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On Pebbling Jahangir Graph

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Abstract

Given a configuration of pebbles on the vertices of a connected graph G, a pebbling move (or pebbling step) is defined as the removal of two pebbles off a vertex and placing one on an adjacent vertex. The pebbling number, f(G), of a graph G is the least number m such that, however m pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of pebbling moves. In this paper, we determine f(G) for Jahangir graph $J_{2,m}$ *(*$m \geq 8$*).*

Keywords: *Pebbling, Jahangir graph, and graph parameters.*

1 Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research. It was first introduced into the literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of pebbling results in [3]. There have been many developments since Hulbert's survey appeared.

Given a graph G, distribute k pebbles (indistinguishable markers) on its vertices in some configuration C. Specifically, a configuration on a graph G is a function from V(G) to $N \cup \{0\}$ representing an arrangement of pebbles on G. For our purposes, we will always assume that G is connected.

A pebbling move (or pebbling step) is defined as the removal of two pebbles from some vertex and the placement of one of these pebbles on an adjacent vertex. Define the pebbling number, f(G), to be the minimum number of pebbles such that regardless of their initial configuration, it is possible to move to any root vertex v, a pebble by a sequence of pebbling moves. Implicit in this definition is the fact that if after moving to vertex v one desires to move to another root vertex, the pebbles reset to their original configuration.

Fact 1.1 [7,8]. For any vertex v of a graph G, $f(v,G) \geq n$ where $n=|V(G)|$.

Proof. Consider the configuration $C:V-\{v\}\rightarrow N$ defined by $C(w)=1$ for all $w\in V-\{v\}$. Then the size $|C|$ of the configuration is n-1. In this configuration we cannot move a pebble to v. Hence $f(v,G) \geq |V(G)|$.

Fact 1.2 [7]. The pebbling number of a graph G satisfies $f(G) \geq \max \{2^{diam(G)}, |V(G)|\}$.

Proof. Let $w\in V(G)$ be a vertex at a distance diam(G) from the target vertex v. Place $2^{diam(G)}-1$ pebbles at w. clearly we cannot move any pebble to v. Thus $f(G) \geq \max \{2^{diam(G)}, |V(G)|\}$.

There are few other interesting results in the pebbling number of graphs. Hulbert [3] has written an excellent survey article on pebbling. We earnestly request the interested readers to refer to it for further study.

We also request the readers to read [2] in which Moews has studied the pebbling number of product of trees.

A. Lourdusamy, S. Samuel Jayaseelan and T. Mathivanan conjectured in [5] that the pebbling number of Jahangir graph $J_{2,m}$ is 2m+10. We prove this result in this paper. For $3 \leq m \leq 7$, we state the results from [5].

Theorem 1.1[5] *For the Jahangir graph* $J_{2,3}$, $f(J_{2,3}) = 8$.

Theorem 1.2[5] *For the Jahangir graph* $J_{2,4}$, $f(J_{2,4}) = 16$.

Theorem 1.3[5] *For the Jahangir graph* $J_{2,5}$, $f(J_{2,5}) = 18$.

Theorem 1.4[5] *For the Jahangir graph* $J_{2,6}$, $f(J_{2,6}) = 21$.

Theorem 1.5[5] *For the Jahangir graph* $J_{2,7}$, $f(J_{2,7}) = 23$.

We now proceed to determine the pebbling number for $J_{2,m}$ (m ≥ 8).

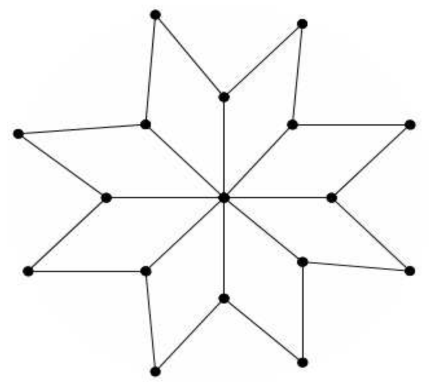

Figure 1: $J_{2,8}$

2 Pebbling Number of Jahangir Graph $J_{2,m}$ ($m \geq 8$)

Definition 2.1 [6] *Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm + 1$ vertices, that is, a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.*

Example 2.2 Figure1 shows Jahangir graph $J_{2,8}$. The Figure1 appears on Jahangir's tomb in his mausoleum. It lies in 5 kilometer north- west of Lahore, Pakistan, across the River Ravi.

