ASYMPTOTIC FORMULAS AND CRITICAL EXPONENTS FOR TWO-PARAMETER NONLINEAR EIGENVALUE PROBLEMS

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We study the nonlinear two-parameter problem

\[-u''(x) + \lambda u(x)^q = \mu u(x)^p,\]

\[u(x) > 0, \ x \in (0, 1), \ u(0) = u(1) = 0.\]

Here, \(1 < q < p\) are constants and \(\lambda, \mu > 0\) are parameters. We establish precise asymptotic formulas with exact second term for variational eigencurve \(\mu(\lambda)\) as \(\lambda \to \infty\). We emphasize that the critical case concerning the decaying rate of the second term is \(p = (3q - 1)/2\) and this kind of criticality is new for two-parameter problems.

1. Introduction

We consider the following nonlinear two-parameter problem:

\[-u''(x) + \lambda u(x)^q = \mu u(x)^p, \quad x \in I = (0, 1),\]

\[u(x) > 0, \ x \in I,\]

\[u(0) = u(1) = 0,\]

where \(1 < q < p\) and \(\lambda, \mu > 0\) are parameters.

The purpose of this paper is to establish the asymptotic formulas for the eigencurve \(\mu = \mu(\lambda)\) with the exact second term as \(\lambda \to \infty\) by using a variational method. We also establish the critical relationship between \(p\) and \(q\) from a viewpoint of the decaying rate of the second term of \(\mu(\lambda)\).

The study of two-parameter eigenvalue problems began with the oscillation theory and has been investigated by many authors. We refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references therein. One of the main problems in this area is to analyze the structure of the solution set \(\{(\lambda, \mu, u)\}\) of (1.1), and the effective approach to this problem is to study the structure of the set \(S_{\lambda, \mu} := \{ (\lambda, \mu, \|u\|_{p+1}) \} \subset \mathbb{R}^3\) for large \(\lambda\). In Shibata [7], by using a standard variational framework (see Section 2), the variational eigencurve \(\mu = \mu(\lambda)\) was defined to
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analyze $S_{\lambda, \mu}$ and the following asymptotic formula for $\mu(\lambda)$ as $\lambda \to \infty$ was established:

$$\mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} + o(\lambda^{(p+3)/(2p-q+3)})$$

(1.2)

where

$$C_1 = \left( \frac{(p+1)(q+3)}{(p+3)(q+1)} \right)^{\frac{1}{2p-q}} \frac{\sqrt{\pi(q+1)}}{2} \left( \frac{p+1}{q+1} \right)^{(q+3)/(2p-q)} \frac{\Gamma\left((q+3)/2(p-q)\right)}{\Gamma\left((p+3)/2(p-q)\right)} \right)^{2(p-q)/(2p-q+3)},$$

(1.3)

By this formula, we understood the first term of $\mu(\lambda)$ as $\lambda \to \infty$. However, the remainder estimate of $\mu(\lambda)$ has not been obtained. The purpose here is to obtain the exact second term of $\mu(\lambda)$ as $\lambda \to \infty$. We emphasize that the second term depends deeply on the relationship between $p$ and $q$, and the critical case is $p = (3q - 1)/2$. More precisely, if $p = (3q - 1)/2$, then the asymptotic behavior of the second term of $\mu(\lambda)$ is completely different from that of the case where $p \neq (3q - 1)/2$. As far as we know, this kind of criticality is new for two-parameter problems and great interest by itself. Finally, it should be mentioned that the asymptotic behavior of such eigencurve is also effected by the variational framework (cf. [6, 8]).

