Gen. Math. Notes, Vol. 31, No. 1, November 2015, pp. 18-41

ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

The Local Bifurcation and the Hopf Bifurcation for Eco-Epidemiological System with One Infectious Disease

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(Received: 31-7-15 / Accepted: 19-10-15)

Abstract

In this paper, we established the conditions of the occurrence of local bifurcation (such as saddle-node, transcritical and pitchfork) with particular emphasis on the Hopf bifurcation near of the positive equilibrium point of eco-epidemiological mathematical model consisting of prey-predator model involving SIS infectious disease in prey population are established. After the study and analysis, of the observed incidence transcritical bifurcation near equilibrium point E_0 as well as the occurrence of saddle-node bifurcation at equilibrium points E_1, E_2. It is worth mentioning, there are no possibility occurrence of the pitchfork bifurcation at each point $E_i i= 0,1,2$. Finally, some numerical simulations are used to illustration the occurrence of local bifurcation of this model.

Keywords: *Eco-epidemiological model, Equilibrium Points, Local bifurcation, Hopf bifurcation.*

1 Introduction

Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiology, Physiology, Ecology, Immunology, Genetics, Physics are some of those disciplines. In fact, both mathematical ecology and mathematical epidemiology are distinct major fields of study in biology. But there are some commonalities between them. Recently, these two major fields of study are merged and renamed as a new field of study called eco-epidemiology. On the other hand eco-pidemiology is the branch of biomathematics that understands the dynamics of disease spread on the predator-prey system, whereas considered interaction between predators and their prey is a complex phenomenon in ecology. Many researchers, especially in the last two decades, have proposed and studied number of eco-epidemiological models involving two or more interacting species have already been performed in this particular direction, see for example [1-3] and the references there in.

Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family, such as the integral curves of a family of vector fields, and the solutions of a family of differential equations. Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behavior, for example, see [4-6]. The bifurcation occurs in both continuous systems (described by ODEs, DDEs or PDEs), see for example [7-12] and discrete systems (described bymaps), see for example [12-17]. Henri Poincaré [18] was first introduced the name "bifurcation" in 1885 in the first paper in mathematics showing such a behavior also later named various types of stationary points and classified them. Perko L. [19] established the conditions of the occurrence of local bifurcation (such as saddle-node, transcritical and pitchfork). However,the necessary condition for the occurrence of the Hopf bifurcation presented by Hirsch M.W. and Smale S. [20] while, Haque M. and Venturino E. [21] Explained the sufficient condition for the occurrence of the Hopf bifurcation in addition to them, nots see for example [22,23, 24]. R. Latief Tayeh and R. Kamel Naji [25] had previously studied local bifurcation (such as saddle-node, transcritical and pitchfork) and Hopf bifurcation around each of the equilibrium points of prey-predator model involving SI infection disease in both the prey and predator species.

In this paper, we will establish the conditions of the occurrence of local bifurcation and Hopf bifurcations around each of the equilibrium points of a mathematical model proposed by Karrar Q., Azhar A. and Raid N. [26].

2 Model Formulation [26]

An eco-epidemiological mathematical model consisting of prey-predator model involving SIS infectious disease in prey population, is proposed and analyzed in [26].

$$\left.\begin{aligned}
\frac{dS}{dT} &= rS\left(1-\frac{S+I}{K}\right) - c_1SP_1 - \lambda_1SI - \Theta_1S + \alpha I\\
\frac{dI}{dT} &= \lambda_1SI + \Theta_1S - c_2IP_1 - \gamma_1I - \alpha I\\
\frac{dP_1}{dT} &= -\lambda_2P_1P_2 - \Theta_2P_1 + e_1c_1SP_1 + (1-m)\, e_2c_2IP_1 - \gamma_2P_1 + \beta P_2\\
\frac{dP_2}{dT} &= \lambda_2P_1P_2 + \Theta_2P_1 + m\, e_2c_2IP_1 - \gamma_2P_2 - \gamma_3P_2 - \beta P_2.
\end{aligned}\right\} \quad (1)$$

Where $0<e_i<1$; $i = 1,2$ represent the conversion rate constants and $0< m < 1$ represents the infection rate of susceptible predator that predation the infected prey. This model consists of a prey, whose total population density at time T is denoted by N(T), interacting with predator whose total population density at time T is denoted by P(T). Note that, there is an SIS epidemic disease in prey population divides the prey population into two classes namely S(T) that represents the density of susceptible prey species at time T and I(T) which represents the density of infected prey species at time T. Therefore at any time T, we have N (T) = S (T) + I (T). Also, The disease is transmitted from a prey to predator during attacking of predator to prey, which divides the predator population into two classes namely $P_1(T)$ that represents the density of susceptible predator species at time T and $P_2(T)$ which represents the density of infected predator species at time T. Therefore at any time T, we have $P(T)=P_1(T) + P_2(T)$. All the parameters are moreover assumed to be positive and described as given in [26].

Now, for further simplification of the system (2), the following dimensionless variables are used in [26].

$$t = r\,T,\ x = \frac{S}{k},\ \ y = \frac{I}{k},\ \ z = \frac{c_1}{r}P_1,\ \ w = \frac{c_1}{r}P_2.$$

Thus, system (2) can be turned into the following dimensionless form:

$$\left.\begin{aligned}
\frac{dx}{dt} &= x\,(1 - x - (1+u_1)y - z - u_2) + u_3y = f_1(x, y, z, w)\\
\frac{dy}{dt} &= y\,\big(u_1x - u_4z - (u_3 + u_5)\big) + u_2x = f_2(x, y, z, w)(2)\\
\frac{dz}{dt} &= z\,\big(-u_6w + u_8x + u_9(1-m)\,y - (u_7 + u_{10})\big) + u_{11}w = f_3(x, y, z, w)
\end{aligned}\right\}$$

$\frac{dw}{dt} = u_6 zw + (u_7 + u_9 my)z - (u_{10} + u_{11} + u_{12})w = f_4(x, y, z, w)$.

Here:

$u_1 = \frac{\lambda_1 k}{r}$, $u_2 = \frac{\theta_1}{r}$, $u_3 = \frac{\alpha}{r}$, $u_4 = \frac{c_2}{c_1}$, $u_5 = \frac{\gamma_1}{r}$, $u_6 = \frac{\lambda_2}{c_1}$, $u_7 = \frac{\theta_2}{r}$, $u_8 = \frac{e_1 c_1 k}{r}$,

$u_9 = \frac{e_2 c_2 k}{r}$, $u_{10} = \frac{\gamma_2}{r}$, $u_{11} = \frac{\beta}{r}$, $u_{12} = \frac{\gamma_3}{r}$.

With, $x(0) \geq 0$, $y(0) \geq 0$, $z(0) \geq 0$, $w(0) \geq 0$and it is observed that the number ofparameters have been reduced from Sixteen in the system (1) to Thirteen in the system (2).Obviously the interaction functions of the system (2) are continuous andhave continuous partial derivatives on the following positive fourdimensional space:$R_+^4 = \{(x, y, z, w) \in R^4 : x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, w(0) \geq 0\}$. Therefore these functions are Lipschitzian on R_+^4, and hence the solution of the system (2) exists and is unique.Further, in the following theorem, the boundedness of the solution of the system (2) in R_+^4 is established by [26].

Theorem 1: *All the solutions of system (2) which initiate in the*R_+^4*are uniformly bounded.*

3 The Stability Analysis of Equilibrium Points of System (2) [26]

It is observed that, system (2) has at most three biologically feasible equilibrium points $E_i=(x, y, z, w)$; $i = 0, 1, 2$; which are mentioned with their existence conditions in [26] as in the following:

1. The Vanishing Equilibrium Point: $E_0 = (0,0,0,0)$ always exists and E_0 is locally asymptotically stable in the Int.R_+^4. If the following conditions hold

$u_2 > 1 + \frac{u_3}{u_5}$(3.a)

However, it is (a saddle point) unstable otherwise. More details see [26].

2. The Predator Free Equilibrium Point: $E_1 = (\hat{x}, \hat{y}, 0, 0)$ exists uniquely in the Int. R_+^4 if and only if the following conditions are hold.

$$u_2 > 1 + \frac{u_3}{u_5} \quad (3.a)$$

$$\frac{u_3}{1 + u_1} < \hat{x} < 1 - u_2 \quad (3.b)$$

Where

$\hat{y} = \frac{1-(\hat{x}+u_2)}{(1+u_1)-(\frac{u_3}{\hat{x}})}$, $(1+u_1) \neq \frac{u_3}{\hat{x}}$.