Remark 2.3 *Let v_{2m+1} be the label of the center vertex and $v_1, v_2, \ldots, v_{2m}$ be the label of the vertices that are incident clockwise on cycle C_{2m} so that $deg(v_1) = 3$.*

Theorem 2.4 *For the Jahangir graph $J_{2,m}$ ($m \geq 8$), $f(J_{2,m}) = 2m + 10$.*

Proof: If m is even, then consider the following configuration C_1 such that $C_1(v_2)=0$, $C_1(v_{m+2})=15$, $C_1(v_{m-2})=3$, $C_1(v_{m+6}) = 3$, and $C_1(x)=1$ where $x \notin N[v_2]$, $x \notin N[v_{m+2}]$, $x \notin N[v_{m-2}]$, and $x \notin N[v_{m+6}]$.
If m is odd, then consider the following configuration C_2 such that $C_2(v_2)=0$, $C_2(v_{m+1})=15$, $C_2(v_{m-3})=3$, $C_2(v_{m+5}) = 3$, and $C_2(x)=1$where $x \notin N[v_2]$, $x \notin N[v_{m+1}]$, $x \notin N[v_{m-3}]$, and $x \notin N[v_{m+5}]$.

Then, we cannot move a pebble to v_2. The total number of pebbles placed in both configurations is

$$15 + 2(3) + (m - 4)(1) + (m - 8)(1)$$

$$= 2m + 9.$$

Therefore, $f(J_{2,m}) \geq 2m + 10$.

Now, consider the distribution of 2m + 10 pebbles on the vertices of $J_{2,\ m}$. Consider the sets

$S_1 = \{v_1, v_3, v_5, \ldots, v_{2m-3}, v_{2m-1}\}$ and

$S_2 = \{v_2, v_4, v_6, \ldots, v_{2m-2}, v_{2m}\}$

Case (i) Let the target vertex be v_{2m+1}.

Suppose we cannot move a pebble to the target vertex. Then the vertices of S_1 contain at most one pebble each and the vertices of S_2 contain at most three pebbles each. Also, note that, no two consecutive vertices of S_2 contain two or three pebbles each and if a vertex of S_2 contains two or three pebbles then the neighbors of that vertex contain zero pebbles. Also, if a vertex of S_1 contains a pebble then the neighbors of that vertex contain at most one pebble each. Thus, in any distribution, $J_{2,m}$ contains at most 2m pebbles so that a pebble could not be moved to v_{2m+1}–a contradiction to the total number of pebbles placed over the vertices of $J_{2,m}$. Hence we can move a pebble to the target vertex.

Case (ii) Let the target vertex be a vertex of S_1. Without loss of generality, let v_1 be the target vertex.

Suppose we cannot move a pebble to the target vertex. Then the neighbors v_2, v_{2m}, and v_{2m+1} of v_1 contain at most one pebble each. The vertices of $S_1 - \{v_1\}$ contain at most three pebbles each and the vertices of $S_2 - \{v_2, v_{2m}\}$ contain at most seven pebbles each. Also, note that, no two vertices of $S_1 - \{v_1\}$ contain two or three pebbles each and if a vertex of S_1-$\{v_1\}$ contains two or three pebbles then the neighbors of that vertex contain at most three pebbles each. Also, no two vertices of $S_2 - \{v_2, v_{2m}\}$ contain four or more pebbles each and if a vertex of S_2-$\{v_2, v_{2m}\}$ contains four or more pebbles then the neighbors of that vertex contain at most one pebble each.

Suppose v_{2m+1} has a pebble on it. Then clearly the vertices of S_1-$\{v_1\}$ contain at most one pebble each and if a vertex of S_1-$\{v_1\}$ contains a pebble then the neighbors of that vertex contain at most one pebble each. Also, the vertices of S_2-$\{v_2, v_{2m}\}$ contain at most three pebbles each and if a vertex of S_2-$\{v_2, v_{2m}\}$ contains two or three pebbles then the neighbors of that vertex contain zero pebbles. Thus in any distribution, $J_{2,\ m}$ contains at most (m-1)+(m-2)+1+1+1=2m pebbles so that a pebble could not be moved to v_1-a contradiction to the total number of pebbles placed over the vertices of $J_{2,\ m}$. So, assume that v_{2m+1} has zero pebbles on it.