2. Main results

We explain notations before stating our results. Let $H^1_0(I)$ be the usual real Sobolev space. Let $\|u\|_r$ denote the usual $L^r$-norm. For $u \in H^1_0(I)$,

$$E_\lambda(u) := \frac{1}{2} \|u'\|^2 + \frac{1}{q+1} \lambda \|u\|^{q+1}_{q+1},$$

$$M_\gamma := \{u \in H^1_0(I) : \|u\|_{p+1} = \gamma\},$$

(2.1)

where $\gamma > 0$ is a fixed constant. For a given $\lambda > 0$, we call $\mu(\lambda)$ the variational eigenvalue when the following conditions are satisfied:

$$(\lambda, \mu(\lambda), u_\lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \times M_\gamma \quad \text{satisfies (1.1)},$$

$$E_\lambda(u_\lambda) = \inf_{u \in M_\gamma} E_\lambda(u).$$

(2.2)

Then $\mu(\lambda)$ is obtained as a Lagrange multiplier and is represented explicitly as follows:

$$\mu(\lambda) = \frac{\|u_\lambda'\|^2 + \lambda\|u_\lambda\|^{q+1}_{q+1}}{\gamma^{p+1}}.$$
Indeed, multiply the equation in (1.1) by $u_\lambda$. Then integration by parts yields
\[ \|u_\lambda\|_{2}^{2} + \lambda \|u_\lambda\|_{q+1}^{q+1} = \mu(\lambda) \|u_\lambda\|_{p+1}^{p+1} = \mu(\lambda) \gamma^{p+1}. \] (2.4)

This implies (2.3). The existence of $\mu(\lambda)$ for a given $\lambda > 0$ is ensured in [7, Theorem 2.1] and $\mu(\lambda)$ is continuous for $\lambda > 0$ (cf. [7, Theorem 2.2]). Finally, let

\[ K_1 := \left( \sqrt{2} \left( \frac{q+1}{p+1} \right)^{(q-1)/(2(p-q))} \Gamma(1/(q+1)) \Gamma((q-1)/2(q+1)) \right)^{2(q+1)/(q-1)} \times C_1^{q-1)/(2(p-q))}, \]
\[ K_2 := \frac{1}{2} \int_{0}^{1} \frac{s^{(2p-3q-1)/2} (1-s^{p+1})}{(1-s^{p-q})^{3/2}} \, ds, \]
\[ K_3 := \frac{2^{(2p-2q+3)/(q+1)}}{q+1} \int_{0}^{1} \frac{y^{(2p-2q+3)/(q+1)}}{(1+y)^{2(p+2)/(q+1)}(1-y)^{(2p-2q+2)/(q+1)}} \, dy, \]
\[ J_0 = \frac{\sqrt{\pi}}{p-q+3} \Gamma((q+3)/2(p+q)) \Gamma((p+3)/2(p-q)). \]

Now, we state our results.

**Theorem 2.1.** (1) Assume $p > (3q-1)/2$. Then the following asymptotic formula holds as $\lambda \to \infty$:
\[ \mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \left( 1 + C_2 (1 + o(1)) \lambda^{-2(p+1)(q+1)/(2p-q+3)(q-1)} \right), \] (2.6)
where
\[ C_2 = K_1 \left( 1 - \frac{2(p-q)K_2}{(2p-q+3)J_0} \right). \] (2.7)

(2) Assume $p < (3q-1)/2$. Then as $\lambda \to \infty$,
\[ \mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \left( 1 - C_3 (1 + o(1)) \lambda^{-(p+1)/(q-1)} \right), \] (2.8)
where
\[ C_3 = \frac{2(p-q)K_3 K_1^{(2p-q+3)/(2q+1)}}{(2p-q+3)J_0 K_1^{(2p-q+3)/(2q+1)}}. \] (2.9)

(3) Assume $p = (3q-1)/2$. Then as $\lambda \to \infty$,
\[ \mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \left( 1 - C_4 (1 + o(1)) \lambda^{-2(p+1)(q+1)/(2p-q+3)(q-1)} \log \lambda \right), \] (2.10)
where

\[ C_4 = \frac{2(p - q)(p + 1)}{(q - 1)(2p - q + 3)^2} K_1. \]  

