Gen. Math. Notes, Vol. 7, No. 1, November 2011, pp. 33-40
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Application of Fractional Calculus Operators to Related Areas

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(Received: 16-7-11/ Accepted: 14-10-11)

Abstract

In this paper a new function called as K-function, which is an extension of the generalization of the Mittag-Leffler function[10,11] and its generalized form introduced by Prabhakar[20], is introduced and studied by the author in terms of some special functions and derived the relations that exists between the K-function and the operators of Riemann-Liouville fractional integrals and derivatives.

Keywords: *Fractional calculus, Riemann- Liouville fractional integrals and derivatives.*

1 Introduction

Fractional Calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last three decades Fractional Calculus has been applied to almost every field of Mathematics like Special Functions etc., Science, Engineering and Technology. Many applications of Fractional Calculus can be found in Turbulence and Fluid Dynamics, Stochastic Dynamical System, Plasma Physics and Controlled Thermonuclear Fusion, Non-linear Control Theory, Image Processing, Non-linear Biological Systems and Astrophysics.

The Mittag-Leffler function has gained importance and popularity during the last one decade due mainly to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences. This function is introduced and studied by MittagLeffler[10,11] in terms of the power series

$$E_\alpha(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r + 1)}, \quad (\alpha > 0) \tag{1.1}$$

A generalization of this series in the following form

$$E_{\alpha,\beta}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r + \beta)}, \quad (\alpha, \beta > 0) \tag{1.2}$$

has been studied by several authors notably by Mittag-Leffler[10,11],Wiman[13], Agrawal[15], Humbert and Agrawal[8] and Dzrbashjan[1,2,3]. It is shown in [5] that the function defined by (1.1) and (1.2) are both entire functions of order $\rho = 1$ and type $\sigma = 1$. A detailed account of the basic properties of these two functions are given in the third volume of Bateman manuscript project[4] and an account of their various properties can be found in [2,12].

The multiindex Mittag-Leffler function is defined by Kiryakova[9] by means of the power series

$$E_{(\frac{1}{\rho i}),(\mu i)}(x) = \sum_{r=0}^{\infty} \varphi_r z^r = \sum_{r=0}^{\infty} \frac{x^r}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{r}{\rho_j})} \tag{1.3}$$

where $m > 1$ is an integer, ρ_j and μ_j are arbitrary real numbers.
The multiindex Mittag-Leffler function is an entire function and also gives its asymptotic, estimate, order and type see Kiryakova[9].

A generalization of (1.1) and (1.2) was introduced by Prabhakar [20] in terms of the series representation

$$E_{\alpha,\beta}^{\gamma}(x) = \sum_{r=0}^{\infty} \frac{(\gamma)_n x^r}{r!\Gamma(\alpha r + \beta)} \quad , (\alpha, \beta, \gamma \in C, \mathrm{Re}(\alpha) > 0)$$

(1.4)

where $(\gamma)_n$ is Pochammer's symbol defined by

$$(\gamma)_n = \gamma(\gamma+1).......((\gamma+(n-1)), n \in N, \gamma \neq 0.$$

It is an entire function of order $\rho = [\mathrm{Re}(\alpha)]^{-1}$.

An interesting generalization of (1.2) is recently introduced by Kilbas and Saigo[16] in terms of a special entire function of the form

$$E_{\alpha,m,l}(x) = \sum_{r=0}^{\infty} c_r x^r, \tag{1.5}$$

where

$$c_r = \prod_{i=0}^{r-1} \frac{\Gamma[\alpha(im+l)+1]}{\Gamma[\alpha(im+l+1)+1]}$$

and an empty product is to be interpreted as unity. Certain properties of this function associated with fractional integrals and derivatives[12].
The present paper is organized as follows; In section 2, we give the definition of the K-function and its relation with another special functions, namely generalization of the Mittag-Leffler function[11] and its generalized form introduced by Prabhakar[20] and other special functions. In section 3, relations that exists between the K-function and the operators of Riemann-Liouville fractional calculus are derived.

