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On Some New Almost Double Lacunary Δ^m-Sequence Spaces Defined by Orlicz Functions

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Abstract

In this paper we introduce a new concept for almost double lacunary Δ^m - sequence spaces defined by Orlicz function and give inclusion relations.The results here in proved are analogous to those by Ayhan Esi [General Mathematics (2009),2(17) 53-66].

Keywords: *Lacunary Sequence, Differene Double sequence; Orlicz Function,Strongly almost convergence.*

1 Introduction

Let l_∞, c and c_0 be the spaces of bounded, convergent and null sequences $x = (x_k)$, with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N}$.
A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent[15] if all Banach limits of $x = (x_k)$ coincide. in [15], it was shown that

$$\hat{c} = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s \right\}.$$

In [16,17], Maddox defined a sequence $x = (x_k)$ to be strongly convergent to a number L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0, \text{ uniformly in } s$$

By a lacunary sequence $\theta = (k_r)$, r=0,1,2,... where $k_o = 0$, we mean an increasing sequence of non negative integers $h_r = (k_r - k_{r-1}) \to \infty (r \to \infty)$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.
The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al.[3] as follows

$$N_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

The double lacunary sequence was defined by E.Savas and R.F.Patterson[20] as follows:
The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequence of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, h_s^- = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

The following intervals are determined by θ.

$$I_r = \{(k_r) : k_{r-1} < k < k_r\}, I_s = \{(l) : l_{s-1} < l < l_s\}$$

$$I_{r,s} = \{(k,l) : k_{r-1} < k < k_r \text{ and } l_{s-1} < l < l_s\},$$

$q_r = \frac{k_r}{k_{r-1}}, q_s^- = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r q_s^-$. We will denote the set of all lacunary sequences by $N_{\theta_{r,s}}$.

Let $x = (x_{kl})$ be a double sequence that is a double infinite array of elements x_{kl}. The space of double lacunary strongly convergent sequence is defined as follows:

$$N_{\theta_{r,s}} = \left\{ x = (x_k l) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |x_{kl} - L| = 0 \text{ for some } L \right\} (see[20]).$$

Double sequences have been studied by V.A.Khan[8,9,10,11], Moricz and Rhoades[19] and many others.

In [12], Kizmaz defined the sequence spaces
$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\}$ for $Z = l_\infty, c, c_0$, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. After Et. and Colak [1] generalized the difference sequence spaces to the sequence spaces $Z(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in Z\}$ for $Z = l_\infty, c, c_0$, where $m \in \mathbb{N}, \Delta^0 x = (x_k)$,
$\Delta x = (x_k - x_{k+1}), \Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$, and so that

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v}.$$

An *Orlicz Function* is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

An Orlicz function M satisfies the $\Delta_2 - condition$ ($M \in \Delta_2$ for short) if there exist constant $k \geq 2$ and $u_0 > 0$ such that

$$M(2u) \leq KM(u)$$

whenever $|u| \leq u_0$.

An Orlicz function M can always be represented (see[13])in the integral form
$M(x) = \int_0^x q(t)dt$, where q known as the kernel of M, is right differentiable for $t \geq 0, q(t) > 0$ for $t > 0, q$ is non-decreasing and $q(t) \to \infty$ as $t \to \infty$.

Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1,$$

since M is convex and $M(0) = 0$.
Lindesstrauss and Tzafriri [14] used the idea of Orlicz sequence space;

$$l_M := \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is Banach space with the norm the norm

$$\|x\|_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$. Orlicz function has been studied by V.A.Khan[4,5,6,7] and many others.

The purpose of this paper is to introduce and study a concept of lacunary almost generalized Δ^m-convergence function and to examine these new sequence spaces which also generalize the well known Orlicz sequence space l_M and strongly summable sequence $[C,1,p], [C,1,P]_0$ and $[C,1,p]_\infty$(see[18]).

