EXISTENCE OF A POSITIVE SOLUTION FOR AN NTH ORDER BOUNDARY VALUE PROBLEM FOR NONLINEAR DIFFERENCE EQUATIONS

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ABSTRACT. The nth order eigenvalue problem:
\[ \Delta^n x(t) = (-1)^{n-k} \lambda f(t, x(t)), \quad t \in [0, T], \]
\[ x(0) = x(1) = \cdots = x(k - 1) = x(T + k + 1) = \cdots = x(T + n) = 0, \]
is considered, where \( n \geq 2 \) and \( k \in \{1, 2, \ldots, n - 1\} \) are given. Eigenvalues \( \lambda \) are determined for \( f \) continuous and the case where the limits \( f_0(t) = \lim_{n \to 0^+} f(t, u) \) and \( f_\infty(t) = \lim_{n \to \infty} f(t, u) \) exist for all \( t \in [0, T] \). Guo’s fixed point theorem is applied to operators defined on annular regions in a cone.

1. Introduction

Define the operator \( \Delta \) to be the forward difference
\[ \Delta u(t) = u(t + 1) - u(t), \]
and then define
\[ \Delta^i u(t) = \Delta(\Delta^{i-1} u(t)), \quad i \geq 1. \]

For \( a < b \) integers define the discrete interval \([a, b] = \{a, a+1, \ldots, b\} \). Let the integers \( n, T \geq 2 \) be given, and choose \( k \in \{1, 2, \ldots, n - 1\} \). Consider the nth order nonlinear difference equation
\[ \Delta^n x(t) = (-1)^{n-k} \lambda f(t, x(t)), \quad t \in [0, T], \]
satisfying the boundary conditions
\[ x(0) = x(1) = \cdots = x(k - 1) = x(T + k + 1) = \cdots = x(T + n) = 0. \]

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We determine eigenvalues $\lambda$ that yield a solution to (1) and (2), where
\[ (A)f : [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+ \]
is continuous, where $\mathbb{R}^+$ denotes the nonnegative reals,

(B) For all $t \in [0, T]$, $f_0(t) = \lim_{u \to 0^+} \frac{f(t, u)}{u}$ and $f_\infty(t) = \lim_{n \to \infty} \frac{f(t, u)}{u}$ both exist.

We apply Guo’s fixed point theorem using cone methods, Guo and Lakshmikantham [14], and Krasnosel’skiĭ [19], to accomplish this. This method was first applied to differential equations in the landmark paper by Erbe and Wang [12]. Our proof will follow along the lines of those in Henderson [16], Lauer [17], and Merdivenci [20], additionally utilizing techniques from Peterson [21], Hartman [15], Eloe and Kaufmann [11], Agarwal and Wong [6,7], Agarwal and Henderson [1], and Agarwal, Henderson and Wong [2]. A key to applying this fixed point theorem involves discrete concavity of solutions of the boundary value problem in conjunction with a lower bound on an appropriate Green’s function. Extensive use of the results by Eloe [8] concerning a lower bound for the Green’s function will be made. Related results for nth order differential equation may be found in Agarwal and Wong [3,4], Eloe and Henderson [9,10], and Fang [13].

2. Preliminaries

Let $G(t, s)$ be the Green’s function for the disconjugate boundary value problem
\[ Lx(t) \equiv \Delta^n x(t) = 0, t \in [0, T], \]
and satisfying (2), where, as shown in Kelly and Peterson [18], $G(t, s)$ is the unique function satisfying:

(a) $G(t, s)$ is defined for all $t \in [0, T + n], s \in [0, T]$.
(b) $LG(t, s) = \delta_{ts}$ for all $t \in [0, T], s \in [0, T]$ where $\delta_{ts} = 1$ if $t = s$, $\delta_{ts} = 0$ if $t \neq s$.
(c) For all $s \in [0, T]$, $G(t, s)$ satisfies the boundary conditions (2) in $t$.

We will use $G(t, s)$ as the kernel of an integral operator preserving a cone in a Banach space. This is the setting for our fixed point theorem.

Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be such that $\mathcal{P}$ is closed and non-empty. Then $\mathcal{P}$ is a cone provided (i) $au + bv \in \mathcal{P}$ for all $u, v \in \mathcal{P}$ and for all $a, b \geq 0$, and (ii) $u, -u \in \mathcal{P}$ implies $u = 0$.

Applying the following fixed point theorem from Guo, Guo and Lakshmikantham [14], will yield solutions of (1), (2) for certain $\lambda$.

**Theorem 1.** Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Let $\Omega_1$ and $\Omega_2$ be two bounded open sets in $\mathcal{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let
\[
H : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}
\]
be a completely continuous operator such that, either
(i) \( \| Hx \| \leq \| x \| , x \in \mathcal{P} \cap \partial \Omega_1 \), and \( \| Hx \| \geq \| x \| , x \in \mathcal{P} \cap \partial \Omega_2 \), or

(ii) \( \| Hx \| \geq \| x \| , x \in \mathcal{P} \cap \partial \Omega_1 \), and \( \| Hx \| \leq \| x \| , x \in \mathcal{P} \cap \partial \Omega_2 \).

Then \( H \) has a fixed point in \( \mathcal{P} \cap (\Omega_2 \setminus \Omega_1) \).

We now apply Theorem 1 to the eigenvalue problem (1), (2), following along the lines of methods incorporated by Henderson [16]. Note that \( x(t) \) is a solution of (1), (2) if, and only if,

\[
x(t) = (-1)^{n-k} \lambda \sum_{s=0}^{T} G(t, s)f(s, x(s)), \quad t \in [0, T].
\]

Hartman [15] extensively studied the boundary value problem (1), (2), with \( (-1)^{n-k} \lambda f(t, u) \geq 0 \). We begin by stating three lemmas from Hartman.

**Lemma 1.** Let \( G(t, s) \) denote the Green’s function of (3), (2). Then

\[
(-1)^{n-k} G(t, s) \geq 0, \quad (t, s) \in [k, T+k] \times [0, T].
\]

**Lemma 2.** Assume that \( u \) satisfies the difference inequality

\[
(-1)^{n-k} \Delta^n u(t) \geq 0, \quad t \in [0, T],
\]

and the homogeneous boundary conditions, (2). Then \( u(t) \geq 0, \ t \in [0, T+k] \).

**Lemma 3.** Suppose that the finite sequence \( u(0), \ldots, u(j) \) has \( N_j \) nodes and the sequence \( \Delta u(0), \ldots, \Delta u(j-1) \) has \( M_j \) nodes. Then \( M_j \geq N_j - 1 \).

Eloe [8] employed these three lemmas to arrive at the following theorem that gives a lower bound for the solution to the class of boundary value problems studied by Hartman.

**Theorem 2.** Assume that \( u \) satisfies the difference inequality

\[
(-1)^{n-k} \Delta^n u(t) \geq 0, \quad t \in [0, T],
\]

and the homogeneous boundary conditions, (2). Then for \( t \in [k, T+k] \),

\[
(-1)^{n-k} u(t) \geq \frac{\nu!}{[(T+1)\cdots(T+\nu)]} \| u \| ,
\]

where \( \| u \| = \max_{t \in [k, T+k]} |u(t)| \) and \( u = \max\{k, n-k\} \).

We remark that Agarwal and Wong [5] have recently sharpened the inequality of Theorem 2. However, this sharper inequality is of little consequence for this work.

Eloe also contributed the following corollary.

**Corollary 1.** Let \( G(t, s) \) denote the Green’s function for the boundary value problem, (3), (2). Then for all \( s \in [0, T], t \in [k, T+k] \),

\[
(-1)^{n-k} G(t, s) \geq \frac{\nu!}{[(T+1)\cdots(T+\nu)]} \| G(\cdot, s) \| ,
\]

where \( \| G(\cdot, s) \| = \max_{t \in [k, T+k]} |G(t, s)| \) and \( \nu = \max\{k, n-k\} \).
To fulfill the hypotheses of Theorem 1, let
\[ \mathcal{B} = \{ u : [0, T+n] \to \mathbb{R} \mid u(0) = u(1) = \cdots = u(k-1) = u(T+k+1) = \cdots = u(T+n) = 0 \}, \]
with \( \|u\| = \max_{t \in [t,T+k]} |u(t)| \). Now, \((\mathcal{B}, \|\cdot\|)\) is a Banach space.