While $\hat{x}$ represents a positive root of the following second order polynomial equation

$A_1 x^2 + A_2 x + A_3 = 0$

Where

$A_1 = u_1 > 0$
$A_2 = -(u_1 + u_2 + u_3 + u_5) < 0$
$A_3 = (u_3 + u_5) - (u_2 u_5)$.

And it is locally asymptotically stable if the following conditions are satisfied:

$$\left.\begin{array}{l} \hat{x}^2 < \frac{(u_1+u_2+2(u_3+u_5))\hat{x} + (u_5(1+u_1)+u_3)\hat{y} + u_5(u_2-1) - u_3}{2u_1} \\ \hat{x} < \min\{a, b\}. \end{array}\right\} \quad (3.c)$$

Where

$$a = \frac{(u_7+2u_{10}+u_{11}+u_{12}) - u9(1-m)\hat{y}}{u_8}$$

$$b = \frac{u_7(u_{10}+u_{12}) + u_{10}(u_{10}+u_{11}+u_{12}) - u_9\hat{y}\big((1-m)(u_{10}+u_{12}) + u_{11}\big)}{u_8(u_{10}+u_{11}+u_{12})} .$$

However, it is (a saddle point) unstable otherwise.More details see [26].

3. Finally, the Positive (Coexistence) Equilibrium Point: $E_2 = (x^*, y^*, z^*, w^*)$ exists and it is locally asymptotically stable, as shown in [26].

4 The Local Bifurcation Analysis of System (2)

In this section, the effect of varying the parameter values on the dynamical behavior of the system (2) around each equilibrium points is studied. Recall that the existence of non hyperbolic equilibrium point of system (2) is the necessary but not sufficient condition for bifurcation to occur. Therefore, in the following theorems an application to the Sotomayor's theorem [19] for local bifurcation is adapted.

Now, according to Jacobian matrix of system (2) given by Eq. (4.1) in [26], it is clear to verify that for any nonzero vector $V = (v_1, v_2, v_3, v_4)^T$ we have :

$$D^2F(V,V) = \begin{bmatrix} -2v_1\left(v_1+(1+u_1)v_2+v_3\right) \\ 2v_2\left(u_1v_1-u_4v_3\right) \\ 2v_3\left(u_8v_1+u_9(1-m)v_2-u_6v_4\right) \\ 2v_3\left(u_9mv_2+u_6v_4\right) \end{bmatrix} \tag{4.a}$$

and $D^3F(V,V,V) = (0,0,0,0)^T$.

So, according to Sotomayor'stheorem the pitchfork bifurcation does not occur at each point E_i,i = 0, 1, 2.

4.1 The Local Bifurcation Analysis Near E_0

Theorem 2: *Assume that the following condition holds:*

$$\mu_1 \neq \mu_2 \tag{4.b}$$

Where

$\mu_1 = \frac{u_1}{u_5}(v_1^{[0]})^2\psi_2^{[0]}+(\frac{u_5u_8+ u_9(1-m)}{u_5}) v_1^{[0]}v_3^{[0]}\psi_3^{[0]}+(\frac{u_7+ u_{10}}{u_7})v_3^{[0]}\psi_3^{[0]}$

$(\frac{u_9m}{u_5} v_1^{[0]} + \frac{u_6(u_7+u_{10})}{u_{11}} v_3^{[0]})$.

$\mu_2 = (\frac{u_3+u_5}{u_3}) v_1^{[0]}\psi_2^{[0]} ((\frac{u_5+u_1+1}{u_5}) v_1^{[0]}+ v_3^{[0]}) +(\frac{u_4}{u_5})v_1^{[0]}v_3^{[0]}\psi_2^{[0]} + \frac{u_6(u_7+u_{10})}{u_{11}}$

$(v_3^{[0]})^2\psi_3^{[0]}$.

Then, the system (2) near the vanishing equilibrium point E_0 with the parameter $u_2^* = 1+\frac{u_3}{u_5}$ has:

1. No saddle- node bifurcation.
2. Transcritical bifurcation.

Proof: According to the Jacobian matrix J_0given by Eq.(4.2) in [26], the system (2) at the equilibrium point E_0has zero eigenvalue (sayλ_{0x}= 0) at $u_2 = u_2^*$, and the Jacobian matrix J_0 with$u_2 = u_2^*$ becomes:

$$J_0^* = J_0(u_2^*) = \begin{bmatrix} -\frac{u_3}{u_5} & u_3 & 0 & 0 \\ u_2^* & -(u_3+u_5) & 0 & 0 \\ 0 & 0 & -(u_7+u_{10}) & u_{11} \\ 0 & 0 & u_7 & -(u_{10}+u_{11}+u_{12}) \end{bmatrix}$$

Now, let $V^{[0]} = (v_1^{[0]}, v_2^{[0]}, v_3^{[0]}, v_4^{[0]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{0x} = 0$. Thus $(J_0^* - \lambda_{0x} I) V^{[0]} = 0$, which gives:

$v_2^{[0]} = \frac{1}{u_5} v_1^{[0]}$, $v_4^{[0]} = \frac{(u_7+u_{10})}{u_{11}} v_3^{[0]}$ and $v_1^{[0]}$, $v_3^{[0]}$ are any nonzero real numbers.

Let $\psi^{[0]} = (\psi_1^{[0]}, \psi_2^{[0]}, \psi_3^{[0]}, \psi_4^{[0]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{0x} = 0$ of the matrix J_0^{*T}. Then we have $(J_0^{*T} - \lambda_{0x} I)\psi^{[0]} = 0$. By solving this equation for $\psi^{[0]}$ we obtain

$\psi^{[0]} = \left(\frac{(u_3+u_5)}{u_3}\psi_2^{[0]}, \psi_2^{[0]}, \psi_3^{[0]}, \frac{(u_7+u_{10})}{u_7}\psi_3^{[0]}\right)^T$, where $\psi_2^{[0]}$ and $\psi_3^{[0]}$ are any nonzero real numbers.

Now, consider:

$\frac{\partial F}{\partial u_2} = F_{u_2}(X, u_2) = (\frac{\partial F_1}{u_2}, \frac{\partial F_2}{u_2}, \frac{\partial F_3}{u_2}, \frac{\partial F_4}{u_2})^T = (-x, x, 0, 0)^T$.

So, $F_{u_2}(E_0, u_2^*) = (0, 0, 0, 0)^T$ and hence $(\psi^{[0]})^T F_{u_2}(E_0, u_2^*) = 0$.

Thus, according to Sotomayor'stheorem for local bifurcation,the saddle-nod bifurcation can't occur. While the first condition of transcritical bifurcation is satisfied. Now, since

$$DF_{u_2}(X, u_2) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Where $DF_{u_2}(X, u_2)$ represents the derivative of $F_{u_2}(X, u_2)$ with respect to $X = (x, y, z, w)^T$. Further, it is observed

$$DF_{u_2}(E_0, u_2^*)V^{[0]} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{[0]} \\ \dfrac{1}{u_5} v_1^{[0]} \\ v_3^{[0]} \\ \dfrac{(u_7 + u_{10})}{u_{11}} v_3^{[0]} \end{bmatrix} = \begin{bmatrix} -v_1^{[0]} \\ v_1^{[0]} \\ 0 \\ 0 \end{bmatrix}$$

$(\psi^{[0]})^{T}\left[DF_{u_2}(E_0, u_2^*)V^{[0]}\right] = \left(\frac{(u_3+u_5)}{u_3}\psi_2^{[0]}, \psi_2^{[0]}, \psi_3^{[0]}, \frac{(u_7+u_{10})}{u_7}\psi_3^{[0]}\right)$

$(-v_1^{[0]}, v_1^{[0]}, 0, 0)^{T}$

$= -\frac{u_5}{u_3} v_1^{[0]} \psi_2^{[0]} \neq 0.$

Moreover, by substituting E_0 , u_2^* and $V^{[0]}$ in (4.a) we get:

$$D^2F(E_0, u_2^*)(V^{[0]}, V^{[0]}) = \begin{bmatrix} -2v_1^{[0]}\left(v_1^{[0]} \dfrac{(u_5 + u_1 + 1)}{u_5} + v_3^{[0]}\right) \\ \dfrac{2}{u_5} v_1^{[0]}\left(u_1 v_1^{[0]} - u_4 v_3^{[0]}\right) \\ 2v_3^{[0]}\left(v_1^{[0]} \dfrac{(u_5 u_8 + u_9(1-m))}{u_5} - \dfrac{u_6(u_7 + u_{10})}{u_{11}} v_3^{[0]}\right) \\ 2v_3^{[0]}\left(\dfrac{u_9 m}{u_5} v_1^{[0]} + \dfrac{u_6(u_7 + u_{10})}{u_{11}} v_3^{[0]}\right) \end{bmatrix}.$$

Hence, it is obtain that:

$(\psi^{[0]})^{T}\left[D^2F(E_0, u_2^*)\left(V^{[0]}, V^{[0]}\right)\right] = 2(\mu_1 - \mu_2).$

According to condition (4.b) we obtain that:

$(\psi^{[0]})^{T}\left[D^2F(E_0, u_2^*)\left(V^{[0]}, V^{[0]}\right)\right] \neq 0$.

