Gen. Math. Notes, Vol. 30, No. 2, October 2015, pp.16-30
ISSN 2219-7184; Copyright ©ICSRS Publication, 2015
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A Coefficient Functional for Transformations of Starlike and Convex Functions of Complex Order

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(Received: 12-8-15 / Accepted: 26-9-15)

Abstract

We define two subclasses of analytic functions say $S_b^\gamma(\phi)$ & $C_b^\gamma(\phi)$. We obtain the sharp upper bound for the coefficient functional $\mid b_{2k+1} - \mu b_{k+1}^2 \mid$ corresponding to the k^{th} root transformation of f and function defined through convolution and fractional derivatives. We also investigated the Fekete-Szegö problem for the inverse function of f and for the function $\frac{z}{f(z)}$. The results of this paper generalize the work of earlier researchers in this direction.

Keywords: *Analytic function, Starlike function, Convex function, k^{th} root transformation, Fekete-Szegö inequality.*

1 Introduction

Let $\mathcal{A}$ be the class of all functions $f(z)$ analytic in the open unit disk $\Delta = [z \in C : |z| < 1]$ normalized by $f(0) = 0$ and $f'(0) = 1$. The functions in the class $\mathcal{A}$ are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n; \ \forall z \in \Delta \tag{1.1}$$

Let $\mathbb{S}$, be the subclass of $\mathcal{A}$, consisting of univalent functions. For a univalent function $f(z)$ of the form (1), the k^{th} root transformation is defined by

$$F(z) = [f(z^k)]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{nk+1} z^{nk+1} \tag{1.2}$$

Let B_o be the family of analytic functions $w(z)$ in Δ with $w(0) = 0$ and $|w(z)| \leq 1$. The functions in the class B_o are called as Schwartz functions.

If f is analytic in Δ, g is analytic and univalent in Δ and $f(0) = g(0)$ with $f(\Delta) = g(\Delta)$ then we say that f is subordinate to g and we write it as $f \prec g$. If $f \prec g$ then there exists a Schwartz function $w(z)$ in B_o such that$f(z) = g(w(z))$.

During the last century a lot of work has been done in the direction of finding inequalities and also finding upper bounds for $a_2, a_3, a_4, a_5, ..., a_n$ and $|a_3 - \mu a_2^2|$ for the function f in some subclasses of A, for some real or complex μ. A classical result of Fekete and Szegö [6] determines the maximum value of $|a_3 - \mu a_2^2|$ as a function of real parameter μ for the subclass S of A. This is known as Fekete-Szegö inequality[6]. There are now several results in the literature each of them dealing with $|a_3 - \mu a_2^2|$ for the functions in various subclasses of S.

Several authors [1-6,8-9,11-13] have investigated the Fekete- Szegö coefficient functional for the functions in various subclasses of analytic , univalent and multivalent functions.

For any two functions f analytic in $|z| < R_1$ and g analytic in $|z| < R_2$ with two power series expansion $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, the convolution or Hadamard product of f and g is defined as

$$(f \star g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \tag{1.3}$$

and $(f \star g)$ is analytic in $|z| < R_1 R_2$.

If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in S$ is analytic and univalent in the open disc Δ then Fekete-Szegö [6] have shown that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3, & \text{if} \quad \mu \geq 1; \\ 1 + 2e^{\frac{-2\mu}{1-\mu}}, & \text{if } 0 \leq \mu \leq 1; \\ 3 - 4\mu, & \text{if } \mu \leq 0 \ . \end{cases}$$

The result is sharp in the sense that for each μ there is a function in the class under consideration for which the equality holds.Keogh and Merks [7] found the Fekete - Szegö inequality for the function f in the class K of close-to-convex functions. They have shown that, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n;\ \forall\ z \in K$ and if μ is a real number then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if} \quad \mu \leq \frac{1}{3}; \\ \frac{1}{3} + \frac{4}{9\mu}, & \text{if } \frac{1}{3} \leq \mu \leq \frac{2}{3}; \\ 1, & \text{if } \mu \leq 0\ ; \\ 4\mu - 3, & \text{if } \mu \geq 1\ . \end{cases}$$