Suppose v_2 has a pebble on it. Then the vertex v_3 contains at most one pebble and the vertex v_4 contains at most three pebbles. If one of the vertices of S_1-$\{v_1, v_3\}$, say v_9, contains two or three pebbles, then the vertices of S_1-$\{v_1, v_3, v_9\}$ contain at most one pebble each. Thus, the path $v_8v_9v_{10}$ contains at most six pebbles. Also, note that, no two consecutive vertices of S_2-$\{v_2, v_{2m}\}$ contain two or three pebbles each. Also, the path $v_4v_3v_2v_1v_{2m}$ contains at most five pebbles. Clearly, the vertices of S_1-$\{v_1, v_3, v_7, v_9, v_{11}, v_{13}\}$ contains at most one pebble each and the vertices of S_2-$\{v_2, v_4, v_8, v_{10}, v_{12}, v_{2m}\}$ contain at most three pebbles each. This implies that the vertices of S_2-$\{v_2, v_4, v_8, v_{10}, v_{12}, v_{2m}\}$ contain at least 2m+10-(m-6+1+6+1+3+1)=m+4=(m-6)+10. So we can move a pebble to v_1 easily. If we cannot move a pebble to v_1 then the graph $J_{2,m}$ contains at most 2m+1 pebbles-a contradiction to the total number of pebbles placed. So assume that all the vertices of S_1-{v_1, v_3} contain at most one pebble each. If v_{10} contains four or more (at most seven) pebbles, then other vertices of S_2-{v_2, v_{10}, v_{2m}} contain at most three pebbles each and if a vertex of S_2-{v_2, v_{10}, v_{2m}} contains two or three pebbles then the neighbors of that vertex contain zero pebbles. Suppose, we cannot move a pebble to v_1 then the graph $J_{2,m}$ contains at most $1+3+1+7+3\left(\frac{m-4}{2}\right)+\frac{m-4}{2}=$ 12+2(m-4) = 2m+4, a contradiction to the total number of pebbles placed. So, assume that all the vertices of S_2-{v_2, v_{2m}} contain at most three pebbles each. Here also if we cannot move a pebble to v_1 then the total number of pebbles placed over the vertices of the graph $J_{2,m}$ is at most $3+3+3+3\left(\frac{m-3}{2}\right)+\frac{m-3}{2}$=9+2(m-3)=2m+6 - a contradiction to the total number of pebbles placed. So, assume that v_2 has zero pebbles on it. In a similar way we may assume that v_{2m} contains zero pebbles.

Suppose v_3 has two or three pebbles on it. Then the vertices of S_1-{v_1, v_3} contain at most one pebble each and the vertices of S_2-{v_2, v_{2m}} contain at most three pebbles each. Also no two consecutive vertices of S_2-{v_2, v_{2m}} contain two or three pebbles each and if a vertex of S_2-{v_2, v_{2m}} contains two or three pebbles then the neighbors of that vertex contain zero pebbles. If we cannot move a pebble to v_1 then the total number of pebbles placed over the vertices of the graph $J_{2,m}$ is at most $5+3\left(\frac{m-3}{2}\right)+\frac{m-3}{2}$= 2m-1 – a contradiction to the total number of pebbles placed. So, assume that v_3 contains at most one pebble. In a similar way we may assume that the vertices of S_1-{v_1, v_3} contain at most one pebble each.

Suppose v_{10} contains four or more pebbles on it. Then the path $v_8v_9v_{10}v_{11}v_{12}$ contains at most nine pebbles. Also, note that, no two consecutive vertices of S_2-{v_2, v_{2m}, v_8, v_{10}, v_{12}} contain two or three pebbles each and if a vertex of S_2-{v_2, v_{2m}, v_8, v_{10}, v_{12}} contains two or three pebbles then the neighbors of that vertex contain zero pebbles. So, we can always move a pebble to v_1, since the total number of pebbles over the vertices of S_2-{v_2, v_{2m}, v_8, v_{10}, v_{12}} is at least 2m+10-(m-3-9)=m+12. If we cannot move a pebble to v_1 then the graph $J_{2,m}$ contains at most $9+3\left(\frac{m-5}{2}\right)+\frac{m-5}{2}$=9+2m-10=2m-1 pebbles-a contradiction. So, assume that v_{10} contains at most three pebbles. In a similar way we may assume that the vertices of S_2-{v_2, v_{2m}, v_{10}} contain at most three pebbles each. If we cannot move a pebble to v_1 then the total number of pebbles placed over the vertices of the graph $J_{2,m}$ is at most $3+3+3+3\left(\frac{m-3}{2}\right)+\frac{m-3}{2}$=2m+3. This is a contradiction to the total number of pebbles placed. Thus, we can always move a pebble to v_1.

Case (iii) Let the target vertex be a vertex of S_2. Without loss of generality, let v_2 be the target vertex.