Let M be an Orlicz function and $p = (p_k)$ be any bounded sequence of strictly positive real numbers. Ayhan Esi[2] defined the following sequence spaces:

$$[\hat{c}, M, p](\Delta^m) = \Bigg\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} \left[M\left(\frac{|\Delta^m x_{k+m} - L|}{\rho} \right) \right]^{p_k} = 0,$$

$$\text{uniformly in } m \text{ for some } \rho > 0 \text{ and } L > 0 \Bigg\}.$$

$$[\hat{c}, M, p]_0(\Delta^m) = \Bigg\{ x = (x_k) : \lim_{n\to\infty} \sum_{k=1}^{n} \left[M\left(\frac{|\Delta^m x_{k+m}|}{\rho} \right) \right]^{p_k} = 0,$$

$$\text{uniformly in } m, \text{for some } \rho > 0 \Bigg\}.$$

$$[\hat{c}, M, p]_\infty(\Delta^m) = \Bigg\{ x = (x_k) : \sup_{n,m} \frac{1}{n} \sum_{k=1} n \left[M\left(\frac{|\Delta^m x_{k+m}|}{\rho} \right) \right]^{p_k} < \infty, \text{for some } \rho > 0 \Bigg\}.$$

If $x = (x_k) \in [\hat{c}, M, p](\Delta^m)$, we say that $x = (x_k)$ is lacunary almost Δ^m-convergent to L with respect an Orlicz function M.

The folowing inequality will be used throughout the paper

$$|x_{kl} + y_{kl}|^{p_{kl}} \leq K(|x_{kl}|^{p_{kl}} + |y_{kl}|^{p_{kl}}) \qquad [1.1]$$

where x_{kl} and y_{kl} are complex numbers, $K = \max(1, 2^{H-1})$ and $H = \sup_{k,l} p_{kl} < \infty$.

2 Main Results

In the following paper we introduce and examine the following spaces defined by Orlicz function.

Definition 2.1. Let M be an Orlicz function and $p = (p_{kl})$ be any bounded sequence of strictly positive real numbers. We have

$$[\hat{c}_2, M, p]^{\theta}(\Delta^m) = \left\{ x = (x_{kl}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n} - L|}{\rho} \right) \right]^{p_{kl}} = 0, \right.$$

$$\left. \text{uniformly in } m \text{ and } n, \text{for some } \rho > 0 \text{ and } L > 0 \right\}.$$

$$[\hat{c}_2, M, p]_0^{\theta}(\Delta^m) = \left\{ x = (x_{kl}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho} \right) \right]^{p_{kl}} = 0, \right.$$

$$\left. \text{uniformly in } m \text{ and } n, \text{for some } \rho > 0 \right\}.$$

$$[\hat{c}_2, M, p]_{\infty}^{\theta}(\Delta^m) = \left\{ x = (x_{kl}) : \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho} \right) \right]^{p_{kl}} < \infty, \text{for some } \rho > 0 \right\}.$$

where

$$\Delta^m x = (\Delta^m x_{kl}) = (\Delta^{m-1} x_{kl} - \Delta^{m-1} x_{k,l+1} - \Delta^{m-1} x_{k+1,l} + \Delta^{m-1} x_{k+1,l+1}),$$

$$(\Delta^1 x_{kl}) = (\Delta x_{kl}) = (x_{kl} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}),$$

$$\Delta^0 x = (x_{k,l}), \quad \text{for all } \ k, l \in N,$$

and also this generalized difference double notion has the following binomial representation:

$$\Delta^m x_{kl} = \sum_{i=0}^{m} \sum_{j=0}^{m} (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{k+i,l+j}$$

.

If $x = (x_{kl}) \in [\hat{c}_2, M, p]^{\theta}(\Delta^m)$, we say that $x = (x_{kl})$ is double lacunary almost Δ^m-convergent to L with respect an Orlicz function M.

In this section we prove some results involving the double sequence spaces $[\hat{c}_2, M, p]^\theta(\Delta^m)$, $[\hat{c}_2, M, p]_0^\theta(\Delta^m)$ and $[\hat{c}_2, M, p]_\infty^\theta(\Delta^m)$.

Theorem 2.1. Let M be an Orlicz function and $p = (p_{kl})$ be a bounded sequence of strictly real numbers. Then $[\hat{c}_2, M, p]^\theta(\Delta^m)$, $[\hat{c}_2, M, p]_0^\theta(\Delta^m)$ and $[\hat{c}_2, M, p]_\infty^\theta(\Delta^m)$ are linear spaces over the set of complex numbers $\mathbb{C}$.