Definition 1.1. *Let $\phi(z)$ be a univalent, analytic function with positive real part on Δ with $\phi(0) = 1, \phi'(0) > 0$ where $\phi(z)$ maps Δ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Such a function ϕ has a series expansion of the form $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...$ with $B_1 > 0,\ B_2 \geq 0$ and $B_n s$ are real.*

Ma and Minda [8] introduced the unified class $S^{\star}(\phi)$ consisting of functions $f \in \mathcal{A}$ satisfying $\frac{zf'(z)}{f(z)} \prec \phi(z)$. They also investigated the corresponding class $\mathcal{C}(\phi)$ consisting of functions $f \in \mathcal{A}$ satisfying $\{1 + \frac{zf''(z)}{f'(z)}\} \prec \phi(z)$. They have obtained subordination results, distortion, growth and rotation theorems.They have also obtained estimates for the first few coefficients and determined bounds for the Fekete-Szegö functional. A function $f \in S^{\star}(\phi)$ is said to be a starlike function with respect to ϕ . A function $f \in \mathcal{C}(\phi)$ is said to be convex function with respect to ϕ .

Ravichandran et.al [12] have further generalized the classes by defining $S_b^{\star}(\phi)$ to be the class of function $f \in \mathbb{S}$ for which

$$1 + \frac{1}{b}[\frac{zf'(z)}{f(z)} - 1] \prec \phi(z),$$

and $\mathcal{C}_b(\phi)$ to be the class of functions $f \in \mathbb{S}$ for which

$$1 + \frac{1}{b}[\frac{zf''(z)}{f'(z)}] \prec \phi(z).$$

Here b is a non-zero complex number.

Recently Ali et.al [1] has obtained the sharp upper bounds for the Fekete-Szegö coefficient functional $|b_{2k+1} - \mu b_{k+1}^2|$ associated with the k^{th} root transformation of the function f belonging to the following classes. They have also investigated a similar problem for the function $\frac{z}{f(z)}$.

$$R_b(\phi) = \{f \in \mathcal{A} : 1 + \frac{1}{b}[f'(z) - 1] \prec \phi(z)\},$$

$$S^\star(\alpha, \phi) = \{f \in \mathcal{A} : [\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f'(z)}] \prec \phi(z)\},$$

$$L(\alpha, \phi) = \{f \in \mathcal{A} : \{[\frac{zf'(z)}{f(z)}]^\alpha\, [1 + \frac{zf''(z)}{f'(z)}]^{1-\alpha}\} \prec \phi(z)\},$$

$$M(\alpha, \phi) = \{f \in \mathcal{A} : (1-\alpha)\{\frac{zf'(z)}{f(z)}\} + \alpha\{1 + \frac{zf''(z)}{f'(z)}\} \prec \phi(z)\},$$

where $z \in \Delta$, $b \in \mathcal{C} - \{0\}$ and $\alpha \geq 0$. Functions in the class $L(\alpha, \phi)$ are called as logarithmic $\alpha-$ convex functions with respect to ϕ and the functions in the class $M(\alpha, \phi)$ are called as $\alpha-$ convex functions with respect to ϕ. They have obtained the sharp upper bounds for the Fekete - Szegö coefficient functional associated with the k^{th} root transformation of the function f belonging to the above mentioned classes.
Recently Annamalai et.al [3] have considered the following classes of Janowski α -spirallike functions

(1) $S^*_\alpha(A, B)$ if and only if

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = p(z)\cos\alpha + i\sin\alpha \;\; z \in \Delta.$$

(2) $C_\alpha(A, B)$ if and only if

$$e^{i\alpha}(1 + \frac{zf''(z)}{f'(z)}) = p(z)\cos\alpha + i\sin\alpha, \;\; z \in \Delta.$$

They have studied the k^{th} root transformations for the functions in the class $S^\star_\alpha(A, B)$ and $C_\alpha(A, B)$.

Motivated by the above mentioned work in the present paper, we define two subclasses of analytic functions of complex order and obtain the Fekete - Szegö coefficient inequality associated with the k^{th} root transformation of the function f and for the function f defined through convolution and fractional derivatives in these classes. We also study Fekete - Szegö inequality for the inverse function of f and for the function $\frac{z}{f(z)}$. The results obtained in this paper will generalize the work of earlier researchers in this direction.