(2.11)

We briefly explain the idea of the proof. Put

\[ \nu(\lambda) = \lambda^{(p - 1)/2(p - q)} \mu(\lambda)^{(1 - q)/2(p - q)}, \]

\[ w_\lambda(t) = \left( \frac{\mu(\lambda)}{\lambda} \right)^{1/(p - q)} u_\lambda(x), \quad t = \nu(\lambda) \left( x - \frac{1}{2} \right). \]  

(2.12)

Then it follows from (1.1) that \( w_\lambda \) satisfies

\[ -w_\lambda''(t) = w_\lambda(t)^p - w_\lambda(t)^q, \quad t \in I_{\nu(\lambda)} := \left( -\frac{1}{2} \nu(\lambda), \frac{1}{2} \nu(\lambda) \right), \]

\[ w_\lambda(t) > 0, \quad t \in I_{\nu(\lambda)}, \]

\[ w_\lambda \left( \pm \frac{1}{2} \nu(\lambda) \right) = 0. \]  

(2.13)

Then by [7, Lemma 5.1],

\[ \nu(\lambda) \longrightarrow \infty \]  

(2.14)

as \( \lambda \to \infty \). Put \( z_\lambda = \frac{w_\lambda}{\|w_\lambda\|_\infty} \). Then it is easy to see from (2.3) that

\[ \mu(\lambda) = \frac{\lambda^{(p + 3)/2(p - q)} \mu(\lambda)^{-(q + 3)/2(p - q)} \left( \|w_\lambda\|_2^2 + \|w_\lambda\|_{q+1}^{q+1} \right)}{\gamma^{p+1}} \]

\[ = \frac{\lambda^{(p + 3)/2(p - q)} \mu(\lambda)^{-(q + 3)/2(p - q)} \|w_\lambda\|_{p+1}^{p+1}}{\gamma^{p+1}} \]

\[ = \frac{\lambda^{(p + 3)/2(p - q)} \mu(\lambda)^{-(q + 3)/2(p - q)} \|w_\lambda\|_{\infty}^{p+1} \|z_\lambda\|_{p+1}^{p+1}}{\gamma^{p+1}}. \]  

(2.15)

Therefore, it is crucial to study the asymptotic behavior of \( \|w_\lambda\|_\infty \) and \( \|z_\lambda\|_{p+1} \)

as \( \lambda \to \infty \).

3. Asymptotic behavior of \( \|w_\lambda\|_\infty \)

In this section, we study the asymptotic behavior of \( \|w_\lambda\|_\infty \) as \( \lambda \to \infty \). We put

\[ \|w_\lambda\|_\infty = \left( \frac{p + 1}{q + 1} (1 + \epsilon(\lambda)) \right)^{1/(p - q)}. \]  

(3.1)

Then by [7, (5.10), Lemma 5.2], we know that \( \epsilon(\lambda) > 0 \) and \( \epsilon(\lambda) \to 0 \) as \( \lambda \to \infty \).
Lemma 3.1. The following equality holds for $\lambda > 0$:

$$
\nu(\lambda) = \sqrt{2(q + 1)} \left( \frac{p + 1}{q + 1} (1 + \epsilon(\lambda)) \right)^{-(q-1)/(2(p-q))} L(\epsilon(\lambda)), \tag{3.2}
$$

where

$$
L(\epsilon) = \int_0^1 \frac{1}{m(\epsilon, s)} ds,
$$

$$
m(\epsilon, s) = \sqrt{s^{q+1} - s^{p+1} + \epsilon (1 - s^{p+1})} \quad (\epsilon > 0).
$$