Proof. Let $x = (x_{kl}), y = (y_{kl}) \in [\hat{c}_2, M, p]_0^\theta(\Delta^m)$ and $\alpha, \beta \in \mathbb{C}$. Then there exists positive ρ_1 and ρ_2 such that

$$\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho_1} \right) \right]^{p_{kl}} = 0, \text{ uniformly in } m \text{ and } n$$

and

$$\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m y_{k+m,l+n}|}{\rho_2} \right) \right]^{p_{kl}} = 0, \text{ uniformly in } m \text{ and } n$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non decreasing convex function, by using equation [1.1], we have

$$\begin{aligned}
&\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m \alpha x_{k+m,l+n} + \beta y_{k+m,l+n}|}{\rho_3} \right) \right]^{p_{kl}} \\
&= \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m \alpha x_{k+m,l+n}|}{\rho_3} + \frac{|\beta \Delta^m (y_{k+m,l+n})|}{\rho_3} \right) \right]^{p_{kl}} \\
&\leq K \frac{1}{h_{rs}} \sum_{(k,l)\in I_{r,s}} \frac{1}{2^{p_{kl}}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho_1} \right] + K \frac{1}{h_{rs}} \sum_{(k,l)\in I_{r,s}} \frac{1}{2^{p_{kl}}} \left[M\left(\frac{|\Delta^m (y_{k+m,l+n})|}{\rho_2} \right) \right]^{p_{kl}} \\
&\leq K \frac{1}{h_{rs}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho_1} \right) \right] + K \frac{1}{h_{rs}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m (y_{k+m,l+n})|}{\rho_2} \right) \right]^{p_{kl}} \\
&\to 0 \text{ as } r, s \to \infty \text{ uniformly in } m \text{ and } n.
\end{aligned}$$

So $\alpha x + \beta y \in [\hat{c}_2, M, p]_0^\theta(\Delta^m)$. Hence $[\hat{c}_2, M, p]_0^\theta(\Delta^m)$ is a linear space. The proof for the cases $[\hat{c}_2, M, p]^\theta(\Delta^m)$ and $[\hat{c}_2, M, p]_\infty^\theta(\Delta^m)$ are similar to the above proof.

Theorem 2.2. For any Orlicz function M on a bounded double sequence $p = (p_{kl})$ of strictly positive real numbers, $[\hat{c}_2, M, p]_0^\theta(\Delta^m)$ is a topological linear space paranormed by

$$h(x_{kl}) = \sup_k |x_{k1}| + \sup_l |x_{1l}| + \inf \left\{ \rho^{\frac{p_{kl}}{H}} : \left(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho} \right) \right]^{p_{kl}} \right)^{\frac{1}{H}} \leq 1 \right\}$$

where $H = \max(1, \sup_{k,l} p_{kl}) \leq \infty$.

Proof. Clearly $h(x_{kl}) \geq 0$, for all $x = (x_{kl}) \in [\hat{c}, M, p]_0^{\theta}(\Delta^m)$
Since $M(0) = 0$, we get $h(0) = 0$
$h(-(x_{kl})) = h(x_{kl})$.
Let $(x_{kl}), (y_{kl}) \in [\hat{c}_2, M, p]_0^{\theta}(\Delta^m)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\left(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}} \leq 1$$

and

$$\left(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m y_{k+m,l+n}|}{\rho}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}} \leq 1$$

for each r, s, m and n.
Let $\rho = \rho_1 + \rho_2$. Then we have,

$$\left(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n} + y_{k+m,l+n}|}{\rho}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}}$$
$$\leq \left(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n}| + |\Delta^m y_{k+m,l+n}|}{\rho_1 + \rho_2}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}}$$
$$\leq \left(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[\frac{\rho_1}{\rho_1 + \rho_2} M\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho_1}\right)\right.\right.$$
$$\left.\left. + \frac{\rho_2}{\rho_1 + \rho_2} M\left(\frac{|\Delta^m y_{k+m,l+n}|}{\rho_2}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}}$$