Let \( \sigma = \nu! / ((T+1) \cdots (T+\nu)) \),
and define a cone
\[ \mathcal{P} = \{ u \in \mathcal{B} \mid u(t) \geq 0 \text{ on } [0, T+n] \text{ and } \min_{t \in [k,T+k]} u(t) \geq \sigma \|u\| \}. \]

Also choose \( \tau, \eta \in [k, T+k] \) such that
\[ (-1)^{n-k} \sum_{s=k}^{T} G(\tau, s)f_\infty(s) = \max_{t \in [k,T+k]} \sum_{s=k}^{T} G(t, s)f_\infty(s), \]
\[ (-1)^{n-k} \sum_{s=k}^{T} G(\eta, s)f_0(s) = \max_{t \in [k,T+k]} (-1)^{n-k} \sum_{s=k}^{T} G(t, s)f_0(s). \]

3. Main Results

**Theorem 3.** Assume conditions (A) and (B) are satisfied. Then, for each \( \lambda \) satisfying
\[ \frac{1}{\sigma(-1)^{n-k} \sum_{s=k}^{T} G(\tau, s)f_\infty(s)} < \lambda < \frac{1}{\sum_{s=k}^{T} \|G(:, s)\|f_0(s)} \]
there exists at least one solution of (1), (2) in \( \mathcal{P} \).

**Proof.** Let \( \lambda \) be given as in Theorem 3. Let \( \epsilon > 0 \) be such that
\[ \frac{1}{\sigma(-1)^{n-k} \sum_{s=k}^{T} G(\tau, s)(f_\infty(s) - \epsilon)} \geq \lambda \geq \frac{1}{\sum_{s=0}^{T} \|G(:, s)\|(f_0(s) + \epsilon)}. \]

Define a summation operator \( H : \mathcal{P} \to \mathcal{B} \) by
\[ Hx(t) = (-1)^{n-k} \lambda \sum_{s=0}^{T} G(t, s)f(s, x(s)), \quad x \in \mathcal{P}. \]

We seek a fixed point of \( H \) in the cone \( \mathcal{P} \). By the nonnegativity of \( f \) and \((-1)^{n-k}G, Hx(t) \geq 0 \) on \([0, T+n] \), and from the properties of \( G, Hx \)
satisfies the boundary conditions. Now if we choose \( x \in \mathcal{P} \), we have

\[
Hx(t) = (-1)^{n-k} \lambda \sum_{s=0}^{T} G(t, s) f(s, x(s)) \\
\leq \lambda \sum_{s=0}^{T} \|G(\cdot, s)\| f(s, x(s)), t \in [k, T + k].
\]

So

\[
\|Hx\| = \max_{t \in [k, T + k]} |Hx(t)| \leq \lambda \sum_{s=0}^{T} \|G(\cdot, s)\| f(s, x(s)).
\]

Hence, if \( x \in \mathcal{P} \), \((-1)^{n-k} G(t, s) \geq \sigma \|G(\cdot, s)\|\), for \( t \in [k, T + k] \) and \( s \in [0, T] \), and thus,

\[
\min_{t \in [k, T + k]} Hx(t) = \min_{k, T + k} (-1)^{n-k} \lambda \sum_{s=0}^{T} G(t, s) f(s, x(s)) \\
\geq \sigma \lambda \sum_{s=0}^{T} \|G(\cdot, s)\| f(s, x(s)) \\
\geq \sigma \|Hx\|.
\]

Thus \( H : \mathcal{P} \rightarrow \mathcal{P} \). Additionally, \( H \) is completely continuous.

