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The Derivation of a Goldstein Formula

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Abstract

This technical note presents the derivation of an integral function credited to Goldstein [2] in 1932 and recently employed in the authors' previous work [1] in Archive of Applied Mechanics. The particular form of this improper integral is developed using techniques involving contour integration and the calculus of residues.

Keywords: *Bingham Number, Slip Flow, Inversion Theorem, Laplace Transform.*

1 Introduction

The problem of axially-symmetric slip flow generated by an infinite cylinder undergoing impulsive motion was recently investigated by Crane and McVeigh [1]. In accounting for momentum slip close to the cylinder wall, they obtained the non-dimensional shear stress analytically in terms of the Bingham number, Bn, in the cases where the cylinder moved under both uniform velocity and acceleration. In denoting the non-dimensional variables of axial velocity, cylinder radius and time by U, R and T, respectively, they presented the unsteady Navier Stokes momentum equation as follows:

$$\frac{\partial U}{\partial T} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right) \tag{1}$$

subject to, for $T > 0$

$$U_{R=1+} = 1 + \frac{\lambda}{2}\left(\frac{\partial U}{\partial R}\right)_{R=1+}, \quad U \to 0 \text{ as } R \to \infty \tag{2}$$

and, for $T > 0$:

$$U = 0 \text{ for } R > 1 \tag{3}$$

where λ is an empirically-derived slip-length parameter. In this work, the Laplace transform of $f(T)$ is the function $\bar{f}(p)$; taken to be:

$$\mathcal{L}\{f(T)\} = \int_0^\infty \exp(-pT)f(T)dT = \bar{f}(p)$$

Now, investigating radiating heat flow from an infinite region of constant initial temperature and bounded internally by a circular cylinder, Goldstein [2], derived the transform:

$$\bar{\Psi}(p) = \frac{1}{p}\left[1 + \frac{K_0(\sqrt{p})}{\hat{\mu}\sqrt{p}K_0'(\sqrt{p}) - K_0(\sqrt{p})}\right] \tag{4}$$

where K_0 denotes the modified Bessel function of the second kind of order 0, and in the work herein, Crane and McVeigh [1] specify $\hat{\mu} = 2\lambda$. The associated inverse is thus:

$$\Psi(T) = \frac{4}{\hat{\mu}\pi^2}\int_0^\infty \frac{\exp(-b^2T)}{b}\left[\frac{1}{(bJ_1 + J_0/\hat{\mu})^2 + (bY_1 + Y_0/\hat{\mu})^2}\right]db \tag{5}$$

where J_0 and J_1 are cylindrical Bessel functions of the first kind of order 0 and 1, respectively and where Y_0 and Y_1 denote the cylindrical Bessel functions of the first kind having order 0 and 1. Accordingly, Crane and McVeigh [1], give:

$$Bn = \frac{2}{\lambda}\Psi(T) \qquad \text{(uniform velocity)} \tag{6}$$

and

$$Bn = \frac{2}{T\lambda}\int_0^T \Psi(T)dT \qquad \text{(uniform acceleration)} \tag{7}$$

2 Derivation

From (4), the complex inversion integral is:

$$\Psi(T) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{p}\left[1 + \frac{K_0(\sqrt{p})}{\hat{\mu}\sqrt{p}K_0'(\sqrt{p}) - K_0(\sqrt{p})}\right]\exp(pt)dp, \quad t > 0 \tag{8}$$

The integration in (8) is to be performed along a line, $p = \gamma$, in the complex plane where p is a point having coordinates $(x + iy)$. The real number, γ, is to be so large that all singularities of the integrand lie to the left of the line (γ-i∞, γ+i∞). Since $p = 0$ is a branch point of the integrand, the adjoining Bromwich contour is chosen as the integration path (Fig. 1). This comprises

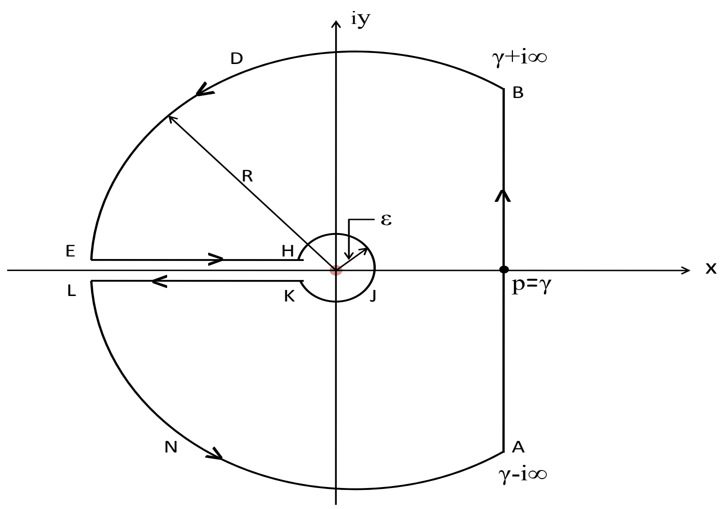

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Figure 1: The modified Bromwich contour

the line AB ($p = \gamma + iy$), the arcs BDE and LNA of a circle of radius R and centre at $(0, 0)$, and the arc HJK of a circle of radius, ϵ, with centre at $(0, 0)$. Set

$$\Psi(T) = \int_{AB} + \int_{BDE} + \int_{EH} + \int_{HJK} + \int_{KL} + \int_{LNA} \tag{9}$$