By Minkowski's Inequality

$$\leq \left(\frac{\rho_1}{\rho_1 + \rho_2}\right)\left(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho_1}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}}$$
$$+ \left(\frac{\rho_1}{\rho_1 + \rho_2}\right)\left(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m y_{k+m,l+n}|}{\rho_2}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}} \leq 1$$

Since ρ_1 and ρ_2 are non negative, so we have

$$h(x_{kl} + y_{kl}) = \sup_k |x_{k1} + y_{k1}| + \sup_l |x_{1l} + y_{y_1 l}| + \inf\left\{\rho^{\frac{p_{kl}}{H}} : \left(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{kl} + y_{kl}|}{\rho}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}} \leq 1\right\}$$

$$\leq \sup_k |x_{k1}| + \sup_l |x_{1l}| + \inf\left\{\rho_1^{\frac{p_{kl}}{H}} : \left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{|\Delta^m x_{kl}|}{\rho_1}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}} \leq 1\right\}$$

$$+\sup_k |y_{k1}| + \sup_l |y_{1l}| + \inf\left\{\rho_2^{\frac{p_{kl}}{H}} : \left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{|\Delta^m y_{kl}|}{\rho_2}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}} \leq 1\right\}$$

$$= h(x_{kl}) + y(_{kl})$$

This implies that

$$h(x_{kl} + y_{kl}) \leq h(x_{kl}) + y(_{kl}).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition

$$h(\lambda(x_{kl})) = \sup_k |\lambda(x_{k1})| + \sup_l |\lambda(x_{1l})| + \inf\left\{\rho^{\frac{p_{kl}}{H}} : \left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{|\Delta^m \lambda x_{k+m,l+n}|}{\rho}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}} \leq 1\right\}$$

$$= |\lambda| \sup_k |x_{k1}| + |\lambda| \sup_l |x_{1l}| + \inf\left\{(|\lambda| t)^{\frac{p_{kl}}{H}} : \left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho}\right)\right]^{p_{kl}}\right)^{\frac{1}{H}} \leq 1\right\}$$

where $t = \frac{\rho}{|\lambda|}$
This complets the proof of the theorem.

Theorem 2.3. Let M be an Orlicz function. If $\sup_{k,l}[M(x)]^{p_{kl}} < \infty$ for all fixed $x > 0$ then

$$[\hat{c}_2, M, p]_0^\theta(\Delta^m) \subset [\hat{c}_2, M, p]_\infty^\theta(\Delta^m).$$

Proof. Let $x = (x_{kl}) \in [\hat{c}_2, M, p]_0^\theta(\Delta^m)$. Then there exists some positive ρ_1 such that

$$\lim_{k,l} \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho_1}\right)\right]^{p_{kl}} = 0, \text{ uniformly in } m \text{ and } n$$

Define $\rho = 2\rho_1$. Since M is non decreasing and convex, by using (1) we have

$$\sup_{r,s,m,n} \tfrac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\tfrac{|\Delta^m x_{k+m,l+n}|}{\rho}\right)\right]^{p_{kl}}$$

$$= \sup_{r,s,m,n} \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{|\Delta^m x_{k+m,l+n} - L + L|}{\rho}\right)\right]^{p_{kl}}$$

$$\le K \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[\frac{1}{2^{p_{kl}}} M\left(\frac{|\Delta^m x_{k+m,l+n} - L|}{\rho_1}\right)\right]^{p_{kl}}$$

$$+K \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[\frac{1}{2^{p_{kl}}} M\left(\frac{|L|}{\rho_1}\right)\right]^{p_{kl}}$$

$$\le K \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n} - L|}{\rho_1}\right)\right]^{p_{kl}}$$

$$+K \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|L|}{\rho_1}\right)\right]^{p_{kl}} < 1$$

Hence $x = (x_{kl}) \in [\hat{c}_2, M, p]_\infty^\theta(\Delta^m)$.
This complets the proof.

Theorem 2.4. Let $0 < \inf p_{k,l} = h \le p_{k,l} = H \le \infty$ and M, M_1 be Orlicz functions satisfying Δ_2-condition, then we have $[\hat{c}_2, M_1, p]_0^\theta(\Delta^m) \subset [\hat{c}_2, M_0 M_1, p]_0^\theta(\Delta^m)$, $[\hat{c}_2, M_1, p]^\theta(\Delta^m) \subset [\hat{c}_2, M_0 M_1, p]^\theta(\Delta^m)$ and $[\hat{c}_2, M_1, p]_\infty^\theta(\Delta^m) \subset [\hat{c}_2, M_0 M_1, p]_\infty^\theta(\Delta^m)$.