Thus, according to Sotomayor'stheorem system (2) has transcritical bifurcation atE_0 with the parameter $u_2 = u_2^*$. Otherwise, when condition (4.b) does not satisfied,the system (2) has no any type of bifurcation and thiscomplete the proof. ■

4.2 The Local Bifurcation Analysis Near E_1

Theorem 3: *Assume that left the condition (3.b) holds and let the following conditions hold*

$u_3 + u_5 + 2u_1(x^{\wedge})^2 > (u_1 + 2(u_3 + u_5))\, x^{\wedge} + (u_3 + u_5(1 + u_1))\, y^{\wedge}$ (4.c)

$u_2^{\#} + 2x^{\wedge} + (1 + u_1)y^{\wedge} > 1$ (4.d)

$(u_3 + u_5)(u_2^{\#} + 2x^{\wedge} + (1 + u_1)y^{\wedge} - 1) + (u_2^{\#} + u_1 y^{\wedge})((1+u_1)x^{\wedge} - u_3)$
$\neq u_1 x^{\wedge}(1 - (u_2^{\#} + 2x^{\wedge} + (1 + u_1)y^{\wedge}))$ (4.e)

$u_3 + u_5 > u_1 x^{\wedge} + u_4(1 + u_1)y^{\wedge}$ (4.f)

$u_7 + u_{10} < u_8 x^{\wedge} + u_9(1 - m)y^{\wedge}$ (4.g)

$\alpha_1 \neq \alpha_2.$ (4.h)

Where:

$\alpha_1 = p_1 p_2 q_1 q_2 (1 + u_1) q_3 + p_1 p_3 q_1 q_3 u_4 + p_4 q_1 q_2 u_8 + u_6$.

$\alpha_2 = p_1 p_2 q_1 q_2 (1 + q_2) + p_1 p_3 q_1 q_2 q_3 u_1 + p_4(u_9(1 - m) q_1 q_3 + u_6) + u_8 m q_1 q_3.$

Here:

$q_1 = \frac{(u_{10}+u_{11}+u_{12})}{u_7+u_9 m y^{\wedge}}$, $q_2 = \frac{u_4 y^{\wedge} + (u_1 x^{\wedge} - (u_3+u_5)) r_1 r_2}{(u_2^{\#}+u_1 y^{\wedge})}$, $q_3 = \frac{r_1}{r_2}$.

$p_1 = \frac{r_3}{u_{11} r_4}$, $p_2 = u_1 x^{\wedge} - (u_3 + u_5)$, $p_3 = (1 + u_1)x^{\wedge} - u_3$, $p_4 = \frac{(u_{10}+u_{11}+u_{12})}{u_{11}}$.

With:

$r_1 = u_4 y^{\wedge}(u_2^{\#} + 2x^{\wedge} + (1 + u_1)y^{\wedge} - 1) + (u_2^{\#} + u_1 y^{\wedge})x^{\wedge}$.

$r_2 = (u_3 + u_5)(u_2^{\#} + 2x^{\wedge} + (1 + u_1)y^{\wedge} - 1) + (u_2^{\#} + u_1 y^{\wedge})((1+u_1)x^{\wedge} - u_3) +$

$u_1 x^{\wedge}(1 - (u_2^{\#} + 2x^{\wedge} + (1 + u_1)y^{\wedge})).$

$r_3 = u_{11}((u_{10} + u_{12})(u_7 + u_{10} - (u_8 x^{\wedge} + u_9(1 - m)y^{\wedge}) - (u_7 + u_9 m y^{\wedge}))$.

$r_4 = x^{\wedge}(u_3 + u_5 - (u_1 x^{\wedge} + u_4(1 + u_1)y^{\wedge})) + u_3 u_4 y^{\wedge}.$

Then system (2) near the predator free equilibrium point E_1 with the parameter

$u_2^{\#} = \frac{\left(u_3+u_5+2u_1 x^{\wedge 2}\right) - \left(\left(u_1+2(u_3+u_5)\right)x^{\wedge} + \left(u_3+u_5(1+u_1)\right)y^{\wedge}\right)}{u_5+x^{\wedge}}$, has:

1. No transcritical bifurcation.
2. Saddle-node bifurcation.

Proof: According to the Jacobian matrix J_1 given by Eq. (4.5) in [26], the system (2) at the equilibrium point E_1 has zero eigenvalue (say λ_{1y} = 0) at $u_2 = u_2^{\#}$, it is clearly that $u_2^{\#} > 0$ provided that condition (4.c) holds, and the Jacobian matrix J_1 with $u_2 = u_2^{\#}$ becomes:

$J_1^{\#} = J_1(u_2^{\#}) = [j_{ij}]_{4\times4}$, where $j_{ij} = a_{ij}$ for all i, j = 1,2,3,4 except j_{11} & j_{21}

which are given by: $j_{11} = 1 - (u_2^{\#} + 2x^{\wedge} + (1+u_1)y^{\wedge})$ & $j_{21} = u_2^{\#} + u_1 y^{\wedge}$.

Now, let $V^{[1]} = (v_1^{[1]}, v_2^{[1]}, v_3^{[1]}, v_4^{[1]})^T$ be the eigenvector corresponding to the eigenvalue λ_{1y} = 0. Thus $(J_1^{\#} - \lambda_{1y} I)V^{[1]} = 0$, which gives:

$v_1^{[1]} = q_1 q_2 v_4^{[1]}$, $v_2^{[1]} = -q_1 q_3 v_4^{[1]}$ & $v_3^{[1]} = q_1 v_4^{[1]}$ here $v_4^{[1]}$ is any nonzero real number, according to left the condition (3.b) and (4.d), (4.e) we have $v_2^{[1]}$ exist.

Let $\psi^{[1]} = (\psi_1^{[1]}, \psi_2^{[1]}, \psi_3^{[1]}, \psi_4^{[1]})^T$ be the eigenvector associated with the eigenvalue λ_{1y} = 0 of the matrix $J_1^{\#T}$. Then we have $(J_1^{\#T} - \lambda_{1y} I)\psi^{[1]} = 0$. By solving this equation for $\psi^{[1]}$ we obtain:

$\psi^{[1]} = (p_1 p_2 \psi_4^{[1]}, p_1 p_3 \psi_4^{[1]}, p_4 \psi_4^{[1]}, \psi_4^{[1]})^T$ here $\psi_4^{[1]}$ is any nonzero real number.

It is clear that $\psi_1^{[1]}, \psi_2^{[1]}$ exists under the condition (4.f).

Now, since

$\frac{\partial F}{\partial u_2} = F_{u_2}(X, u_2) = (\frac{\partial F_1}{u_2}, \frac{\partial F_2}{u_2}, \frac{\partial F_3}{u_2}, \frac{\partial F_4}{u_2})^T = (-x, x, 0, 0)^T$, where $X = (x, y, z, w)^T$.

So, $F_{u_2}(E_1, u_2^{\#}) = (-x^{\wedge}, x^{\wedge}, 0, 0)^T$ and hence

$(\psi^{[1]})^T F_{u_2}(E_1, u_2^{\#}) = p_1 x^{\wedge} \psi_4^{[1]}(p_3 - p_2) = p_1 x^{\wedge} \psi_4^{[1]}(5 + x^{\wedge}) \neq 0$, where $p_1 \neq 0$

Under the condition (4.f) & (4.g) .Thus, according to the Sotomayor's theorem for local bifurcation, the transcritical bifurcation can't occur while the first condition of saddle-node bifurcation is satisfied. Further, by substituting E_1 , $u_2^{\#}$ and $V^{[1]}$ in (4.a) we get:

$$D^2F(E_1, u_2^{\#})(V^{[1]}, V^{[1]}) = \begin{bmatrix} -2q_1q_2v_4^{[1]}\left(q_1q_2v_4^{[1]} + q_1v_4^{[1]} - (1+u_1)q_1q_3v_4^{[1]}\right) \\ -2q_1q_3v_4^{[1]}\left(u_1q_1q_2v_4^{[1]} - u_4q_1v_4^{[1]}\right) \\ 2q_1v_4^{[1]}\left(u_8q_1q_2v_4^{[1]} - (u_9(1-m)q_1q_3v_4^{[1]} + u_6v_4^{[1]})\right) \\ 2q_1v_4^{[1]}\left(u_6v_4^{[1]} - u_9mq_1q_3v_4^{[1]}\right) \end{bmatrix}.$$

Hence, it is obtain that:

$(\psi^{[1]})^T[D^2F(E_1,u_2^{\#})(V^{[1]},V^{[1]})] = 2q_1(v_4^{[1]})^2\psi_4^{[1]}(\alpha_1 - \alpha_2)$.