Definition 1.2. *Let ϕ be a function as in Definition* (1.1)*. Let b be a non-zero complex number, γ, α be two real numbers with $0 < \gamma \leq 1$ and $\alpha \geq 0$. Let*

$S_b^\gamma(\phi)$ be the class of functions satisfying the condition

$$1 + \frac{1}{b}[\frac{zf'(z)}{f(z)} - 1] \prec [\phi(z)]^\gamma. \tag{1.4}$$

Here the powers are taken with their principal values.

1. If $\gamma = 1$ then $S_b^1(\phi) = S_b(\phi)$ this class was introduced and studied by Ravichandran et.al [12].
2. If $b = 1, \gamma = 1$ then $S_1^1(\phi) = S^*(\phi)$ this class was introduced and studied by Ma and Minda [8].

Definition 1.3. *Let ϕ be a function as in Definition (1.1). Let b be a non-zero complex number and γ be a real number with $0 < \gamma \leq 1$. Let $C_b^\gamma(\phi)$ be the class of functions $f \in \mathcal{A}$ satisfying the condition*

$$1 + \frac{1}{b}[\frac{zf''(z)}{f'(z)} - 1] \prec [\phi(z)]^\gamma; \tag{1.5}$$

Here the powers are taken with their principal values.

1. $C_b^1(\phi) = C_b(\phi)$ this class was introduced and studied by Ravichandran et.al [12].
2. $C_1^1(\phi) = C(\phi)$ this class was introduced and studied by Ma and Minda [8].
3. $C_1^1(\phi) = M(1, \phi)$ this class was introduced and studied by Ali et.al [1].

The present work is organized as follows:

Preliminaries
Main results
Applications for the functions defined through convolution and fractional derivatives
Applications for the functions defined through $\frac{z}{f(z)}$.
Applications for the inverse of the function $f(z)$.

2 Preliminaries

The following two Lemmas regarding the coefficients of functions are needed to prove our main results. Lemma 2.1 is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda [8]. Lemma 2.2 is due to Keogh and Merks [7].

Lemma 2.1 (12). *If $w \in B_0$ and $w(z) = w_1z + w_2z^2 + w_3z^3 + ...$ then for any real number t*

$$| w_2 - tw_1^2 | \leq \begin{cases} -t, & if\ t \leq -1\ ; \\ 1, & if\ -1 \leq t \leq 1\ ; \\ t, & if\ t \geq 1\ . \end{cases}$$

1. For $t < -1$ and $t > 1$,the equality holds for $w(z) = z$ or one of its rotation.
2. For $-1 < t < 1$ equality holds if and only if $w(z) = z^2$ or one of its rotation.
3. For $t = -1$ equality holds if and only if $w(z) = z[\frac{\lambda+z}{1+\lambda z}], (0 \leq \lambda \leq 1)$ or one of its rotation.
4. For $t = -1$ equality holds if and only if $w(z) = -z[\frac{\lambda+z}{1+\lambda z}]$ or one of its rotation.

Lemma 2.2 (10). *If $w \in B_0$ and $w(z) = w_1z + w_2z^2 + w_3z^3 + ...$ then*

$$| w_2 - tw_1^2 | \leq 2\ max\{1, | t |\}$$

for any complex number t. The result is sharp for the function $w(z) = z^2$ or $w(z) = z$.

3 Main Results

In this section, we obtain the bound for the functional $|b_{2k+1} - \mu b_{k+1}^2|$ corresponding to the k^{th} root transformation for the function f in this class.

Theorem 3.1. *If $f \in S_b^\gamma(\phi)$ and F is the k^{th} root transformation of the function f given by (1.2) then for any complex number μ, we have*

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \frac{|b|\gamma B_1}{2k} max\{1, |\frac{b\gamma B_1}{k}(2\mu - 1) - \frac{B_2}{B_1} - [\frac{\gamma - 1}{2}]B_1|\}, \quad (3.1)$$

and the result is sharp.