Proof. Let $x = (x_{kl}) \in [\hat{c}_2, M_0 M_1, p]_0^\theta(\Delta^m)$. Then we have

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M_1\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho}\right)\right]^{p_{kl}} = 0, \text{ uniformly in } m \text{ and } n.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \epsilon$ for $0 \le t \le \delta$.
Let $y_{k,l} = M_1\left(\frac{|\Delta^m x_{k+m,l+n}|}{\rho}\right)$ for $k, l, m, n \in \mathbb{N}$
We can write

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} [M(y_{k,l})]^{p_{kl}} = \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, y_{k,l}\le\delta} [M(y_{k,l})]^{p_{kl}} + \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, y_{k,l}>\delta} [M(y_{k,l})]^{p_{kl}}$$

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, y_{k,l}\le\delta} [M(y_{k,l})]^{p_{kl}} < \epsilon \qquad [2.1]$$

since M is continuous and $M(t) < \epsilon$ for $t \le \delta$.
For $y_{k,l} > \delta$ we use the fact that

$$y_{k,l} < \frac{y_{k,l}}{\delta} < 1 + \frac{y_{k,l}}{\delta}$$

Since M is non decreasing and convex, it follows that

$$M(y_{k,l}) < M(1 + \delta^{-1} y_{k,l}) = M\left(\frac{2}{2} + \frac{2}{2}\delta^{-1} y_{k,l}\right)$$

$$< \frac{1}{2}M(2) + \frac{1}{2}M(2\delta^{-1}y_{k,l})$$

Since M satisfies Δ_2-condition, there is a constant $K > 2$ such that

$$M(2\delta^{-1}y_{k,l}) \leq \frac{1}{2}K\delta^{-1}y_{k,l}M(2)$$

Hence

$$\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s},y_{k,l}>\delta}[M(y_{k,l})]^{p_{kl}} \leq \max\left(1,\left(\frac{KM(2)}{\delta}\right)\right)\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s},y_{k,l}>\delta}[(y_{k,l})]^{p_{kl}}$$

$$\rightarrow 0 \text{ as } r,s \rightarrow \infty \qquad [2.2]$$

By [2.1] and [2.2] we have $x = (x_{k,l} \in [\hat{c}_2, M_0M_1, p]_0^\theta(\Delta^m)$.
Similarly we can prove that
$[\hat{c}_2, M_1, p]^\theta(\Delta^m) \subset [\hat{c}_2, M_0M_1, p]^\theta(\Delta^m)$ and $[\hat{c}_2, M_1, p]_\infty^\theta(\Delta^m) \subset [\hat{c}_2, M_0M_1, p]_\infty^\theta(\Delta^m)$.
This complets the proof.

Taking $M_1(x)$ in above theorem we have the following result.

Corollary 2.5. Let $0 < \inf p_{k,l} = h \leq p_{k,l} = H \leq \infty$ and M be Orlicz function satisfying Δ_2-condition, then we have $[\hat{c}_2, p]_0^\theta(\Delta^m) \subset [\hat{c}_2, M, p]_0^\theta(\Delta^m),$ and $[\hat{c}_2, p]_\infty^\theta(\Delta^m) \subset [\hat{c}_2, M, p]_\infty^\theta(\Delta^m)$.

Theorem 2.6. Let M be an Orlicz function. Then the following statements are euivalent:

(i) $[\hat{c}_2, p]_\infty^\theta(\Delta^m) \subset [\hat{c}_2, M, p]_\infty^\theta(\Delta^m)$.

(ii) $[\hat{c}_2, p]_0^\theta(\Delta^m) \subset [\hat{c}_2, M, p]_\infty^\theta(\Delta^m)$.

(iii) $\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{t}{\rho}\right)\right]^{p_{kl}} < \infty \quad (t, \rho > 0)$.