According to condition (4.h) we obtain that:

$(\psi^{[1]})^T[D^2F(E_1,u_2^{\#})(V^{[1]},V^{[1]})] \neq 0$, and hence system (2) hassaddle-node bifurcation at E_1 with the bifurcation point given by $u_2^{\#}$ and thiscomplete the proof.■

4.3 The Local Bifurcation Analysis Near E_2

In order to study the local bifurcation analysis near the positive equilibrium point $E_2 = (x^*, y^*, z^*, w^*)$ of system (2) in the Int. R_+^4. Note the following, according to the Jacobian matrix J_2given by Eq. (4.13) in [26], the characteristic equation of J_2, can be written as:

$$\lambda^4 + C_1\lambda^3 + C_2\lambda^2 + C_3\lambda + C_4 = 0 \tag{4.i}$$

Where the coefficients:

$\boldsymbol{C_1} = -(b_{44} + b_{11} + b_{22})$,

$\boldsymbol{C_2} = b_{44}(b_{11} + b_{22} + b_{33}) + b_{11}(b22 + b_{33}) + b_{22}b_{33} + b_{23}b_{42} + b_{31}b_{13} - (b_{34}b_{43} + b_{33} + b^2_{12})$,

$\boldsymbol{C_3} = b_{34}(b23\, b_{42} + b_{11}b_{43} + b_{22}b_{43}) + b_{44}(b_{23}b_{42} + b^2_{12} + b_{31}b_{13}) + b_{11}b_{23}b_{42} + b_{12}b_{21}b_{33} + b_{22}b_{31}b_{13} - (b_{11}(b_{22}b_{44} + b_{33}b_{44} + b_{22}b_{33}) + b_{11}b_{22}b_{33} + b_{21}b_{32}b_{13})$,

$\boldsymbol{C_4} = b_{34}(b_{23}b_{42} + b_{12}b_{21}b_{43} - b_{23}b_{42}(u_2 + 2x^* + (1+u_1)y^* + z^*) - b_{31}b_{42}b_{21} - b_{11}b_{22}b_{43}) + b_{44}(b_{11}b_{22}b_{33} + b_{21}b_{32}b_{13} - b_{11}b_{23}b_{42} - b_{12}b_{21}b_{33} - b_{12}b_{31}b_{23} - b_{22}b_{31}b_{13}) - b_{12}b_{23}b_{31}$

Note that, according to the elements of J_2, it is easy to verify that:

$C_1 = k_1 - k_2$
$C_2 = k_3 - k_4$
$C_3 = k_5 - k_6$
$C_4 = k_7 - k_8$

Further:

$\Delta_1 = C_1C_2 - C_3$.

$= k_1k_3 + k_2k_4 + k_6 - (k_1k_4 + k_2k_3 + k_5)$,

and

$\Delta_2 = C_3(C_1C_2 - C_3) - C_1^2C_4$.

Where:

k_1 $= u_3 + u_5 + u_{10} + u_{11} + u_{12} + 2x^* + (1+u_1)\, y^* + (1+u_4)\, z^*$.

k_2 $= 1 + u_1x^* + u_6 z^*$.

k_3 $= u_6z^*(1+ u_7+u_9my^*+ u_6w^*)+(u_{10} + u_{11} + u_{12})(u_2+2x^*+(1+u_1)\, y^*+(1+ u_4)\, z^*+ u_3 + u_5 + u_7 + u_{10} + u_6w^*) + (u_8x^*+u_9(1-m)\, y^*)(u_6z^*+u_1x^*) + (u_2+2x^*+(1+u_1)\, y^*+ z^*)(u_4 z^*+ u_3+ u_5 + u_7 + u_{10} + u_6w^*) + u_1x^*(1+ u_6 z^*) + 2u_3(1+u_1)x^*+ (u_3 + u_5+ u_4 z^*)(u_7 + u_{10} + u_6w^*)$.

k_4 $= (u_{10} + u_{11} + u_{12})(1+ u_1x^*+ u_8x^*+u_9(1-m)\, y^*) +u_6z^*(u_2+2x^*+(1+u_1)\, y^*+(1+ u_4)\, z^*+ u_3 +u_5+ u_7 +u_{10}+ u_6w^*)+ (u_8x^*+u_9(1-m)\, y^*)(u_2+2x^*+(1+u_1)\, y^*+(1+ u_4)\, z^*+ u_3 + u_5) + u_1x^*(u_2+2x^*+(1+u_1)\, y^*+ z^*+ u_7 + u_{10}+ u_6w^*) +(u_3 + u_5+ u_4 z^*) + u_{11}(u_7 + u_9my^*+ u_6w^*) + ((1+u_1)\, x^*)^2 + u_4u_9my^*+ u_8x^*z^* + u_3^2$.

k_5 $= u_{11}(u_7+u_9my^*+u_6w^*)(1+u_1x^*) + u_6z^*(u_4u_9my^*z^* + (u_7+u_9my^*+ u_6w^*)(u_2+2x^*+(1+u_1)\, y^*+(1+u_4)z^*+u_3+u_5) + ((1+u_1)\, x^*)^2 + u_3^2) + (u_{10} + u_{11} + u_{12})(u_4u_9my^*z^*+2u_3(1+u_1)\, x^* + u_8x^*z^*) + u_3(u_2+u_1y^*)(u_8x^*+u_9(1-m)\, y^*) + u_4u_9my^*z^*(u_2+2x^*+(1+u_1)\, y^*+ z^*+1) + (u_2+u_1y^*)(u_7 + u_{10} + u_6w^*)(1+u_1)\, x^*+ u_8x^*z^*(u_3 + u_5+ u_4 z^*) + m_4 + m_2 + m_1(u_2+2x^*+(1+u_1)\, y^*+ z^*)$.

k_6 $= u_{11}(u_4u_9my^*z^*+ (u_7 + u_9my^* + u_6w^*)(u_2+2x^*+(1+u_1)\, y^* + (1+ u_4)z^*+ u_3 +u_5)) + (u_{10}+u_{11}+u_{12})((1+u_1)x^*)^2 + u_3^2) + u_6z^*(u_4u_9my^*z^*+2u_3(1+u_1)\, x^*+ u_8x^*z^*+ (u_7+u_9my^*+u_6w^*)(1 + u_1x^*))+(u_2+u_1y^*)(u_8x^*+u_9(1-m)\, y^*)(1+u_1)x^* + u_3(u_2+u_1y^*)(u_7 + u_{10}+ u_6w^*) + u_8u_1x^{*2}z^* + m_3 + m_1 + m_2(u_2+2x^*+(1+u_1)\, y^*+ z^*)$.

k_7 $= u_{11}(m_5 + m_8(u_{10}+u_{12})) + u_6z^*(m_6 + m_7(u_{10}+u_{12})) + u_4u_8(1+u_1)\, x^*y^*z^*$

k_8 $=u_{11}(m_6 + m_7(u_{10}+u_{12})) + u_6z^*(m_5 + m_8(u_{10}+u_{12}))\, u_3u_4u_8y^*z^*$

Here:

m_1 $= (u_{10}+ u_{11}+ u_{12})(u_3 + u_5+ u_4 z^*+ u_7 + u_{10} + u_6w^*)+ u_6z^*(u_8x^*+u_9(1-m)\, y^* + u_1x^*) +u_1x^*(u_8x^*+u_9(1-m)\, y^*) +(u_3 + u_5+ u_4 z^*)(u_7 + u_{10} + u_6w^*)$

m_2 $=u_1x^*(2u_{10}+ u_{11}+ u_{12}+ u_7 +u_6w^*)+u_6 z^*(u_3 + u_5+ u_4 z^*)+(u_8x^*+u_9(1-m)\, y^*)(u_7 +u_{10} +u_6w^*+u_3 +u_5+u_4 z^*)$.