Proof: If $f \in S_b^\gamma(\phi)$ then there exists a Schwartz's function $w(z)$ in B_0 with $w(0) = 0$ and $|w(z)| \leq 1$ such that

$$1 + \frac{1}{b}\frac{zf'(z)}{f(z)} - 1 = [\phi(w(z))]^\gamma. \quad (3.2)$$

Since,

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (3a_3 - a_2)z^2 + ... \tag{3.3}$$

Consider

$$[\phi(w(z))]^{\gamma} = 1 + (\gamma B_1 w_1)z + (\gamma B_1 w_2 + \gamma B_2 w_1^2 + \frac{\gamma(\gamma-1)}{2} B_1^2 w_1^2)z^2 + ... \tag{3.4}$$

From equations (3.2), (3.3) and (3.4) we get

$$1 + \frac{a_2}{b} z + \frac{(2a_3 - a_2^2)}{b} z^2 + + ... = 1 + (\gamma B_1 w_1)z + (\gamma B_1 w_2 + \gamma B_2 w_1^2 + \frac{\gamma(\gamma-1)}{2} B_1^2 w_1^2)z^2 + ... \tag{3.5}$$

Upon equating the coefficients of z and z^2 on both sides we get

$$a_2 = b\gamma B_1 w_1, \tag{3.6}$$

$$a_3 = \frac{b\gamma}{2}[B_1 w_2 + B_2 w_1^2 + [\frac{(\gamma-1)}{2}]B_1^2 w_1^2 + b\gamma B_1^2 w_1^2]. \tag{3.7}$$

If $F(z)$ is the k^{th} root transformation of $f(z)$ then

$$F(z) = \{f(z^k)\}^{\frac{1}{k}} = z + (\frac{a_2}{k})z^{k+1} + [\frac{a_3}{k} - \frac{(k-1)}{2k^2}a_2^2]z^{2k+1} + ... = z + \sum_{n=1}^{\infty} b_{nk+1} z^{nk+1}.$$

Upon equating the coefficients of z^{k+1}, z^{2k+1} ,we get

$$b_{k+1} = \frac{1}{k}a_2, \tag{3.8}$$

$$b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}(\frac{k-1}{k^2})a_2^2. \tag{3.9}$$

From equations (3.6),(3.7), (3.8) and (3.9), it follows:

$$b_{k+1} = \frac{b\gamma B_1 w_1}{k}, \tag{3.10}$$

$$\begin{aligned} b_{2k+1} &= \frac{b\gamma B_1}{2k}[w_2 + \frac{B_2}{B_1}w_1^2 + [\frac{(\gamma-1)}{2}]B_1 w_1^2 + b\gamma B_1 w_1^2] \\ &- [1 - \frac{1}{k}]b\gamma B_1 w_1^2. \end{aligned} \tag{3.11}$$

For any complex number μ, consider

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{b\gamma B_1}{2k}\{w_2 - t w_1^2\}, \tag{3.12}$$

where

$$t = \frac{b\gamma B_1}{k}(2\mu - 1) - \frac{B_2}{B_1} - [\frac{\gamma - 1}{2}]B_1.$$

Taking modulus on both sides of the equation (3.12) and applying Lemma 2.2 on right hand side, we get the result as in (3.1). This proves the result of the theorem . The result is sharp and followed by

$$|b_{2k+1} - \mu b_{k+1}^2| = \begin{cases} \frac{|b|\gamma B_1}{2k}, & \text{If } w(z) = z^2 \ ; \\ \frac{|b|\gamma B_1}{2k}\{|\frac{b\gamma B_1}{k}(2\mu - 1) - \frac{B_2}{B_1} - [\frac{\gamma-1}{2}]B_1|\}, & \text{If } w(z) = z. \end{cases}$$

Restricting μ to be real and taking $b = 1$, we now obtain the coefficient inequality for the function f in the class $S_b^\gamma(\phi)$.