and since the only singularity, $p = 0$, of the integrand is not inside the contour, the integral on the left is zero by Cauchy's theorem. Further, it is readily shown that, as R tends to infinity, the integrals along BDE and LNA vanish in the limit. Along the inner circle, HJK, where $p = \epsilon \exp(i\theta)$, then, on taking the limit as ϵ becomes vanishingly small:

$$\Psi(T) = \int_{HJK} = i \int_{\pi}^{-\pi} \left[1 - \frac{K_0(0)}{K_0(0)}\right] d\theta = 0 \tag{10}$$

and so,

$$\int_{AB} = -\int_{EH} - \int_{KL} \tag{11}$$

Along the path, EH, where $p = x\exp(i\pi) = -x$:

$$\int_{EH} = \frac{1}{i\pi}\int_{\sqrt{R}}^{\sqrt{\epsilon}} \frac{\exp(-b^2 t)}{b}\left[1 + \frac{K_0(ib)}{i\hat{\mu}bK_0'(ib) - K_0(ib)}\right] db \tag{12}$$

Introducing the identities:

$$ib = b\exp(\frac{1}{2}\pi) \text{ and } K_0'(ib) = \frac{1}{2}\pi\left[J_1(b) + iY_1(b)\right]$$

so that, along EH, as $R \to \infty$ and $\epsilon \to 0$:

$$\int_{EH} = \frac{1}{i\pi}\int_{\infty}^{0} \frac{\exp(-b^2 t)}{b}\left[\frac{\hat{\mu}b\left(-J_1 + iY_1\right)}{-\hat{\mu}bJ_1 - J_0 + i\left(Y_0 + \hat{\mu}bY_1\right)}\right] db \tag{13}$$

and, on taking the complex conjugate, then:

$$\int_{EH} =$$

$$\frac{1}{i\pi}\int_{\infty}^{0} \frac{\exp(-b^2 t)}{b}\left[\frac{\hat{\mu}^2 b^2\left(J_1^2 + Y_1^2\right) + \hat{\mu}b\left(J_0J_1 + Y_0Y_1\right) + i\hat{\mu}b\left(J_1Y_0 - J_0Y_1\right)}{2\hat{\mu}b\left(Y_0Y_1 + J_0J_1\right) + \hat{\mu}^2b^2\left(J_1^2 + Y_1^2\right) + J_0^2 + Y_0^2}\right] db \tag{14}$$

Similarly, for the path KL, where $p = x\exp(-i\pi) = -x$.

$$\int_{KL} =$$

$$\frac{1}{i\pi}\int_{0}^{\infty} \frac{\exp(-b^2 t)}{b}\left[\frac{\hat{\mu}^2 b^2\left(J_1^2 + Y_1^2\right) + \hat{\mu}b\left(J_0J_1 + Y_0Y_1\right) + i\hat{\mu}b\left(J_0Y_1 - J_1Y_0\right)}{2\hat{\mu}b\left(Y_0Y_1 + J_0J_1\right) + \hat{\mu}^2b^2\left(J_1^2 + Y_1^2\right) + J_0^2 + Y_0^2}\right] db \tag{15}$$

Denoting the real and imaginary parts of the integrand in (14) by $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$, respectively; likewise, for KL in (15) respectively by $\mathrm{Re}(B)$ and $\mathrm{Im}(B)$, so that (11) can be written:

$$\Psi(T) = -\int_{EH} - \int_{KL}$$

$$= \frac{1}{i\pi}\int_{0}^{\infty} \frac{\exp(-b^2 t)}{b}\left[\mathrm{Re}(A) + \mathrm{Im}(A)\right] db - \frac{1}{i\pi}\int_{0}^{\infty} \frac{\exp(-b^2 t)}{b}\left[\mathrm{Re}(B) + \mathrm{Im}(B)\right] db \tag{16}$$

and so, from (14) and (15), Re(A)=Re(B) and Im(A)=-Im(B); hence:

$$\Psi(T) = \frac{2}{i\pi} \int_0^\infty \frac{\exp(-b^2 t)}{b} \mathrm{Im}(A) db \tag{17}$$

where

$$\mathrm{Im}(A) = \frac{i\hat{\mu}b\,(J_1 Y_0 - J_0 Y_1)}{2\hat{\mu}b\,(Y_0 Y_1 + J_0 J_1) + \hat{\mu}^2 b^2\,(J_1^2 + Y_1^2) + J_0^2 + Y_0^2} \tag{18}$$

Introducing the identities:

$$Y_0' = -Y_1 \quad \text{and} \quad J_0' = -J_1$$

and, using the Wronskian relation:

$$J_0 Y_0' - Y_0 J_0' = 2/\pi b$$

returns (17) as the real-valued function, that is:

$$\Psi(T) = \frac{4\hat{\mu}}{\pi^2} \int_0^\infty \frac{\exp(-b^2 t)}{b} \left[\frac{db}{2\hat{\mu}b\,(Y_0 Y_1 + J_0 J_1) + \hat{\mu}^2 b^2\,(J_1^2 + Y_1^2) + J_0^2 + Y_0^2} \right] \tag{19}$$

and finally, following some algebra, Goldstein's result (5) is recovered; namely:

$$\Psi(T) = \frac{4}{\hat{\mu}\pi^2} \int_0^\infty \frac{\exp(-b^2 t)}{b} \left[\frac{1}{(bJ_1 + J_0/\hat{\mu})^2 + (bY_1 + Y_0/\hat{\mu})^2} \right] db \tag{20}$$

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