m_3 $= u_1x^*(1+ (u_2+2x^*+(1+u_1)\, y^*+ z^*)(u_7+u_{10} + u_6w^*)) + (u_3+u_5+u_4 z^*)(u_8x^*+u_9(1-m)\, y^*)(u_2+2x^*+(1+u_1)y^* + z^*)$.

m_4 = $(u_3+u_5+u_4z^*)\ (1+(u_2+2x^*+(1+u_1)y^*+z^*)\ (u_7+u_{10}+u_6w^*)) + u_1x^*(u_2+2x^*+(1+u_1)\ y^*+ z^*)\ (u_8x^*+u_9(1\text{-}m)\ y^*) + u_9(1\text{-}m)\ (u_2+u_1y^*)\ x^*z^*$.

m_5 = $(u_2+2x^*+(1+u_1)y^*+z^*)\ (u_4u_9my^*z^*+u_1x^*(u_7+u_9my^*+u_6w^*)) +(u_7+u_9my^*+u_6w^*)\ (u_3(u_2+u_1y^*)+(u_3+u_5+u_4 z^*)$.

m_6 = $(u_7+u_9my^*+u_6w^*)\ (u_1x^*+(1+u_1)\ x^*(u_2+u_1y^*) + (u_3+u_5+ u_4z^*)\ (u_2+2x^*+(1+u_1)y^*+z^*)) + u_9mz^*(u_4y^*+u_8z^*(u_2+u_1y^*))$.

m_7 = $\sigma_2 + (u_2+2x^*+(1+u_1)y^*+z^*)\ \sigma_1 + (u_2+u_1y^*)\ \sigma_4 + u_8z^*\sigma_5$

m_8 = $\sigma_1 + (u_2+2x^*+(1+u_1)y^*+z^*)\ \sigma_2 + (u_2+u_1y^*)\ \sigma_3 + u_8z^*\ \sigma_6$.

With:

σ_1 = $u_1x^*(u_8x^*+u_9(1\text{-}m)\ y^*) + (u_3 + u_5+ u_4 z^*)(u_7+u_9my^*+u_6w^*) + u_4u_9my^*z^*$.

σ_2 = $u_1x^*(u_7+u_9my^*+u_6w^*) + (u_3 + u_5+ u_4 z^*)(u_8x^*+u_9(1\text{-}m)\ y^*)$.

σ_3 = $u_3\ (u_7 + u_{10} + u_6w^*) + (1+u_1)\ x^*(u_8x^*+u_9(1\text{-}m)\ y^*)$.

σ_4= $u_9(1\text{-}m)x^*z^* + u_3(u_8x^*+u_9(1\text{-}m)\ y^*) +(1+u_1)\ x^*(u_7+u_9my^*+u_6w^*)$.

σ_5 = $u_4y^*(1+u_1)\ x^* + (u_3 + u_5+ u_4 z^*)\ x^*$, $\sigma_6 = u_1x^{*2} + u_3u_4y^*$.

According to described above, the local bifurcation analysis near the positive equilibrium point E_2 of system (2) can be derived easily as shown in the following theorem.

Theorem 4: *Suppose that the following conditions*

$$x^* > \frac{u_3}{1+u_1} \tag{4.j}$$

$$m_7 > m_8 \ \& \ m_5 < m_6 + (u_{10}+u_{12})\ (m_7-m_8) \tag{4.k}$$

$$u_2^* + 2x^* + (1 + u_1)y^* + z^* > 1 \tag{4.L}$$

$$(u_3+u_5+u_4 z^*)\ (u_2+2x^* + (1+u_1)y^*+z^*-1) + (u_2+u_1y^*)\ ((1+u_1)\ x^* - u_3) \neq u_1x^*(1-(u_2+2x^*+ (1+u_1)\ y^*+z^*)) \tag{4.m}$$

$$z^* > \max\left\{\frac{u_{11}^*}{u_6}\ ,\ \frac{u_{10}+u_{11}^*+u_{12}}{u_6}\right\} \tag{4.n}$$

$$\beta_1 \neq \beta_2 \tag{4.o}$$

where:

$\beta_1 = t_1 h_1(h_1 + (1+u_1)h_2 + 1) + t_2 h_1 h_2 u_1 + u_9 m h_2 + t_3\,(u_8 h_1 + u_9(1-m)h_2 + u_6 h_3)$.

$\beta_2 = t_2 h_2 u_4 + u_6 h_3$.

Here:

$h_1 = \frac{((1+u_1)x^*-u_3)n_1 + x^* n_2}{(1-(u_2+2x^*+(1+u_1)y^*+z^*))n_2}$, $h_2 = \frac{n_1}{n_2}$, $h_3 = \frac{(u_9 m z^* n_1 + (u_7+u_9 m y^* + u_6 w^*)n_2)}{(u_6 z^* - (u_{10}+u_{11}^*+u_{12}))n_2}$.

$$t_1 = \frac{(u_2+u_1y^*)\left((u_6 z^*-(u_{10}+u_{11}^*+u_{12}))(n_3+u_8 z^* n_2) - n_4 u_9 m z^*\right)}{n_2 n_4},$$

$$t_2 = \frac{(u_6 z^*-(u_{10}+u_{11}^*+u_{12}))n_3 - n_4 u_9 m z^*}{(u_{11}^*-u_6 z^*)n_2}$$

$$t_3 = \frac{-(u_6 z^*-(u_{10}+u_{11}^*+u_{12}))}{(u_{11}^*-u_6 z^*)}.$$

With:

$n_1 = u_4 y^*\,(1-(u_2+2x^*+(1+u_1)\,y^*+z^*)) - (u_2+u_1 y^*)\,x^*$.

$n_2 = u_1 x^*\,(1-(u_2+2x^*+(1+u_1)\,y^*+z^*)) + (u_3+u_5+u_4\,z^*)\,(u_2+2x^*+(1+u_1)y^*+z^*-1) + (u_2+u_1 y^*)\,((1+u_1)\,x^*+u_3)$.

$n_3 = u_9(1-m)\,z^*(1-(u_2+2x^*+(1+u_1)\,y^*+z^*)) + u_8 z^*((1+u_1)\,x^*+u_3)$.

$n_4 = (u_6 z^* - u_{11}^*)(u_2+2x^*+(1+u_1)\,y^*+z^*-1)$.

are satisfied. Then for the parameter value

$u_{11}^* = -\frac{\left(u_4 u_8 z^* y^*((1+u_1)x^*-u_3)\right)}{m_5-(m_6+(u_{10}+u_{12})(m_7-m_8))} + u_6\,z^*$, system (2) at the equilibrium point E_2 has:

1. No transcritical bifurcation.
2. Saddle-node bifurcation.

Proof: The characteristic equation of J_2 that given by Eq. (4.i) having zero eigenvalue (say $\lambda = 0$) if and only if $C_4=0$ and then E_2 becomes a nonhyperbolic equilibrium point. Now, by substituting the value of u_{11}^* in C_4 we get:

$(u_{11}^{*} - u_6 z^{*})(m_5 - (m_6 + (u_{10} + u_{12})(m_7 - m_8))) + u_4 u_8 y^{*} z^{*}((1+u_1)x^{*} - u_3) = 0$, where $u_{11}^{*} > 0$ under the conditions (4.j) & (4.k). Clearly the Jacobian matrix of system (2) at the equilibrium point E_2 with parameter $u_{11} = u_{11}^{*}$ becomes:

$J_2^{*} = J_2(u_{11}^{*}) = [c_{ij}]_{4x4}$, where $c_{ij} = b_{ij}$ for all $i, j = 1,2,3,4$ except c_{34} & c_{44} which are given by:

$c_{34} = u_{11}^{*} - u_6 z^{*}$ & $c_{44} = u_6 z^{*} - (u_{10} + u_{11}^{*} + u_{12})$.

Now, let $V^{[2]} = (v_1^{[2]}, v_2^{[2]}, v_3^{[2]}, v_4^{[2]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda = 0$. Thus

$(J_2^{*} - \lambda I)V^{[2]} = 0$, which gives:

$v_1^{[2]} = h_1 v_3^{[2]}$, $v_2^{[2]} = h_2 v_3^{[2]}$, $v_4^{[2]} = -h_3 v_3^{[2]}$ and $v_3^{[2]}$ is any nonzero real number.

It is clear that $v_1^{[2]}$ and $v_2^{[2]}$ exists under the conditions (4.j), (4.L) & (4.m), while $v_4^{[2]}$ exist under the conditions (4.j), (4.L), (4.m) & (4.n).

