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Bertrand Mate of Biharmonic Reeb Curves in 3-Dimensional Kenmotsu Manifold

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Abstract

In this article, we study biharmonic Reeb curves in 3-dimensional Kenmotsu manifold. Moreover, we apply biharmonic Reeb curves in special 3-dimensional Kenmotsu manifold $\mathbb{K}$*. Finally, we characterize Bertrand mate of the biharmonic Reeb curves in terms of their curvature and torsion in special 3-dimensional Kenmotsu manifold* $\mathbb{K}$*.*

Keywords: *Kenmotsu manifold, biharmonic curve, Bertrand curve, Reeb vector field.*

1 Introduction

In the theory of space curves in differential geometry, the associated curves, the curves for which at the corresponding points of them one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve have an important role for the characterizations of space curves. The well-known examples of such curves are Bertrand curves. These special curves are very interesting and characterized as a kind of corresponding relation between two curves such that the curves have the common principal normal i.e., the Bertrand curve is a curve which shares the normal line with another curve. These curves have an important role in the theory of curves.

Let (N, h) and (M, g) be Riemannian manifolds. A smooth map $\phi : N \longrightarrow$

M is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where the section $\mathcal{T}(\phi) := \operatorname{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ.

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_{\phi}\mathcal{T}(\phi) + \operatorname{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this article, we study biharmonic Reeb curves in 3-dimensional Kenmotsu manifold. Moreover, we apply biharmonic Reeb curves in special 3-dimensional Kenmotsu manifold $\mathbb{K}$. Finally, we characterize Bertrand mate of the biharmonic Reeb curves in terms of their curvature and torsion in special 3-dimensional Kenmotsu manifold $\mathbb{K}$.

2 Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold with 1-form η, the associated vector field ξ, $(1,1)$-tensor field ϕ and the associated Riemannian metric g. It is well known that [2]

$$\phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X, Y on M. Moreover,

$$(\nabla_X \phi) Y = -\eta(Y)\phi(X) - g(X, \phi Y)\xi, \qquad X, Y \in \chi(M), \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g, then (M, ϕ, ξ, η, g) is called an Kenmotsu manifold [2].

In Kenmotsu manifolds the following relations hold [2]:

$$(\nabla_X \eta) Y = g(\phi X, \phi Y), 2.7 \tag{1}$$

$$\eta(R(X, Y)Z) = \eta(Y) g(X, Z) - \eta(X) g(Y, Z), 2.8 \tag{2}$$

$$R(X, Y)\xi = \eta(X) Y - \eta(Y) X, 2.9 \tag{3}$$

$$R(\xi, X) Y = \eta(Y) X - g(X, Y)\xi, 2.10 \tag{4}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, 2.11 \tag{5}$$

where R is the Riemannian curvature tensor.

3 Biharmonic Reeb Curves in the 3-Dimensional Kenmotsu Manifold

Let γ be a curve on the 3-dimensional Kenmotsu manifold parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the 3-dimensional Kenmotsu manifold along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, 3.1 \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{6}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1,\ g(\mathbf{N}, \mathbf{N}) = 1,\ g(\mathbf{B}, \mathbf{B}) = 1,\ 3.2 \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned} \tag{7}$$

Lemma 3.1. *(see [13]) If γ is a biharmonic Reeb curve which are either tangent or normal to the Reeb vector field 3-dimensional Kenmotsu manifold, then γ is a helix.*

We consider the special 3-dimensional manifold

$$\mathbb{K} = \left\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\right\},$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$\mathbf{e}_1 = z\frac{\partial}{\partial x},\quad \mathbf{e}_2 = z\frac{\partial}{\partial y},\quad \mathbf{e}_3 = -z\frac{\partial}{\partial z} \tag{3.3}$$

are linearly independent at each point of $\mathbb{K}$. Let g be the Riemannian metric defined by

$$\begin{aligned} g(\mathbf{e}_1, \mathbf{e}_1) &= g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1, 3.4 \\ g(\mathbf{e}_1, \mathbf{e}_2) &= g(\mathbf{e}_2, \mathbf{e}_3) = g(\mathbf{e}_1, \mathbf{e}_3) = 0. \end{aligned} \tag{8}$$

The characterising properties of $\chi(\mathbb{K})$ are the following commutation relations:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0,\quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1,\quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_2.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, \mathbf{e}_3) \text{ for any } Z \in \chi(M)$$

Let ϕ be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1) = -\mathbf{e}_2,\ \phi(\mathbf{e}_2) = \mathbf{e}_1,\ \phi(\mathbf{e}_3) = 0.$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3) = 1,$$

$$\phi^2(Z) = -Z + \ \eta(Z)\mathbf{e}_3,$$
$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(\mathbb{K})$. Thus for $\mathbf{e}_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on $\mathbb{K}$.