Proof. $(i) \Rightarrow$ **(ii)**: It is obvious, since $[\hat{c}_2, p]_0^\theta(\Delta^m) \subset [\hat{c}_2, p]_\infty^\theta(\Delta^m)$.

(ii) $\Rightarrow$ **(iii)**: Let $[\hat{c}_2, p]_0^\theta(\Delta^m) \subset [\hat{c}_2, M, p]_\infty^\theta(\Delta^m)$.
Suppose that (iii) doesnot hold. Then for some $t, \rho > 0$

$$\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{t}{\rho}\right)\right]^{p_{kl}} = \infty.$$

and therefore we can find a subinterval $I_{r(i),s(j)}$ of the set of interval $I_{r,s}$ such that

$$\frac{1}{h_{r(i),s(j)}} \sum_{(k,l)\in I_{r(i),s(j)}} \left[M\left(\frac{(ij)^{-1}}{\rho}\right)\right]^{p_{kl}} > ij \qquad [2.3]$$

$$i = 1, 2, 3, ..., j = 1, 2, 3......$$

Define the double sequence $x = (x_{kl})$ by

$$\Delta^m x_{k+m,l+n} = \begin{cases} (ij)^{-1} & (k,l) \in I_{r(i),s(j)} \\ 0 & (k,l) \notin I_{r(i),s(j)}. \end{cases}$$

$$\text{for all } m, n \in \mathbb{N}$$

Then $x = (x_{kl}) \in [\hat{c}_2, p]_0^\theta(\Delta^m)$ but by equation [2.3] $x = (x_{kl}) \notin [\hat{c}_2, M, p]_\infty^\theta(\Delta^m)$. Which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ **(i)**: Let (iii) hold and $x = (x_{kl}) \in [\hat{c}_2, p]_\infty^\theta(\Delta^m)$.

Suppose that $x = (x_{kl}) \notin [\hat{c}_2, M, p]_\infty^\theta(\Delta^m)$. Then

$$\sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{\Delta^m x_{k+m,l+n}}{\rho}\right)\right]^{p_{kl}} = \infty \qquad [2.4]$$

Let $t = |\Delta^m x_{k+m,l+n}|$ for each k, l and fixed m, n then by equation [2.4]

$$\sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{t}{\rho}\right)\right]^{p_{kl}} = \infty$$

Which contradicts (iii). Hence (i) must hold.

Theorem 2.7. Let $1 \leq p_{kl} \leq \sup p_{kl} < \infty$ and M be an Orlicz function. Then the following statements are equivalent:

(i) $[\hat{c}_2, M, p]_0^\theta(\Delta^m) \subset [\hat{c}_2, p]_0^\theta(\Delta^m)$.

(ii) $[\hat{c}_2, M, p]_0^\theta(\Delta^m) \subset [\hat{c}_2, p]_\infty^\theta(\Delta^m)$.

(iii) $\inf_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{t}{\rho}\right)\right]^{p_{kl}} > 0 \quad (t, \rho > 0)$

Proof. **(i)** $\Rightarrow$ **(ii)**: It is obvious.
(ii) $\Rightarrow$ **(iii)**: Let (ii) hold. Suppose (iii) doesnot hold. Then

$$\inf_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{t}{\rho}\right)\right]^{p_{kl}} = 0 \quad (t, \rho > 0)$$

So we can find a subinterval $I_{r(i),s(j)}$ of the set of interval $I_{r,s}$ such that

$$\frac{1}{h_{r(i),s(j)}} \sum_{(k,l)\in I_{r(i),s(j)}} \left[M\left(\frac{ij}{\rho}\right)\right]^{p_{kl}} < (ij)^{-1} \qquad [2.5]$$

$$i = 1, 2, 3, ..., j = 1, 2, 3, ...$$

Define the double sequence $x = (x_{kl})$ by

$$\Delta^m x_{k+m,l+n} = \begin{cases} (ij)^{-1} & (k,l) \in I_{r(i),s(j)} \\ 0 & (k,l) \notin I_{r(i),s(j)}. \end{cases}$$

$$\text{for all } m, n \in \mathbb{N}$$

Thus by equation [2.5], $x = (x_{kl}) \in [\hat{c}_2, M, p]_0^\theta(\Delta^m)$ but by equation [2.3] $x = (x_{kl}) \notin [\hat{c}_2, p]_\infty^\theta(\Delta^m)$. Which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ **(i)**: Let (iii) hold and suppose that $x = (x_{kl}) \in [\hat{c}_2, M, p]_0^\theta(\Delta^m)$, that is