Let $\psi^{[2]} = (\psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]}, \psi_4^{[2]})^T$ be the eigenvector associated with the eigenvalue $\lambda = 0$ of the matrix J_2^{*T}. Then we have $(J_2^{*T} - \lambda I)\psi^{[2]} = 0$. By solving this equation for $\psi^{[2]}$ we obtain:

$\psi^{[2]} = (-t_1\psi_4^{[2]}, t_2\psi_4^{[2]}, t_3\psi_4^{[2]}, \psi_4^{[2]})^T$ here $\psi_4^{[2]}$ is any nonzero real number. Clearly, according to conditions (4.j), (4.L), (4.m) & (4.n) we have $\psi_1^{[2]}$ and $\psi_2^{[2]}$ exists, while $\psi_3^{[2]}$ exist under the condition (4.n).

Now, since

$\frac{\partial F}{\partial u_{11}} = F_{u_{11}}(X, u_{11}) = (\frac{\partial F_1}{u_{11}}, \frac{\partial F_2}{u_{11}}, \frac{\partial F_3}{u_{11}}, \frac{\partial F_4}{u_{11}})^T = (0, 0, w, -w)^T$, where

$X = (x, y, z, w)^T$.

So, $F_{u_{11}}(E_2, u_{11}^{*}) = (0, 0, w^{*}, -w^{*})^T$

and hence

$(\psi^{[2]})^T F_{u_{11}}(E_2, u_{11}^{*}) = w^{*}\psi_4^{[2]}(t_3 - 1) = w^{*}\psi_4^{[2]}\left(\frac{u_{10}+u_{12}}{u_{11}^{*}-u_6 z^{*}}\right) \neq 0$, under condition (4.n)

So, according to the Sotomayor's theorem for local bifurcation, the transcritical bifurcation can't occur while the first condition of saddle-node bifurcation is satisfied. Further, by substitutingE_2, u_{11}^* and $V^{[2]}$ in (4.a) we get:

$$D^2F(E_2, u_{11}^*)(V^{[2]}, V^{[2]}) = \begin{bmatrix} -2h_1v_3^{[2]}\left(h_1v_3^{[2]} + (1+u_1)h_2v_3^{[2]} + v_3^{[2]}\right) \\ 2h_2v_3^{[2]}\left(u_1h_1v_3^{[2]} - u_4v_3^{[2]}\right) \\ 2v_3^{[2]}\left(u_8h_1v_3^{[2]} + u_9(1-m)h_2v_3^{[2]} + u_6h_3v_3^{[2]}\right) \\ 2v_3^{[2]}\left(u_9mh_2v_3^{[2]} - u_6h_3v_3^{[2]}\right) \end{bmatrix}.$$

Hence, it is obtain that:

$$(\psi^{[2]})^T\left[D^2F(E_2, u_{11}^*)\left(V^{[2]}, V^{[2]}\right)\right] = 2(v_3^{[2]})^2\psi_4^{[2]}(\beta_1 - \beta_2).$$

According to condition (4.o) we obtain that:

$(\psi^{[2]})^T\left[D^2F(E_2, u_{11}^*)\left(V^{[2]}, V^{[2]}\right)\right] \neq 0$, and hence system (2) has saddle-node bifurcation at E_2 with the bifurcation point given byu_{11}^* and thiscomplete the proof . ■

5 The Hopf Bifurcation Analysis of System (2)

In this section, the occurrence of Hopf-bifurcation near the equilibrium pointsof the system (**2**) is investigated as shown in thebelow.

5.1 The Hopf Bifurcation Analysis Near E_2

To discuss the possibility of Hopf bifurcation to occur,it should be noted the following:

The conditions of Hopf bifurcation for $n = 4$ are constructed according to the Haque and Venturino methods [21]. Consider the characteristic equation given by:

$$p_4(\gamma) = \gamma^4 + C_1\gamma^3 + C_2\gamma^2 + C_3\gamma + C_4 = 0$$

Here:

$C_1 = -\,tr\,(J(x^*))$, $C_2 = N_1(J(x^*))$,$C_3 = -N_2(J(x^*))$ and $C_4 = det(J(x^*))$ with $N_1(J(x^*))$ and $N_2(J(x^*))$ represent the sum of the determent of the principal minors of order two and three of $J(x^*)$ respectively. Clearly, the first condition of Hopfbifurcation holds if and only if

$$C_i > 0;\ i = 1,3;\ \Delta_1 = C_1C_2 - C_3 > 0;\ C_1^3 - 4\Delta_1 > 0;\ \Delta_2 = C_3(C_1C_2 - C_3) - C_1^2C_4 = 0,$$

consequently, $C_4 = \frac{C_3(C_1C_2 - C_3)}{C_1^2}$. So, the characteristic equation becomes:

$$p_4(\gamma) = \left(\gamma^2 + \frac{C_3}{C_1}\right)\left(\gamma^2 + C_1\gamma + \frac{\Delta_1}{C_1}\right) = 0. \quad (5.a)$$

Clearly, the roots of Eq.(5.a) are

$$\gamma_{1,2} = \pm\, i\sqrt{\frac{C_3}{C_1}} \text{ and } \gamma_{3,4} = \frac{1}{2}\left(-C_1 \pm \sqrt{C_1^2 - 4\frac{\Delta_1}{C_1}}\right).$$

Now, to verify the transversality condition of Hopf bifurcation, we substitute $\gamma(q) = \delta_1(q) \mp i\delta_2(q)$ into Eq. (5.a), and calculating its derivative with respect to the bifurcation parameter q, $\acute{p}_4(\gamma(q)) = 0$, also comparing the two sides of this equation and then equating their real and imaginary parts, we have:

$$\left.\begin{array}{l} \Psi(q)\acute{\delta}_1(q) - \Phi(q)\acute{\delta}_2(q) + \Theta(q) = 0 \\ \Phi(q)\acute{\delta}_1(q) + \Psi(q)\acute{\delta}_2(q) + \Gamma(q) = 0 . \end{array}\right\} \quad (5.b)$$

Where:

$$\left.\begin{array}{l} \Psi(q) = 4(\delta_1(q))^3 + 3C_1(q)(\delta_1(q))^2 + C_3(q) + 2C_2(q)\,\delta_1(q) - 12\delta_1(q)\delta_2^2(q) - 3C_1(q)(\delta_2(q))^2 \\ \Phi(q) = 12(\delta_1(q))^2\,\delta_2(q) + 6C_1(q)\,\delta_1(q)\delta_2(q) + 2C_2(q)\,\delta_2(q) - 4(\delta_2(q))^3 \\ \Theta(q) = (\delta_1(q))^3\,\acute{C}_1(q) + \acute{C}_3(q)\,\delta_1(q) + \acute{C}_2(q)(\delta_1(q))^2 + \acute{C}_4(q) - 3\acute{C}_1(q)\delta_1(q)(\delta_2(q))^2 - \acute{C}_2(q)(\delta_2(q))^2 \\ \Gamma(q) = 3(\delta_1(q))^2\,\delta_2(q)\acute{C}_1(q) + \acute{C}_3(q)\,\delta_2(q) + 2\acute{C}_2(q)\,\delta_1(q)\delta_2(q) - \acute{C}_1(q)(\delta_2(q))^3. \end{array}\right\} \quad (5.c)$$

Solving the linear system (5.b) by using Cramer's rule for the unknowns $\acute{\delta}_1(q)$ and $\acute{\delta}_2(q)$, gives that:

$$\acute{\delta}_1(q) = -\frac{\Theta(q)\Psi(q) + \Gamma(q)\Phi(q)}{(\Psi(q))^2 + (\Phi(q))^2}\ ; \ \acute{\delta}_1(q) = -\frac{\Gamma(q)\Psi(q) + \Theta(q)\Phi(q)}{(\Psi(q))^2 + (\Phi(q))^2}.$$

Therefore the second necessary and sufficient condition of Hopf bifurcation

$$\frac{d}{dq}(\mathrm{Re}(\gamma))\big|_{q=\tilde{q}} = \acute{\delta}_1(q)\big|_{q=\tilde{q}} \neq 0.$$

Will be satisfied if and only if

$$\Theta(\tilde{q})\Psi(\tilde{q}) + \Gamma(\tilde{q})\Phi(\tilde{q}) \neq 0 \quad (5.d)$$

Finally, according to the above results in the following theorem, the conditions of Hopf bifurcation of the positive equilibrium point E_2 are established.