2 The K-Function

The K-function introduced by the author is defined as follows:

$${}_pK_q^{\alpha,\beta;\gamma}(a_1,\dots,a_p;b_1,\dots,b_q;x) = {}_pK_q^{\alpha,\beta;\gamma}(x) = \sum_{r=0}^{\infty} \frac{(a_1)_r \dots (a_p)_r}{(b_1)_r \dots (b_q)_r} \frac{(\gamma)_r x^r}{r!\Gamma(\alpha r+\beta)} \tag{2.1}$$

where $\alpha, \beta, \gamma \in C, \mathrm{Re}(\alpha) > 0$ and $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols.
The series(2.1) is defined when none of the parameters $b_j s$, $j = 1,2,...,q,$ is a negative integer or zero. If any numerator parameter a_r is a negative integer or zero, then the series terminates to a polynomial in x. From the ratio test it is evident that the series is convergent for all x if p > q + 1. When p = q + 1 and

$|x|=1$, the series can converge in some cases. Let $\gamma=\sum_{j=1}^{p}a_j-\sum_{j=1}^{q}b_j$. It can be shown that when p = q + 1 the series is absolutely convergent for $|x|=1$ if ($R(\gamma)<0$, conditionally convergent for x = -1 if $0\leq R(\gamma)<1$ and divergent for $|x|=1$ if $1\leq R(\gamma)$.

Special cases:

(i) When there is no upper and lower parameter, we get

$$ {}_0K_0^{\alpha,\beta;\gamma}(-;-;x)=\sum_{r=0}^{\infty}\frac{(\gamma)_r\,x^r}{r!\Gamma(\alpha r+\beta)}=E_{\alpha,\beta}^{\gamma}(x) \tag{2.2}$$

which reduces to the generalization of the Mittag-Leffler function[11] and its generalized form introduced by Prabhakar[20].

(ii) If we put $\gamma=1$ in (2.2), we get

$$ {}_0K_0^{\alpha,\beta;1}(-;-;x)=\sum_{r=0}^{\infty}\frac{x^r}{\Gamma(\alpha r+\beta)}=E_{\alpha,\beta}^{1}(x)=E_{\alpha,\beta}(x) \tag{2.3}$$

which is the generalized Mittag-Leffler function[10].
(iii) If we take $\beta=1$ in (2.3), we get

$$ {}_0K_0^{\alpha,1;1}(-;-;x)=\sum_{r=0}^{\infty}\frac{x^r}{\Gamma(\alpha r+1)}=E_{\alpha,1}^{1}(x)=E_{\alpha,1}(x)=E_{\alpha}(x) \tag{2.4}$$

which is the Mittag-Leffler function[10].
(iv) If we take $\alpha=1$ in (2.4), we get

$$ {}_0K_0^{\alpha,1;1}(-;-;x)=\sum_{r=0}^{\infty}\frac{x^r}{\Gamma(\alpha r+1)}=E_{1,1}^{1}(x)=E_{1,1}(x)=E_{1}(x) \tag{2.5}$$

which is the Exponential function[14] denoted by e^x.

3 Relations with Riemann-Liouville Fractional Calculus Operators

In this section we derive relations between the K-function and the operators of Riemann-Liouville Fractional Calculus.

Theorem 3.1 *Let* $\nu > 0; \alpha, \beta, \gamma \in C(\mathrm{Re}(\alpha) > 0)$ *and* I_x^ν *be the operator of Riemann-Liouville fractional integral then there holds the relation:*

$$I_x^\nu \, {}_pK_q^{\alpha,\beta;\gamma}(x) = \frac{x^\nu}{\Gamma(\nu+1)} \, {}_{p+1}K_{q+1}^{\alpha,\beta;\gamma}(a_1, \ldots, a_p, 1; b_1, \ldots, b_q, \nu+1; x) \tag{3.1}$$