Now consider \( f_0(t) \). For each \( t \in [0, T] \), there exists \( k_t > 0 \) such that \( f(t, u) \leq (f_0(t) + \epsilon)u \) for \( 0 < u \leq k_t \). Let \( K_1 = \min_{t \in [0, T]} k_t \). So, for \( x \in \mathcal{P} \) with \( \|x\| = K_1 \), we have

\[
Hx(t) = (-1)^{n-k} \lambda \sum_{s=0}^{T} G(t, s) f(s, x(s)) \\
\leq \lambda \sum_{s=0}^{T} \|G(\cdot, s)\| (f_0(s) + \epsilon) x(s) \\
\leq \lambda \sum_{s=0}^{T} \|G(\cdot, s)\| (f_0(s) + \epsilon) \|x\| \\
\leq \|x\|, \quad t \in [k, T + k].
\]

Therefore, \( \|H(x)\| \leq \|x\| \). Hence, if we set

\[
\Omega_1 = \{u \in \mathcal{B} \|u\| < K_1\}
\]

then

(8) \[
\|Hx\| \leq \|x\| \text{ for all } x \in \mathcal{P} \cap \partial \Omega_1.
\]

Next consider \( f_\infty(t) \). For each \( t \in [0, T] \), there exists \( k_t \) such that \( f(t, u) \geq (f_\infty(t) - \epsilon)u \) for all \( u \geq k_t \). Let \( K_2 = \max_{t \in [0, T]} k_t \) and \( K_2 = \).
max \{2K_1, \frac{1}{\sigma} \hat{K}_2\}. Define
\[ \Omega_2 = \{ u \in \mathcal{B} \| u \| < K_2 \} \]

If \( x \in \mathcal{P} \) with \( \| x \| = K_2 \), then \( \min_{t \in [k,T+k]} x(t) \geq \sigma \| x \| \geq \hat{K}_2 \), and
\[
H \tau = (-1)^{n-k} \lambda \sum_{s=0}^{T} G(\tau,s)f(s,x(s)) \\
\leq (-1)^{n-k} \lambda \sum_{s=0}^{T} G(\tau,s)f(s,x(s)) \\
\geq (-1)^{n-k} \lambda \sum_{s=0}^{T} G(\tau,s)f_\infty(s) - \epsilon) \| x(s) \| \\
\geq \sigma(-1)^{n-k} \lambda \sum_{s=k}^{T} G(\tau,s)(f_\infty(s) - \epsilon) \| x \| \\
\geq \| x \|.
\]

Thus, \( \| Hx \| \geq \| x \| \), and so
\[
\| Hx \| \geq \| x \| \quad \text{for all } x \in \mathcal{P} \cap \partial \Omega_2
\]

So with (8) and (9) we have shown that \( H \) satisfies the first condition of Theorem 1. Thus we can conclude that \( H \) has a fixed point \( u(t) \in \mathcal{P} \cap (\Omega_2 \setminus \Omega_1) \). This fixed point, \( u(t) \), is a solution of (1), (2) corresponding to the given value of \( \lambda \). \( \blacksquare \)

**Theorem 4.** Assume conditions (A) and (B) are satisfied. Then, for each \( \lambda \) satisfying
\[
\frac{1}{\sigma(-1)^{n-k} \sum_{s=k}^{T} G(\eta,s)f_0(s)} < \lambda < \frac{1}{\sum_{s=0}^{T} \| G(\cdot,s) \| f_\infty(s)}
\]

there exists at least solution of (1), (2) in \( \mathcal{P} \).

**Proof.** Let \( \lambda \) be given as stated above. Let \( \epsilon > 0 \) be such that
\[
\frac{1}{\sigma(-1)^{n-k} \sum_{s=k}^{T} G(\eta,s)(f_0(s) - \epsilon)} \leq \lambda \leq \frac{1}{\sum_{s=0}^{T} \| G(\cdot,s) \| (f_\infty(s) + \epsilon)}
\]

Let \( H \) be the cone preserving, completely continuous operator defined in (7).