Theorem 3.2. *If $f \in S_b^\gamma(\phi)$ and F is the k^{th} root transformation of the function f given by (1.2) then for any real number μ and for*

$$\sigma_1 = \frac{k[(B_2 - B_1) + [\frac{(\gamma-1)}{2}]B_1^2] + \gamma B_1^2}{2\gamma B_1^2},$$

$$\sigma_2 = \frac{k[(B_2 + B_1) + [\frac{(\gamma-1)}{2}]B_1^2] + \gamma B_1^2}{2\gamma B_1^2}.$$

We have,

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \begin{cases} \frac{\gamma B_1}{2k}\{\frac{B_2}{B_1} + [\frac{\gamma-1}{2}]B_1 - \frac{b\gamma B_1}{k}(2\mu - 1)\}, & \text{if } \mu \leq \sigma_1; \\ \frac{\gamma B_1}{2k}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{\gamma B_1}{2k}\{\frac{\gamma B_1}{k}(2\mu - 1) - \frac{B_2}{B_1} - [\frac{\gamma-1}{2}]B_1\}, & \text{if } \mu \geq \sigma_2. \end{cases} \tag{3.13}$$

The result is sharp.

Proof: Since $f \in S_b^\gamma(\phi)$ for $b = 1$ from equation (3.12),we have

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{\gamma B_1}{2k}[w_2 - t w_1^2], \tag{3.14}$$

where

$$t = \frac{\gamma B_1}{k}(2\mu - 1) - \frac{B_2}{B_1} - [\frac{\gamma - 1}{2}]B_1.$$

By applying modulus on both sides of (3.14), we have

$$|b_{2k+1} - \mu b_{k+1}^2| = \frac{\gamma B_1}{2k}|w_2 - t w_1^2|. \tag{3.15}$$

By applying Lemma 2.1 on right hand side of (3.15), we get the following cases
Case(i): If $\mu \leq \sigma_1$, then

$$\mu \leq \frac{k[(B_2 - B_1) + [\frac{(\gamma-1)}{2}]B_1^2] + \gamma B_1^2}{2\gamma B_1^2}$$
$$\Rightarrow t \leq -1$$
$$\Rightarrow |w_2 - tw_1^2| \leq -t$$
$$\Rightarrow |b_{2k+1} - \mu b_{k+1}^2| \leq \frac{\gamma B_1}{2k}\{\frac{B_2}{B_1} + [\frac{(\gamma-1)}{2}]B_1 - \frac{\gamma B_1}{k}(2\mu - 1)\} \tag{3.16}$$

Case(ii): If $\sigma_1 \leq \mu \leq \sigma_2$ then

$$\frac{k[(B_2 - B_1) + [\frac{(\gamma-1)}{2}]B_1^2] + \gamma B_1^2}{2\gamma B_1^2} \leq \mu$$
$$\leq \frac{k[(B_2 + B_1) + [\frac{(\gamma-1)}{2}]B_1^2] + \gamma B_1^2}{2\gamma B_1^2},$$
$$\Rightarrow -1 \leq t \leq 1,$$
$$\Rightarrow |b_{2k+1} - \mu b_{k+1}^2| \leq \frac{\gamma B_1}{2k}. \tag{3.17}$$

Case(iii): If $\mu \geq \sigma_2$ then

$$\mu \geq \frac{k[(B_2 + B_1) + [\frac{(\gamma-1)}{2}]B_1^2] + \gamma B_1^2}{2\gamma B_1^2},$$
$$\Rightarrow t \geq 1,$$
$$\Rightarrow |w_2 - tw_1^2| \leq 1,$$
$$\Rightarrow |b_{2k+1} - \mu b_{k+1}^2| \leq \frac{\gamma B_1}{2k}\{\frac{b\gamma B_1}{k}(2\mu - 1) - \frac{B_2}{B_1} - [\frac{\gamma - 1}{2}]B_1\}. \tag{3.18}$$

From (3.15), (3.16),(3.17) and (3.18), we obtain the result as in (3.13). The result is sharp.

1. When $\mu \leq \sigma_1$ the result is sharp when $w(z) = z$ or one of its rotation.
2. When $\sigma_1 \leq \mu \leq \sigma_2$ the result is sharp when $w(z) = z^2$ or one of its rotation.
3. When $\mu \geq \sigma_2$ the result is sharp when $w(z) = z\frac{\lambda+z}{1+\lambda z}$ or one of its rotation.