Suppose we cannot move a pebble to the target vertex. Then, the neighbors v_1, and v_3 of v_2 contain at most one pebble each and the vertices v_4, v_{2m}, and v_{2m+1} contain at most three pebbles each. The vertices of $S_1 – \{v_1, v_3\}$ contain at most seven pebbles each and the vertices of $S_2 – \{v_2, v_4, v_{2m}\}$ contain at most fifteen pebbles each. Also, note that, no two vertices of $S_1 – \{v_1, v_3\}$ contain four or more pebbles each and if a vertex of $S_1 – \{v_1, v_3\}$ contains four or more pebbles then the neighbors of that vertex contain at most seven pebbles each. Also, no two vertices of $S_2 – \{v_2, v_4, v_{2m}\}$ contain eight or more pebbles each and if a vertex of $S_2 – \{v_2, v_4, v_{2m}\}$ contains eight or more pebbles then the neighbors of that vertex contain at most three pebbles each.

Suppose v_3 has a pebble on it. Clearly the neighbors v_4, and v_{2m+1} of v_3 contain at most one pebble each and the path $v_3v_4v_5$ contains at most four pebbles. If v_{2m+1} has a pebble on it, then the vertices of S_1-$\{v_1, v_3\}$ contain at most one pebble each and the vertices of S_2-$\{v_2, v_4\}$ contain at most three pebbles each. Also, note that, no two consecutive vertices of S_2-$\{v_2, v_4\}$ contain two or three pebbles each and if a vertex of S_2-$\{v_2, v_4\}$ contains two or three pebbles then the neighbors of that vertex contain zero pebbles. This implies that the vertices of S_2-$\{v_2, v_4\}$ contain at least 2m+10-(m-2-1-1)=(m-2)+16. So we can move a pebble to v_2 easily. If we cannot move a pebble to v_2 then the total number of pebbles placed over the vertices of the graph $J_{2,m}$ is at most $4+1+3\left(\frac{m-2}{2}\right)+\frac{m-2}{2}=$ 2m+4 – a contradiction. So, assume that v_{2m+1} has zero pebbles on it.

Suppose that, a vertex of S_1-$\{v_1, v_3\}$, say v_9, contains two or three pebbles on it. Then, the path $v_8v_9v_{10}$ contains at most six pebbles. Clearly the vertices of S_2-$\{v_2, v_4, v_8, v_{10}\}$ contain at least 2m+10-(m-3+3+1+1+1+4)=m-4+7 pebbles. If one of the vertices from S_2-$\{v_2, v_4, v_8, v_{10}\}$ contains four or more pebbles then we can move a pebble to v_2 through v_{2m+1} and v_3. So assume that all the vertices of S_2-$\{v_2, v_4, v_8, v_{10}\}$ contain at most three pebbles. Thus, we can move a pebble to v_2 easily. If not, then the total number of pebbles on the vertices of the graph $J_{2,m}$ is at most $6+3+3\left(\frac{m-4}{2}\right)+\frac{m-4}{2}$=2m +1 so that a pebble could not be moved to v_2-a contradiction to the total number of pebbles placed. So, assume that the vertices of S_1-$\{v_1, v_3\}$ contain at most one pebble each.

Suppose that, a vertex of S_2-$\{v_2, v_4\}$, say v_{10} has four or more pebbles on it. Then the path $v_9v_{10}v_{11}$ contains at most seven pebbles. Also, note that, no two vertices of S_2-$\{v_2, v_4, v_{10}\}$ contain two or three pebbles each and if a vertex of S_2-$\{v_2, v_4, v_{10}\}$ of contain two or three pebbles then the neighbors of that vertex contain zero pebbles. If we cannot move a pebble to v_2, then the total number of pebbles on the vertices of the graph $J_{2,m}$ is at most $7+1+3\left(\frac{m-2}{2}\right)+\frac{m-2}{2}$=8+2m-4=2m+4 -a contradiction. So, assume that the vertices of S_2-$\{v_2, v_4\}$ contain at most three pebbles each. Clearly, we can move a pebble to v_2. Otherwise, the total number of pebbles on the vertices of $J_{2,m}$ is at most $3+3+3+2+3\left(\frac{m-5}{2}\right)+\frac{m-5}{2}=11+2m-10=2m+1$ -a contradiction. Thus, assume that v_3 has zero pebbles on it. In a similar way, we may assume that v_1 has zero pebbles on it.