Proof. Multiply the equation in (2.13) by $w'_\lambda$. Then for $t \in I_{\nu(\lambda)}$,

$$
w''_\lambda(t)w'_\lambda(t) + w_\lambda(t)^p w'_\lambda(t) - w_\lambda(t)^q w'_\lambda(t) = 0, \tag{3.4}
$$

which implies that

$$
d dt \left( \frac{1}{2} (w'_\lambda(t))^2 + \frac{1}{p+1} w_\lambda(t)^{p+1} - \frac{1}{q+1} w_\lambda(t)^{q+1} \right) = 0. \tag{3.5}
$$

We know that $w_\lambda(0) = \|w_\lambda\|_\infty$ and $w'_\lambda(0) = 0$ since $u_\lambda(1/2) = \|u_\lambda\|_\infty$ and $u'_\lambda(1/2) = 0$. Then put $t = 0$ to obtain

$$
\frac{1}{2} w'_\lambda(t)^2 + \frac{1}{p+1} w_\lambda(t)^{p+1} - \frac{1}{q+1} w_\lambda(t)^{q+1} \equiv \frac{1}{p+1} \|w_\lambda\|_{p+1}^{p+1} - \frac{1}{q+1} \|w_\lambda\|_{q+1}^{q+1}. \tag{3.6}
$$

Note that $w'_\lambda(t) < 0$ for $t \in (0, \nu(\lambda)/2)$ since $u'_\lambda(x) < 0$ for $x \in (1/2, 1)$. Then it follows from this and (3.1) that for $t \in (0, \nu(\lambda)/2)$,

$$
-z'_\lambda(t) = \|w_\lambda\|_{\infty}^{(q-1)/2} \sqrt{2} \int_0^t \frac{1}{q+1} \sqrt{z_\lambda(t)^{q+1} - z_\lambda(t)^{p+1} + \epsilon(\lambda)(1 - z_\lambda(t)^{p+1})} \tag{3.7}
$$

$$
= \|w_\lambda\|_{\infty}^{(q-1)/2} \sqrt{2} \int_0^t m(\epsilon(\lambda), z_\lambda(t)).
$$

Put $s = z_\lambda$. Then (3.1) and (3.7) yield

$$
\frac{\nu(\lambda)}{2} = \int_0^{\nu(\lambda)/2} \frac{-z'_\lambda(t)}{\sqrt{2/(q+1)\|w_\lambda\|_{\infty}^{(q-1)/2}m(\epsilon(\lambda), z_\lambda(t))}} dt
$$

$$
= \sqrt{\frac{q+1}{2}} \left( \frac{p+1}{q+1} (1 + \epsilon(\lambda)) \right)^{-(q-1)/(2(p-q))} \int_0^1 \frac{1}{m(\epsilon(\lambda), s)} ds. \tag{3.8}
$$

This implies (3.2).
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In order to study the asymptotic behavior of $\epsilon(\lambda)$ as $\lambda \to \infty$, we investigate the asymptotic behavior of $L(\epsilon)$ as $\epsilon \to 0$.

**Lemma 3.2.** For $0 < \epsilon \ll 1$,

$$L(\epsilon) = \frac{\Gamma(1/(q+1))\Gamma((q-1)/2(q+1))}{(q+1)\sqrt{\pi}} \epsilon^{-(q-1)/(2(q+1))} + o(\epsilon^{-(q-1)/(2(q+1))}).$$

(3.9)

**Proof.** Put

$$L_1(\epsilon) := L(\epsilon) - \int_0^{1/\sqrt{q^{q+1}+\epsilon}} \frac{1}{ds}. \quad (3.10)$$