Consider \( f_0(t) \). For each \( t \in [0,T] \) there exists \( k_t > 0 \) such that \( f(t,u) \geq (f_0(t) - \epsilon)u \) for \( 0 < u \leq k_t \). Let \( K_1 = \min_{t \in [0,T]} k_t \). So, for \( x \in \mathcal{P} \) with \( \| x \| = K_1 \),
we have
\[ Hx(\eta) = (-1)^{n-k} \lambda \sum_{s=0}^{T} G(\eta, s) f(s, x(s)) \]
\[ \geq (-1)^{n-k} \lambda \sum_{s=k}^{T} G(n, x) f(x, x(s)) \]
\[ \geq (-1)^{n-k} \lambda \sum_{s=0}^{T} G(\eta, s) (f_0(s) - \epsilon) x(s) \]
\[ \geq \sigma (-1)^{n-k} \lambda \sum_{s=k}^{T} G(\eta, s) (f_0(s) - \epsilon) \|x\| \]
\[ \geq \|x\|. \]

Therefore, \(\|Hx\| \geq \|x\|\). Hence, if we set
\[ \Omega_1 = \{ u \in \mathcal{B} | \|u\| < K_1 \}, \]
\[ (10) \quad \|Hx\| \geq \|x\|, \text{ for all } x \in \mathcal{P} \cap \partial \Omega_1. \]

Next consider \(f_{\infty}(t)\). For each \(t \in [0, T]\) there exists \(k_t > 2K_1\) such that \(f(t, u) \leq (f_{\infty}(t) + \epsilon) u\) for all \(u \geq k_t\). There exists sets \(I, J \subset [0, T]\), with \(I \cup J = [0, T]\), such that for all \(t \in I, f(t, u)\) is bounded as a function of \(u\), and for all \(t \in J, f(t, u)\) is unbounded as a function of \(u\).

Choose \(M > 0\) such that for all positive \(u\) and for all \(t \in I, f(t, u) \leq M\). Let
\[ \kappa_t = \max \left\{ k_t, \frac{M}{f_{\infty}(t) + \epsilon} \right\} \]

For each \(t \in J\) choose \(\kappa_t \geq k_t\) such that \(f(t, u) \leq f(t, \kappa_t)\), for \(0 < u \leq \kappa_t\). Let \(K_2 = \max_{t \in [0, T]} \kappa_t\). By the continuity of \(f\), for all \(t \in J\) there exists \(\mu_t\), where \(\kappa_t \leq \mu_t \leq K_2\), such that \(f(t, u) \leq f(t, \mu_t)\) for all \(0 < u \leq K_2\). Now
\[ Hx(t) = (-1)^{n-k} \lambda \sum_{s=0}^{T} G(t, s) f(s, x(s)) \]
\[ \leq \lambda \sum_{s \in J} \|G(\cdot, s)\| M + \lambda \sum_{s \in I} \|G(\cdot, s)\| f(s, \mu_s) \]
\[ \leq \lambda \sum_{s \in I} \|G(\cdot, s)\| (f_{\infty}(s) + \epsilon) \kappa_s + \lambda \sum_{s \in J} \|G(\cdot, s)\| (f_{\infty}(s) + \epsilon) \mu_s \]
\[ \leq \lambda \sum_{s=0}^{T} \|G(\cdot, s)\| (f_{\infty}(s) + \epsilon) K_2 \]
\[ = \lambda \sum_{s=0}^{T} \|G(\cdot, s)\| (f_{\infty}(s) + \epsilon) \|x\| \]
\[ \leq \|x\| \quad t \in [k, T + k], \]
for \( x \in \mathcal{P} \) with \( \|x\| = K_2 \). Now if we take
\[
\Omega_2 = \{ u \in \mathcal{B} | \|u\| < K_2 \},
\]
then
\[
\|Hx\| \leq \|x\| \text{ for all } x \in \mathcal{P} \cup \partial \Omega_2.
\] (11)

Thus, with (10) and (11), we have shown that \( H \) satisfies the hypotheses to Theorem 1(ii), which yields a fixed point of \( H \) belonging to \( \mathcal{P} \cap (\overline{\Omega_2 \setminus \Omega_1}) \). This fixed point is a solution of (1), (2) corresponding to the given \( \lambda \).\( \blacksquare \)

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