Theorem 5: *Suppose that the conditions (4.j), (4.k) with the following conditions are satisfied:*

$k_1 > k_2 \& k_5 > k_6$ (5.e)

$k_1k_3 + k_2 k_4 + k_6 > k_1k_4 + k_2 k_3 + k_5$ (5.f)

$(k_1-k_2)^2 + 4k_4 > 4k_3$ (5.g)

$(k_1-k_2)^2 b > a$ (5.h)

$\Delta_1 > C_3 \& H_1 \neq H_2$ (5.i)

Where:

$a = (k_5 - k_6)(k_1k_3 + k_2 k_4 + k_6 - (k_1k_4 + k_2 k_3 + k_5)) > 0.$

$b = u_6z^*(m_6 + (u_{10}+u_{12}) (m_7-m_8) - m_5) + u_4u_8y^*z^*((1+u_1) x^* - u_3) > 0.$

$H_1 = \frac{(\Delta_1 - C_3)}{C_1^2} s_3 + \frac{C_3}{C_1} s_1 + m_6 + (u_{10}+u_{12}) (m_7-m_8) - m_5.$

$H_2 = \frac{(\Delta_1 - C_3)}{C_1^2} (s_4 + \frac{C_3}{C_1}) + \frac{C_3}{C_1} s_2.$

Here:

$s_1 = u_{10} + (u_2+2x^*+(1+u_1) y^* + z^*) + (u_3+u_5+u_4 z^*)$.

$s_2 = 1 + u_1x^* + u_9my^* + (u_8x^* + u_9(1-m) y^*)$.

$s_3 = u_1x^* + (u_7 + u_9my^* + u_6w^*) (1+u_1) x^* + (u_4u_9my^*z^* + 2u_3(1+u_1) x^* + u_8 x^*z^*) + (u_2+2x^*+(1+u_1) y^* + z^*) ((u_7 + u_9my^* + u_6w^*) + (u_3+u_5+u_4 z^*))$.

$s_4 = u_4u_9my^*z^* + (u_7 + u_9my^* + u_6w^*)(1 + (u_2+2x^*+(1+u_1) y^* + z^*) + (u_3+u_5+u_4 z^*)) + u_1x^*(u_2+2x^*+(1+u_1) y^* + z^*) + ((1+u_1) x^*)^2 + u_3^2)$.

Then at the parameter value $\tilde{u}_{11} = \frac{a - (k_1-k_2)^2 b}{m_5 - (m_6 + (u_{10}+u_{12})(m_7-m_8))(k_1-k_2)^2}$, the system (2) has a Hopf bifurcation near the point E_2 .

Proof: Consider the characteristic equation of the system (2) at E_2 which is given by Eq. (4.i). Now, to verify the necessary and sufficient conditions for a Hopf bifurcation to occur we need to find a parameter (say $\tilde{u}_{11}$) satisfy that:

$C_i(\tilde{u}_{11}) > 0$; i = 1, 3, $\Delta_1 (\tilde{u}_{11}) > 0, C_1^3(\tilde{u}_{11}) - 4\Delta_1(\tilde{u}_{11}) > 0$, $\Delta_2 (\tilde{u}_{11}) = 0$.

Where C_i ; i = 1, 3represent the coefficients of characteristic Eq.(4.i).Therefore it is observed that Δ_2 =0 gives:

$(k_5 - k_6)(k_1k_3 + k_2 k_4 + k_6 - (k_1k_4 + k_2 k_3 + k_5)) - (k_1-k_2)^2 (k_7 - k_8) =0$

It is easy to verify that, the parameter's value that satisfy the above equation is:

$$\tilde{u}_{11} = \frac{a - (k_1 - k_2)^2 b}{m_5 - \left(m_6 + (u_{10} + u_{12})(m_7 - m_8)\right)(k_1 - k_2)^2},$$

Where $\tilde{u}_{11}$ is a positive parameter under the conditions (4.j), (4.k) and (5.h). Now, at $u_{11} = \tilde{u}_{11}$ the characteristic equation given by Eq.(4.i) can be written as:

$$p_4(\lambda_2) = \left(\lambda_2^2 + \frac{C_3}{C_1}\right)\left(\lambda_2^2 + C_1\lambda_2 + \frac{\Delta_1}{C_1}\right) = 0.$$

Thus, the roots become $\lambda_{2\,x,y} = \pm\, i\sqrt{\frac{C_3}{C_1}}$ and $\lambda_{2\,z,w} = \frac{1}{2}\left(-C_1 \pm \sqrt{C_1^2 - 4\frac{\Delta_1}{C_1}}\right)$. Clearly, at $u_{11} = \tilde{u}_{11}$ there are two pure imaginary eigenvalues ($\lambda_{2\,x}$ & $\lambda_{2\,y}$) and two eigenvalues which are real and negative provided the conditions (5.f) and (5.g) holds .

Now for all values ofu_{11}in the neighborhood of $\tilde{u}_{11}$, the roots in general of the following form:

$\lambda_{2\,x} = \omega_1(u_{11}) + i\omega_2(u_{11})$ and $\lambda_{2\,y} = \omega_1(u_{11}) - i\omega_2(u_{11})$; $\lambda_{2\,z,w} = \frac{1}{2}\left(-C_1 \pm \sqrt{C_1^2 - 4\frac{\Delta_1}{C_1}}\right)$. Clearly, $Re(\lambda_{2\,x,y}(u_{11}))|_{u_{11}=\tilde{u}_{11}} = \omega_1(\tilde{u}_{11}) = 0$, that means the first condition of the necessary and sufficient conditions for Hopf bifurcation is satisfied at $u_{11} = \tilde{u}_{11}$.

Now to verify the transversality condition we must prove that$\Theta(\tilde{u}_{11})\Psi(\tilde{u}_{11}) + \Gamma(\tilde{u}_{11})\Phi(\tilde{u}_{11}) \neq 0$,where the form of Θ, Ψ, Γ and Φ are given in Eq.(5.c). Note that for $u_{11} = \tilde{u}_{11}$we have $\omega_1 = 0$ and $\omega_2 = \sqrt{\frac{C_3}{C_1}}$, substitution into (5.c) gives the following simplifications:

$$\Psi(\tilde{u}_{11}) = -2C_3(\tilde{u}_{11})$$

$$\Theta(\tilde{u}_{11}) = \acute{C}_4(\tilde{u}_{11}) - \frac{C_3}{C_1}\acute{C}_2(\tilde{u}_{11})$$

$$\Gamma(\tilde{u}_{11}) = \omega_2(\tilde{u}_{11})(\acute{C}_3(\tilde{u}_{11}) - \frac{C_3}{C_1}\acute{C}_1(\tilde{u}_{11}))$$

$$\Phi(\tilde{u}_{11}) = 2\frac{\omega_2(\tilde{u}_{11})}{C_1}(C_1C_2 - 2C_3).$$

Where:

$\acute{C}_1 = \frac{dC_1}{du_{11}}|_{u_{11}=\tilde{u}_{11}} = 1$

$\acute{C}_2 = \frac{dC_2}{du_{11}}|_{u_{11}=\tilde{u}_{11}} = s_1 - s_2$

$\acute{C}_3 = \frac{dC_3}{du_{11}}|_{u_{11}=\tilde{u}_{11}} = s_3 - s_4$

$\acute{C}_4 = \frac{dC_4}{du_{11}}|_{u_{11}=\tilde{u}_{11}} = m_5 - (m_6 + (u_{10}+u_{12}) (m_7-m_8))$.

Further, by substitution into Eq.(5.d) we get that:

$\Theta(\tilde{q})\Psi(\tilde{q}) + \Gamma(\tilde{q})\Phi(\tilde{q}) = H_1 - H_2 \neq 0$ yields, under the condition (5.i).

So, we obtain that the Hopf bifurcation occurs around the equilibrium point E_2 at the parameter $u_{11} = \tilde{u}_{11}$ and the proof is complete. ■

6 Numerical Simulation Analysis of System (2)

In this section the dynamical behavior of system (2) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions ofthe positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in Fig. (1).