Suppose v_{2m+1} has two or three pebbles on it. Clearly, the vertices v_4 and v_{2m} contain at most one pebble each and one of the vertices of S_1-$\{v_1, v_3\}$ contain two or three pebbles. If v_7 has two or three pebbles, then the path $v_6v_7v_8$ contains at most six pebbles. Also, all the vertices of S_2-$\{v_2, v_4, v_6, v_8, v_{2m}\}$ contain at most three pebbles each and no two consecutive vertices of S_2-$\{v_2, v_4, v_{2m}\}$ contain two or three pebbles each. If we cannot move a pebble to v_2, then the total number of pebbles on the vertices of $J_{2,m}$ is at most $1+1+6+3\left(\frac{m-5}{2}\right)+\frac{m-5}{2}+3=11+2m-10=2m+1$-a contradiction. So, assume that the vertices of S_1-$\{v_1, v_3\}$ contain at most one pebble each.

Suppose one of the vertices of S_2-$\{v_2, v_4, v_{2m}\}$, say v_8, contain four or more (at most seven) pebbles on it. If we cannot move a pebble to v_2, then the total number of pebbles on the vertices of the graph $J_{2,m}$ is at most $7+1+1+3\left(\frac{m-4}{2}\right)+\frac{m-4}{2}=2m+4$ -a contradiction. So, assume that the vertices of S_2-$\{v_2, v_4, v_{2m}\}$ contain at most three pebbles each. If we cannot move a pebble to v_2, then the total number of pebbles on the vertices of $J_{2,m}$ is at most $9+3\left(\frac{m-6}{2}\right)+\frac{m-6}{2}+1+1+3=2m+2$ -a contradiction. So, assume that v_{2m+1} contains at most one pebble. In a similar way, we may assume that v_4 and v_{2m} contain at most one pebble each.

Suppose v_9 has four or more (at most seven) pebbles on it. Then one of the vertices of S_1-$\{v_1, v_3, v_9\}$ contains two or three pebbles. This implies that the vertices of S_2-$\{v_2, v_4, v_{2m}\}$ contain at least m+4 pebbles. If v_9 contains six or seven pebbles then we can move a pebble to v_2 easily. Otherwise, the vertices of S_2-$\{v_2, v_4, v_{2m}\}$ contain at least m+6 pebbles. If we cannot move two pebbles to v_{2m+1} then the total number of pebbles on the vertices of the graph $J_{2,m}$ is at most $5+3+4+3\left(\frac{m-5}{2}\right)+\frac{m-5}{2}=2m+2$ -a contradiction. So, assume that the vertices of S_1-$\{v_1, v_3\}$ contain at most three pebbles each. Clearly, at most three vertices of S_1-$\{v_1, v_3\}$ receive two or three pebbles each. This implies that the vertices of S_2-$\{v_2, v_4, v_{2m}\}$ contain at least 2m+10-(m-3+9)=m+4 pebbles. If we cannot move a pebble to v_2, then the total number of pebbles on the vertices of the graph $J_{2,m}$ is at most $6+5+5+1+1+3\left(\frac{m-7}{2}\right)+\frac{m-7}{2}=2m+4$-a contradiction. So, assume that the vertices of S_1-$\{v_1, v_3\}$ contain at most one pebble each.

Suppose that, a vertex of S_2-$\{v_2, v_4, v_{2m}\}$, say v_8, has eight or more (at most fifteen) pebbles on it. Then the path $v_7v_8v_9$ contains at most fifteen pebbles. If any two vertices of S_2-{v_2, v_4, v_8, v_{2m}} contain four or more pebbles (at most seven) or if a vertex of S_2-{v_2, v_4, v_8, v_{2m}} contains four or more (at most seven) and both the neighbors contain one pebble each then we can move a pebble to v_2 easily. If we cannot move a pebble to v_2, then the total number of pebbles on the vertices of $J_{2,m}$ is at most $15+1+1+3\left(\frac{m-4}{2}\right)+\frac{m-4}{2}=2m+9$ -a contradiction. So, assume that the vertices of S_2-{v_2, v_4, v_{2m}} contain at most seven pebbles each. Clearly, at most three vertices of S_2-{v_2, v_4, v_{2m}} contain four or more pebbles. If we cannot move a pebble to v_2, then the total number of pebbles on the vertices of $J_{2,m}$ contains at most $4+4+4+1+1+3\left(\frac{m-5}{2}\right)+\left(\frac{m-5}{2}\right)=14+2m-10=2m+4$ pebbles-a contradiction to the total number of pebbles placed. Thus, we can always move a pebble to v_2 while using 2m+10 pebbles on the vertices of $J_{2,m}$.

Thus, $f(J_{2,m}) \le 2m + 10$.

Therefore, $f(J_{2,m}) = 2m + 10$.

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