Remark 3.3. *1. In view of the Alexander result that $f \in C_b^\gamma(\phi)$ if and only if $zf'(z) \in S_b^\gamma(\phi)$, the estimate for $|b_{2k+1} - \mu b_{k+1}^2|$ for a function in $C_b^\gamma(\phi)$ can be obtained from the corresponding estimates in Theorems 3.1 and 3.2 for the function in $S_b^\gamma(\phi)$. The details are omitted here.*

2. *For $k = 1$ the k^{th} root transformations of f reduces to the given function f itself. Thus estimate given in equation (3.1) and (3.13) is an extension of the corresponding results for the Fekete-Szegö functional corresponding to functions starlike with respect to ϕ of complex order γ studied by Ram Reddy and Sharma [13].*

4 Coefficient Inequalities for the Functions Defined through Convolution

We now derive our result for the function f in the class $S^{\gamma}_{b,g}(\phi)$

Theorem 4.1. *If $f \in S^{\gamma}_{b,g}(\phi)$ and F is the k^{th} root transformation of f given by (2)then for any complex number μ,*

$$|b_{2k+1} - \mu b^2_{k+1}| \leq \frac{|b|\gamma B_1}{2kg_3} max\{1, |\frac{b\gamma B_1 g_3}{kg_2^2}(2\mu - 1) - \frac{B_2}{B_1} - [\frac{\gamma - 1}{2}]B_1|\}, \quad (4.1)$$

and the result is sharp.

Proceeding in a way similar to Theorem 3.1 for the function $(f * g)(z)$ one can obtain this result.

Restricting μ to be real and taking $b = 1$, we now obtain the coefficient inequality for the function f in the class $S^{\gamma}_{b,g}(\phi)$

Theorem 4.2. *If $f \in S^{\gamma}_{b,g}(\phi)$ and F is the k^{th} root transformation of the function f given by (1.2) then for any real number μ and for*

$$\sigma_1 = \frac{kg_2^2[(B_2 - B_1) + [\frac{(\gamma-1)}{2}]B_1^2] + \gamma B_1^2 g_3}{2\gamma B_1^2 g_3},$$
$$\sigma_2 = \frac{kg_2^2[(B_2 + B_1) + [\frac{(\gamma-1)}{2}]B_1^2] + \gamma B_1^2 g_3}{2\gamma B_1^2 g_3}.$$

We have,

$$|b_{2k+1} - \mu b^2_{k+1}| \leq \begin{cases} \frac{\gamma B_1}{2kg_3}\{\frac{B_2}{B_1} + [\frac{(\gamma-1)}{2}]B_1 - \frac{\gamma B_1 g_3}{kg_2^2}(2\mu - 1)\}, & \text{if } \mu \leq \sigma_1; \\ \frac{\gamma B_1}{2kg_3}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{\gamma B_1}{2kg_3}\{\frac{\gamma B_1 g_3}{kg_2^2}(2\mu - 1) - \frac{B_2}{B_1} - [\frac{\gamma-1}{2}]B_1\}, & \text{if } \mu \geq \sigma_2. \end{cases} \quad (4.2)$$

The result is sharp.

Proceeding in a way similar to Theorem 3.2 for the function $(f * g)(z)$ one can obtain this result.

5 Applications to Functions Defined by Fractional Derivatives

We now obtain our result for the function in the class $S^{\gamma,\rho}_{b,g}(\phi)$
For fixed $g \in A$, let $S^{\gamma}_{b,g}(\phi)$ be the class of functions $f \in A$ for which $(f * g) \in S^{\gamma}_{b,g}(\phi)$.

Definition 5.1. *Let $f(z)$ be analytic in a simply connected region of the $z-$plane containing the origin. The fractional derivative of f of order ρ is defined by*

$$D^{\rho}_{z} f(z) = \frac{1}{\Gamma(1-\rho)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\rho}} d\zeta (0 \leq \rho < 1),$$

where the multiplicity of $(z-\zeta)^{\rho}$ is removed by requiring that $log(z-\zeta)$ is real for $(z-\zeta) > 0$.