Now, we consider biharmonicity of curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$.

Theorem 3.4. *(see [13]) Let $\gamma : I \longrightarrow \mathbb{K}$ be a unit speed biharmonic Reeb curve which are either tangent or normal to the Reeb vector field 3-dimensional Kenmotsu manifold $\mathbb{K}$. Then, the parametric equations of γ are*

$$\begin{aligned} x(s) &= \frac{C_1 \sin^5\varphi}{\kappa^2 + \sin^4\varphi\cos^2\varphi} e^{-\cos\varphi s}\left(\frac{\kappa}{\sin^2\varphi}\cos\left(\frac{\kappa}{\sin^2\varphi}s + \sigma\right)\right. \\ &\quad \left. + \cos\varphi\sin\left(\frac{\kappa}{\sin^2\varphi}s + \sigma\right)\right) + C_2, \\ y(s) &= \frac{C_1 \sin^5\varphi}{\kappa^2 + \sin^4\varphi\cos^2\varphi} e^{-\cos\varphi s}\left(-\cos\varphi\cos\left(\frac{\kappa}{\sin^2\varphi}s + \sigma\right)3.14\right. \\ &\quad \left. + \frac{\kappa}{\sin^2\varphi}\sin\left(\frac{\kappa}{\sin^2\varphi}s + \sigma\right)\right) + C_3, \\ z(s) &= C_1 e^{-\cos\varphi s}, \end{aligned} \tag{9}$$

where C, C_1, C_2, C_3 are constants of integration.

4 Bertrand Mate of Biharmonic Reeb Curves in the Special Three-Dimensional Kenmotsu Manifold $\mathbb{K}$

A curve $\gamma : I \longrightarrow \mathbb{K}$ with $\kappa \neq 0$ is called a Bertrand curve if there exist a curve $\gamma_{\mathcal{B}} : I \longrightarrow \mathbb{K}$ such that the principal normal lines of γ and $\gamma_{\mathcal{B}}$ at $s \in I$ are equal. In this case $\gamma_{\mathcal{B}}$ is called a Bertrand mate of γ.

On the other hand, let $\gamma : I \longrightarrow \mathbb{K}$ be a Bertrand curve parametrized by arc length. A Bertrand mate of γ is as follows:

$$\gamma_{\mathcal{B}}(s) = \gamma(s) + \lambda \mathbf{N}(s), \quad \forall s \in I, \tag{4.1}$$

where λ is constant.

Theorem 4.1. *Let $\gamma : I \longrightarrow \mathbb{K}$ be a biharmonic curve parametrized by arc length. If $\gamma_{\mathcal{B}}$ is a Bertrand mate of γ, then the parametric equations of $\gamma_{\mathcal{B}}$ are*

$$\begin{aligned}
x_{\mathcal{B}}(s) &= \frac{\lambda \sin\varphi}{\kappa}\left(\frac{\kappa}{\sin^2\varphi}\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma)+\cos\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma)\right)\left(\bar{C}_1 s+\bar{C}_2\right) \\
&\quad +\frac{C_1\sin^3\varphi}{\kappa}e^{-\cos\varphi s}(-\cos\sigma\cos(\frac{\kappa}{\sin^2\varphi}s)+\sin\sigma\sin(\frac{\kappa}{\sin^2\varphi}s))+C_2, \qquad (4.2) \\
y_{\mathcal{B}}(s) &= \frac{\lambda \sin\varphi}{\kappa}\left(-\frac{\kappa}{\sin^2\varphi}\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma)+\cos\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma)\right)\left(\bar{C}_1 s+\bar{C}_2\right) \\
&\quad \frac{C_1\sin^3\varphi}{\kappa}e^{-\cos\varphi s}(\sin\sigma\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma)+\cos\sigma\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma))+C_3, \\
z_{\mathcal{B}}(s) &= \frac{\lambda}{\kappa}\left(\bar{C}_1 s+\bar{C}_2\right)+C_1 e^{-\cos\varphi s},
\end{aligned}$$

where $\sigma, \bar{C}_1, \bar{C}_2, C_1, C_2, C_3$ are constants of integration.