Note that, from now onward the red, blue, sky blue and green colors are used to describing the trajectories of the susceptible prey *x*, infected prey *y*, susceptible predator *z* and infected predator *w* respectively.

$$\left.\begin{array}{l} u_1 = 0.5,\ u_2 = 0.1,\ u_3 = 0.1, u_4 = 0.5,\ u_5 = 0.5, \\ u_6 = 0.3,\ u_7 = 0.2,\ u_8 = 0.5,\ u_9 = 0.5,\ u_{10} = 0.1, \\ u_{11} = 0.3,\ u_{12} = 0.2,\ m= 0.6. \end{array}\right\} \qquad (6.1)$$

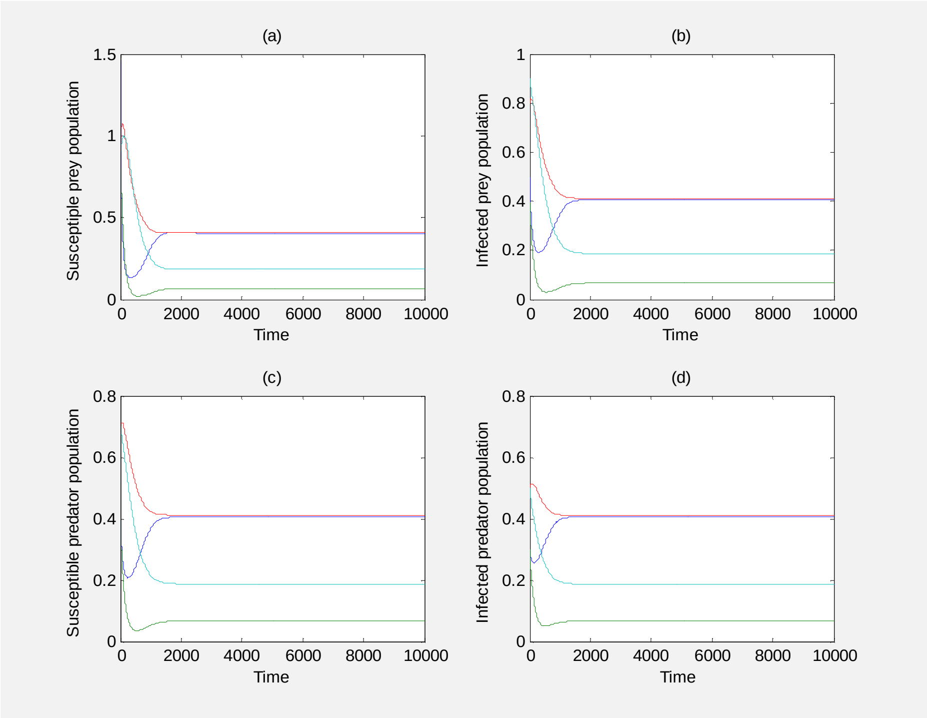

Fig. 1: Time series of the solution of system (2) that started from four different initial points (1.5, 0.8, 0.9, 0.9), (0.5, 0.4, 0.8, 0.9), (0.4, 0.4, 0.7, 0.7) and (0.3, 0.3, 0.5, 0.5) for the data given by Eq. (6.1). (a) trajectories of x as a function of time, (b) trajectories of y as a function of time, (c) trajectories of z as a function of time, (d) trajectories of w as a function of time.

Clearly, figure (1) shows that system (2) has a globally asymptotically stable as the solution of system (2) approaches asymptotically to the positive equilibrium pointE_2 = (0.41, 0.4, 0.18, 0.06) starting from four different initial points and this is confirming our obtained analytical results, see [26].

Now, in order to discuss the effect of the parameters values of system (2) on the dynamical behavior of the system, the system is solved numerically for the data given in Eq. (6.1) with varying one parameter each time. It is observed that varying the parameters values u_i; $i = 1,3,4,5,6,9,11,12$ and m, do not have any effect on the dynamical behavior of system (2) and the solution of the system still approaches to positive equilibrium point $E_2 = (x^*, y^*, z^*, w^*)$.

However, we note that varying the infection rates of susceptible prey and predator u_2 and u_7, respectively keeping other parameters fixed as given in Eq. (6.1), leads to occurrence of local bifurcation as shown in Fig. (2).

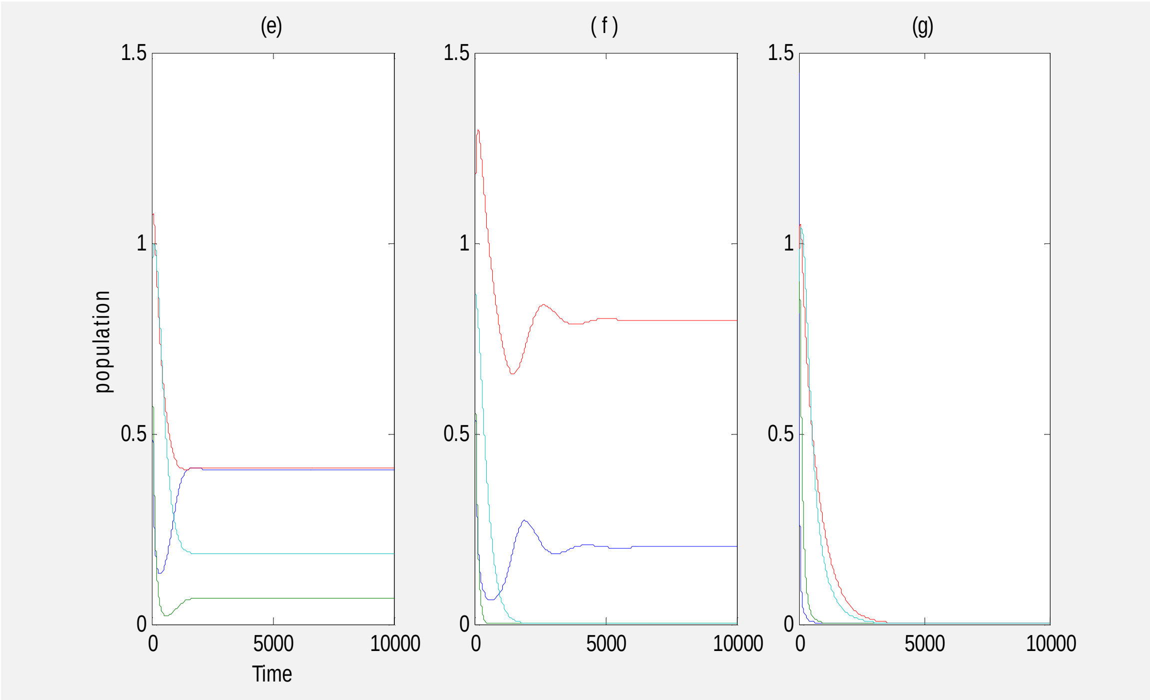

Fig. 2: Time series of the solution of system (2) for the data given by Eq. (6.1)with varying in the values ofu_2 andu_7, which summarized in the following table (1).

Table 1: Numerical behaviors and local bifurcation of system (2) as varying in some parameters with keeping to the rest of the parameters fixed as in Eq. (6.1)

Parameter varied in system (2)	Numerical behavior of system (2)	Local bifurcation of system (2)
$0.1 \leq u_2 \leq 0.9$	Approaches to the positive stable pointInt..R^4_+	Saddle-node bifurcation
$0.001 \leq u_2 \leq 0.0099$ $0.001 \leq u_7 < 0.0036$	Approaches to the stable point E_1	Saddle-node bifurcation
$1.2 \leq u_2 < 34.94$	Approaches to the stable point E_0	Transcritical bifurcation

Clearly, figure (2) show that the occurrence of local bifurcation (such as saddle-node and transcritical) of system (2) and the used values in table (1) satisfy the stability conditions of the equilibrium point of system (2).

7 Conclusion and Discussion

In this paper, we established the conditions of the occurrence of local bifurcation (such as saddle-node, transcritical and pitchfork) with particular emphasis on the Hopf bifurcation near of thepositive equilibrium point of eco-epidemiological mathematical model involving SIS infectious disease in prey population whereas, this disease passed from a prey to predator through attacking of predator to prey . The dynamical behavior of system (2) has been investigated local bifurcation as well as Hopf bifurcation. Further, it is observed that the system (2) near the vanishing equilibrium point (E_0) with the parameter $u_2^* = 1+\frac{u_3}{u_5}$, has transcritical bifurcation. While the system (2) near the predator free equilibrium point(E_1) with the parameter $u_2^{\#} = \frac{\left(u_3+u_5+2u_1\hat{x}^2\right)-\left(\left(u_1+2(u_3+u_5)\right)\hat{x}+\left(u_3+u_5(1+u_1)\right)\hat{y}\right)}{u_5+\hat{x}}$, has Saddle-node bifurcation. Also the system (2) near the positive equilibrium point (E_2) at the parameter

$$u_{11}^*=-\frac{\left(u_4u_8z^*y^*\left((1+u_1)x^*-u_3\right)\right)}{m_5-\left(m_6+(u_{10}+u_{12})(m_7-m_8)\right)}+u_6z^*,$$ has saddle-node bifurcation.

Note that, the system (2)at each point E_i $i= 0,1,2$ has no pitchfork bifurcation. Finally, the conditions of occurrence of the Hopf-bifurcation near the positive equilibrium point (E_2)are given.

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