Definition 5.2. *Using Definition 5.1 and its known extensions involving the fractional derivatives and fractional integral, Owa and Srivastava [10] introduced the operator*

$$(\Omega^{\rho} f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\rho)}{\Gamma(n-\rho+1)} a_n z^n. \tag{5.1}$$

This operator is known as the Owa-Srivastava operator. In terms of the Owa-Operator Ω^{ρ} defined by (5.2) , we now introduce the class $S^{\gamma}_{b,g}(\phi)$ in the following way :
$S^{\gamma,\rho}_{b,g}(\phi)$ is a special case of the class $S^{\gamma}_{b,g}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\rho)}{\Gamma(n-\rho+1)} z^n. \tag{5.2}$$

From equation (5.2), and for $g_n > 0$ we can obtain the coefficient estimates for function in the class $S^{\gamma,\rho}_{b,g}(\phi)$ from the corresponding estimates for functions in the class $S^{\gamma}_{b,g}(\phi)$ when g corresponds to Owa-Operator given in (5.1), we obtain

$$g_2 = \frac{\Gamma(3)\Gamma(2-\rho)}{\Gamma(3-\rho)} = \frac{2}{(2-\rho)}, \tag{5.3}$$

$$g_3 = \frac{\Gamma(4)\Gamma(2-\rho)}{\Gamma(4-\rho)} = \frac{6}{(2-\rho)(3-\rho)}. \tag{5.4}$$

For g_2 and g_3 given by (5.3) and (5.4), respectively Theorems (4.1) and (4.2) reduces to the following results.

Theorem 5.3. *If $f \in S_{b,g}^{\gamma,\rho}(\phi)$ and F is the k^{th} root transformation of f given by (1.2) then for any complex number μ,*

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \frac{|b|\gamma B_1(2-\rho)(3-\rho)}{12k} max\{1, |\frac{3\gamma B_1(2-\rho)}{2k(3-\rho)}(2\mu - 1) - \frac{B_2}{B_1} - \frac{\gamma - 1}{2}B_1|\}, \tag{5.5}$$

and the result is sharp.

Restricting μ to be real and taking $b = 1$, we now obtain the coefficient inequality for the function f in the class $S_{b,g}^{\gamma,\rho}(\phi)$.

Theorem 5.4. *If $f \in S_{b,g}^{\gamma,\rho}(\phi)$ and $g_n > 0$ and F is the k^{th} root transformation of f given by (1.2) then for any real number μ and*

$$\sigma_1 = \frac{k(3-\rho)[(B_2 - B_1) + [\frac{(\gamma-1)}{2}]B_1^2]}{3\gamma B_1^2(2-\rho)} + \frac{\gamma B_1^2}{2},$$
$$\sigma_2 = \frac{k(3-\rho)[(B_2 + B_1) + [\frac{(\gamma-1)}{2}]B_1^2]}{3\gamma B_1^2(2-\rho)} + \frac{\gamma B_1^2}{2}.$$

We have,

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \begin{cases} \frac{3\gamma B_1}{k(2-\rho)(3-\rho)}\{\frac{B_2}{B_1} + [\frac{\gamma-1}{2}]B_1 - \frac{3\gamma B_1(2-\rho)}{2k(3-\rho)}(2\mu - 1)\}, & \text{if } \mu \leq \sigma_1; \\ \frac{3\gamma B_1}{k(2-\rho)(3-\rho)}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{3\gamma B_1}{k(2-\rho)(3-\rho)}\{\frac{3\gamma B_1(2-\rho)}{2k(3-\rho)}(2\mu - 1) - \frac{B_2}{B_1} - p\frac{\gamma-1}{2}]B_1\}, & \text{if } \mu \geq \sigma_2. \end{cases} \tag{5.6}$$

The result is sharp.

6 Coefficient Functional Associated with $\frac{z}{f(z)}$

In this section we obtain the bounds for the Fekete-Szegö coefficient functional pertaining to the k^{th} root transformation associated with the function G defined as

$$G(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} d_n z^n,$$

where $f \in S_b^\gamma(\phi)$.

Theorem 6.1. *If $f \in S_b^\gamma(\phi)$ and $G(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} d_n z^n$, then for any complex number μ, we have*

$$| d_2 - \mu d_1^2 | \leq \frac{| b | \gamma B_1}{2} max\{1, | (2\mu - 1)b\gamma B_1 + \frac{B_2}{B_1} + [\frac{(\gamma-1)}{2}]B_1 |\}, \tag{6.1}$$

and the result is sharp.

Proceeding in a way similar to Theorem 3.1 for the function $\frac{z}{f(z)}$ one can obtain this result.

Restricting μ to be real and taking $b = 1$, we now obtain the coefficient inequality for the function f in the class $S_b^\gamma(\phi)$.