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|\Delta^m x_{k+m,l+n}}{\rho}\right)\right]^{p_{kl}} = 0 \text{ uniformly in } m \text{ and, for some } \rho > 0. \qquad [2.6]$$

Again suppose that $x = (x_{kl}) \in [\hat{c}, p]_0^\theta(\Delta^m)$. Then for some $\epsilon > 0$ and a subinterval $I_{r(i),s(j)}$ of the set interval $I_{r,s}$, we have $|\Delta^m x_{k+m,l+n}| \geq \epsilon$ for all $k, l \in \mathbb{N}$ and some $i \geq i_0, j \geq j_0$. Then, from the properties of the Orlicz function, we can write

$$M\left(\frac{|\Delta^m x_{k+m,l+n}}{\rho}\right)^{p_{kl}} \geq M\left(\frac{\epsilon}{\rho}\right)^{p_{kl}}$$

and cosequently by equation [2.6]

$$\lim_{r,s\to\infty} \frac{1}{h_{r(i),s(j)}} \sum_{(k,l)\in I_{r(i),s(j)}} \left[M\left(\frac{t}{\rho}\right)\right]^{p_{kl}} = 0$$

Which contradicts (iii). Hence (i) must hold.

Theorem 2.8. Let $0 < p_{k,l} \leq q_{k,l}$ for all $k, l \in \mathbb{N}$ and $\left(\frac{q_{k,l}}{p_{k,l}}\right)$ be bounded. Then,

$$[\hat{c}_2, M, q]^\theta(\Delta^m) \subset [\hat{c}_2, p]^\theta(\Delta^m).$$

Proof. Let $x \in [\hat{c}_2, M, q]^\theta(\Delta^m)$
Write

$$t_{k,l} = \left[M\left(\frac{\Delta^m x_{k+m,l+n}}{\rho} \right) \right]^{p_{kl}}$$

and $\lambda_{k,l} = \frac{p_{k,l}}{q_{k,l}}$.
Since $0 < p_{k,l} \leq q_{k,l}$, therefore $0 < \lambda_{k,l} \leq 1$.
Take $0 < \lambda \leq \lambda_{k,l}$.
Define

$$u_{k,l} = \begin{cases} t_{k,l} & t_{k,l} \geq 1 \\ 0 & t_{k,l} < 1. \end{cases}$$

$$v_{k,l} = \begin{cases} 0 & t_{k,l} \geq 1 \\ t_{k,l} & t_{k,l} < 1. \end{cases}$$

So $t_{k,l} = u_{k,l} + v_{k,l}$ and
$t_{k,l}^{\lambda_{k,l}} = u_{k,l}^{\lambda_{k,l}} + v_{k,l}^{\lambda_{k,l}}$
Now it follows that
$u_{k,l}^{\lambda_{k,l}} \leq u_{k,l} \leq t_{k,l}$ and $v_{k,l}^{\lambda_{k,l}} \leq v_{k,l}^{\lambda}$
Therfore

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} t_{k,l}^{\lambda_{k,l}} \leq \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} t_{k,l} + \left[\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} v_{k,l} \right]^{\lambda}$$

Hence $x \in [\hat{c}_2, M, p]^\theta(\Delta^m)$.

By using above theorem it is easy to prove the following result.

Corollary 2.9(i). If $0 < \inf p_{k,l} \leq p_{k,l} \leq 1$ for all $k, l \in \mathbb{N}$ then,

$$[\hat{c}_2, M, p]^\theta(\Delta^m) \subset [\hat{c}_2, p]^\theta(\Delta^m).$$

. **(ii).** If $0 \leq p_{k,l} \leq \sup p_{k,l} \leq \infty$ for all $k, l \in \mathbb{N}$ then,

$$[\hat{c}_2, M]^\theta(\Delta^m) \subset [\hat{c}_2, M, p]^\theta(\Delta^m).$$

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