strings: A course for mathematicians, *Princeton*, (1996-97), 727-805.

[8] R.S. Hamilton, Three-manifolds with positive Ricci curvature, *J. Diff. Geom*, 17(2) (1982), 255-306.

[9] D. Levi and D.P. Winternitz, Nonclassical symmetry reduction: Example of the Boussinesq equation, *Journal of Physics A*, 22(1989), 2915-2924.

[10] S. Lie, On integration of a class of linear partial differential equations by means of definite integrals, *Arch. for Math.*, 6(1881), 328-368, translation by N.H. Ibragimov.

[11] M. Nadjafikhah and M. Jafari, Symmetry reduction of the two-dimensional Ricci flow equation, *Geometry*, Article ID 373701(2013), 6 pages.

[12] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, (1986).

[13] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, (1982).

[14] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain Three-manifolds, *math. DG/0307245*, (2003).