Theorem 6.2. *If $f \in S_b^\gamma(\phi)$ and $G(z) = \frac{z}{f(z)} = 1 + \sum\limits_{n=1}^{\infty} d_n z^n$, then for any real number μ, and for*

$$\sigma_1 = \frac{\gamma B_1^2 - (B_1 + B_2) + [\frac{\gamma-1}{2}]B_1^2}{2\gamma B_1^2},$$
$$\sigma_2 = \frac{\gamma B_1^2 + (B_1 - B_2) - [\frac{\gamma-1}{2}]B_1^2}{2\gamma B_1^2},$$

we have

$$\mid d_2 - \mu d_1^2 \mid \leq \begin{cases} (1-2\mu)\gamma B_1 - \frac{B_2}{B_1} - [\frac{\gamma-1}{2}]B_1, & \text{if } \mu \leq \sigma_1; \\ 1, & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ (2\mu-1)\gamma B_1 - \frac{B_2}{B_1} + [\frac{\gamma-1}{2}]B_1, & \text{if } \mu \geq \sigma_2, \end{cases} \tag{6.2}$$

and the result is sharp.

Proceeding in a way similar to Theorem 3.2 for the function $\frac{z}{f(z)}$ one can obtain this result.

7 Coefficient Inequality for the Inverse of the Function $f(z)$

Theorem 7.1. *If $f \in S_b^\gamma(\phi)$ and $f^{-1}(w) = w + \sum\limits_{n=2}^{\infty} q_n z^n$ is the inverse function of f with $|w| < r_0$ where r_0 is greater than the radius of the Koebe domain of the class $f \in S_b^\gamma(\phi)$, then for any complex number μ , we have*

$$\mid q_3 - \mu q_2^2 \mid \leq \frac{|b|\gamma\ B_1}{2} \times max\{1, |(2\mu+3)b\gamma\ B_1 - \frac{B_2}{B_1} - (\frac{\gamma-1}{2})B_1|\}, \tag{7.1}$$

and the result is sharp.

Proof: As

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} q_n w^n \tag{7.2}$$

is the inverse function of f,we have

$$f^{-1}\{f(z)\} = f\{f^{-1}(z)\} = z \tag{7.3}$$

From equations (7.2) and (7.3) we have

$$f^{-1}\{z + \sum_{n=2}^{\infty} a_n z^n\} = z. \tag{7.4}$$

Upon equating the coefficient of z and z^2,from equations (6.2) and (6.4), we get

$$q_2 = -a_2, \tag{7.5}$$
$$q_3 = 2a_2^2 - a_3. \tag{7.6}$$

For any complex number μ, consider

$$[q_3 - \mu q_2^2] = -\frac{b\gamma\ B_1}{2}\{w_2 - w_1^2\{(2\mu+3)b\gamma\ B_1 - \frac{B_2}{B_1} - [\frac{\gamma-1}{2}]B_1\}\} \tag{7.9}$$

Taking modulus on both sides and applying Lemma 2.2 on right hand side of (7.9), we obtain the result as in (7.1). This proves the result of the theorem . The result is sharp and followed by

$$|q_3 - \mu q_2^2| = \begin{cases} \frac{|b|\gamma\ B_1}{2}, & \text{If } w(z) = z^2\ ; \\ \frac{|b|\gamma\ B_1}{2}|(2\mu+3)b\gamma\ B_1 - \frac{B_2}{B_1} - (\frac{\gamma-1}{2})B_1|\}, & \text{If } w(z) = z. \end{cases}$$

8 Conclusion

In view of Alexander Theorem one can easily obtain the coefficient functional corresponding to the class defined through convolution, class defined through fractional derivatives , class $\frac{z}{f(z)}$ and for the inverse of the function $f(z)$ for a function f in $C_b^\gamma(\phi)$ and $C^\gamma(\phi)$ using Theorems 4.1, 4.2, 5.3, 5.4, 6.1, 6.2 and 7.1.

Acknowledgements:

1. The authors would like to thank Prof. T. Ram Reddy for his help and guidence throughout the work.

2. This work is partially supported by the U.G.C Major Research Project of the first author File no: 42-24/2013(SR), New Delhi, India.

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