Proof. Following Section 2 of the book by Samko, Kilbas and Marichev[12], the fractional Riemann-Liouville(R-L) integral operator(For lower limit a=0 w. r. t. variable x) is given by

$$I_x^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\, dt \tag{3.2}$$

By virtue of (3.2) and (2.1), we obtain

$$I_x^\nu \, {}_pK_q^{\alpha,\beta;\gamma}(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\alpha-1} \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r}{(b_1)_r \cdots (b_q)_r} \frac{(\gamma)_r t^r}{r!\,\Gamma(\alpha r+\beta)} dt \tag{3.3}$$

Interchanging the order of integration and evaluating the inner integral with the help of Beta function, it gives

$$\begin{aligned} I_x^\nu \, {}_pK_q^{\alpha,\beta;\gamma}(x) &= \frac{x^\nu}{\Gamma(\nu+1)} \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r (1)_r}{(b_1)_r \cdots (b_q)_r (\nu+1)_r} \frac{(\gamma)_r x^r}{r!\,\Gamma(\alpha r+\beta)} \\ &= \frac{x^\nu}{\Gamma(\nu+1)} \, {}_{p+1}K_{q+1}^{\alpha,\beta;\gamma}(a_1, \ldots, a_p, 1; b_1, \ldots, b_q, \nu+1; x) \end{aligned} \tag{3.4}$$

The interchange of the order of integration and summation is permissible under the conditions stated along with the theorem due to convergence of the integrals involved in this process.
This shows that a Riemann-Liouville fractional integral of the K-function is again the K-function with indices p+1, q+1.
This completes the proof of the theorem (3.1).

Theorem 3.2 *Let* $\nu > 0; \alpha, \beta, \gamma \in C(\mathrm{Re}(\alpha) > 0)$ *and* D_x^ν *be the operator of Riemann-Liouville fractional derivative then there holds the relation:*

$$D_x^\nu \, {}_pK_q^{\alpha,\beta;\gamma}(x) = \frac{x^{-\nu}}{\Gamma(1-\nu)} \, {}_{p+1}K_{q+1}^{\alpha,\beta;\gamma}(a_1,\ldots,a_p,1; b_1,\ldots,b_q,1-\nu; x) \tag{3.5}$$

Proof. Following Section 2 of the book by Samko, Kilbas and Marichev[12], the fractional Riemann-Liouville(R-L) integral operator(For lower limit a = 0 w. r. t. variable x) is given by

$$D_x^\nu f(x) = \frac{1}{\Gamma(r-\nu)} \left(\frac{d}{dx}\right)^r \int_0^x (x-t)^{r-\nu-1} f(t)\,dt \tag{3.6}$$

where $r = [\nu] + 1$.

From (2.1) and (3.6) it follows that

$$D_x^\nu \, {}_pK_q^{\alpha,\beta;\gamma}(x) = \frac{1}{\Gamma(r-\nu)} \left(\frac{d}{dx}\right)^r \int_0^x (x-t)^{n-\nu-1} \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r}{(b_1)_r \cdots (b_q)_r} \frac{(\gamma)_r t^r}{r!\Gamma(\alpha r + \beta)} dt \tag{3.7}$$

Interchanging the order of integration and evaluating the inner integral with the help of Beta function, it gives

$$D_x^\nu \, {}_pK_q^{\alpha,\beta;\gamma}(x) = \frac{x^{-\nu}}{\Gamma(1-\nu)} \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r (1)_r}{(b_1)_r \cdots (b_q)_r (1-\nu)_r} \frac{(\gamma)_r x^r}{r!\Gamma(\alpha r + \beta)}$$

$$= \frac{x^{-\nu}}{\Gamma(1-\nu)} \, {}_{p+1}K_{q+1}^{\alpha,\beta;\gamma}(a_1,\ldots,a_p,1; b_1,\ldots,b_q,1-\nu; x) \tag{3.8}$$

This shows that a Riemann-Liouville fractional derivative of the K-function is again the K-function with indices p+1, q+1.
This completes the proof of the theorem (3.2).

4 Conclusion

It is expected that some of the results derived in this survey may find applications in the solution of certain fractional order differential and integral equations arising problems of physical sciences and engineering areas.

Acknowledgements

The author is very thankful to the referees for giving several valuable suggestions in the presentation of the paper.

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