Proof. Assume that $\mathbf{T}$ is

$$\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \tag{4.3}$$

where T_1, T_2, T_3 are differentiable functions on I.

From [13], we obtain

$$\mathbf{T} = \sin\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma)\mathbf{e}_1 + \sin\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma)\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \tag{4.4}$$

Using (3.3) in (4.4), we obtain

$$\mathbf{T} = (z\sin\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma), z\sin\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma), -z\cos\varphi). \tag{4.5}$$

Because, by making use of (3.3), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = \left(T_1' + T_1T_3\right)\mathbf{e}_1 + \left(T_2' + T_2T_3\right)\mathbf{e}_2 + T_3'\mathbf{e}_3. \tag{4.6}$$

From (3.1) and (4.4), we get

$$
\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= \sin\varphi\left(\frac{\kappa}{\sin^2\varphi}\cos\left[\frac{\kappa}{\sin^2\varphi}s+\sigma\right]+\cos\varphi\sin\left[\frac{\kappa}{\sin^2\varphi}s+\sigma\right]\right)\mathbf{e}_1 \\
&+\sin\varphi\left(-\frac{\kappa}{\sin^2\varphi}\sin\left[\frac{\kappa}{\sin^2\varphi}s+\sigma\right]+\cos\varphi\cos\left[\frac{\kappa}{\sin^2\varphi}s+C\right]\right)\mathbf{e}_2.
\end{aligned} \tag{4.7}
$$

Then, by using Frenet formulas (3.1), we get

$$
\begin{aligned}
\mathbf{N} &= \frac{1}{\kappa}\nabla_{\mathbf{T}}\mathbf{T} \\
&= \frac{1}{\kappa}[\sin\varphi\left(\frac{\kappa}{\sin^2\varphi}\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma)+\cos\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma)\right)\mathbf{e}_1 \\
&+\sin\varphi\left(-\frac{\kappa}{\sin^2\varphi}\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma)+\cos\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma)\right)\mathbf{e}_2].
\end{aligned} \tag{4.8}
$$

Finally, we substitute (3.5) and (4.8) into (4.1), we get (4.2). The proof is completed.

Corollary 4.2. *Let $\gamma : I \longrightarrow \mathbb{K}$ be a biharmonic curve parametrized by arc length. If $\gamma_{\mathcal{B}}$ is a Bertrand mate of γ, then the parametric equations of $\gamma_{\mathcal{B}}$ in terms of τ are*

$$
\begin{aligned}
x_{\mathcal{B}}(s) &= \frac{\lambda\sin\varphi}{\sqrt{1-\tau^2}}\left(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}\cos(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma)+\cos\varphi\sin(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma)\right)\left(\bar{C}_1 s+\bar{C}_2\right) \\
&+\frac{C_1\sin^3\varphi}{\sqrt{1-\tau^2}}e^{-\cos\varphi s}(-\cos\sigma\cos(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s)+\sin\sigma\sin(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s))+C_2, \\
y_{\mathcal{B}}(s) &= \frac{\lambda\sin\varphi}{\sqrt{1-\tau^2}}\left(-\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}\sin(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma)+\cos\varphi\cos(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma)\right)\left(\bar{C}_1 s+\bar{C}_2\right) \\
&\frac{C_1\sin^3\varphi}{\sqrt{1-\tau^2}}e^{-\cos\varphi s}(\sin\sigma\cos(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma)+\cos\sigma\sin(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma))+C_3, \\
z_{\mathcal{B}}(s) &= \frac{\lambda}{\sqrt{1-\tau^2}}\left(\bar{C}_1 s+\bar{C}_2\right)+C_1 e^{-\cos\varphi s},
\end{aligned}
$$

where $\sigma, \bar{C}_1$, $\bar{C}_2$, C_1, C_2, C_3 are constants of integration.

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