Put $s = \epsilon^{1/(q+1)} \tan^{2/(q+1)} \theta$. Then

$$\int_0^{1/\sqrt{q^{q+1}+\epsilon}} \frac{1}{ds} = 2 \epsilon^{-(q-1)/(2(q+1))} \int_0^{\tan^{-1}(1/\sqrt{\epsilon})} \sin^{-(q-1)/(q+1)} \theta \cos^{-2(q+1)} \theta d\theta$$

$$= \frac{2}{q+1} (1 + o(1)) \epsilon^{-(q-1)/(2(q+1))} \int_0^{\pi/2} \sin^{-(q-1)/(q+1)} \theta \cos^{-2(q+1)} \theta d\theta$$

$$= \frac{1}{q+1} (1 + o(1)) \epsilon^{-(q-1)/(2(q+1))} B\left(1/(q+1), \frac{q-1}{2(q+1)}\right)$$

$$= \frac{1}{q+1} (1 + o(1)) \epsilon^{-(q-1)/(2(q+1))} \frac{\Gamma(1/(q+1)) \Gamma((q-1)/2(q+1))}{\Gamma(1/2)}$$

$$= \frac{1}{q+1} (1 + o(1)) \epsilon^{-(q-1)/(2(q+1))} \frac{\sqrt{\pi}}{\sqrt{\pi}}. \quad (3.11)$$

We use here the formula

$$2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (m, n > 0), \quad (3.12)$$

where $B(m, n)$ is the beta function. Next, we calculate $L_1(\epsilon)$. Note that for $0 \leq s \leq 1$,

$$m(\epsilon, s) = \sqrt{s^{q+1}(1-s^{p-q}) + \epsilon(1-s^{p+1})} \geq \sqrt{(s^{q+1} + \epsilon)(1-s^{p-q})}. \quad (3.13)$$
By this, we obtain

\[ |L_1(\epsilon)| = \int_0^1 \frac{(1 + \epsilon)s^{p+1}}{m(\epsilon,s)\sqrt{s^{q+1} + \epsilon} \left( m(\epsilon,s) + \sqrt{s^{q+1} + \epsilon} \right)} ds \]

\[ \leq \int_0^1 \frac{(1 + \epsilon)s^{p+1}}{\sqrt{(s^{q+1} + \epsilon) (1 - s^{p-q})} \sqrt{s^{q+1} + \epsilon} \left( \sqrt{(s^{q+1} + \epsilon) (1 - s^{p-q})} + \sqrt{s^{q+1} + \epsilon} \right)} ds \]

\[ \leq (1 + \epsilon) \int_0^1 \frac{s^{p+1}}{(s^{q+1} + \epsilon)^{3/2} \sqrt{1 - s^{p-q} (1 + \sqrt{1 - s^{p-q}})}} ds \]

\[ \leq 2 \int_0^1 \frac{s^{p+1}}{(s^{q+1} + \epsilon)^{3/2} \sqrt{1 - s^{p-q}} \sqrt{1 - s^{p-q}(1 + \sqrt{1 - s^{p-q}})}} ds \]

\[ = 2 \int_0^\delta \frac{s^{p+1}}{(s^{q+1} + \epsilon)^{3/2} \sqrt{1 - s^{p-q}}} ds + 2 \int_\delta^1 \frac{s^{p+1}}{(s^{q+1} + \epsilon)^{3/2} \sqrt{1 - s^{p-q}}} ds \]

\[ := I + II, \quad (3.14) \]

where \( 0 < \delta \ll 1 \) is a fixed constant. Let \( C_{j,\delta} > 0 \ (j = 1, 2, \ldots) \) be constants depending only on \( \delta \). Let \( s = \sin^{2/(p-q)} \theta \). Then

\[ II \leq \frac{2}{\delta^{3(q+1)/2}} \int_0^1 \frac{1}{\sqrt{1 - s^{p-q}}} ds \]

\[ = \frac{2}{\delta^{3(q+1)/2}} \frac{2}{p-q} \int_0^1 \sin^{-1} \left( \frac{1}{\sqrt{1 - s^{p-q}}} \right) \frac{1}{\sin^{-1} \left( \frac{1}{\sqrt{1 - s^{p-q}}} \right)} \frac{1}{\sin^{-1} \left( \frac{1}{\sqrt{1 - s^{p-q}}} \right)} \frac{1}{\sin^{-1} \left( \frac{1}{\sqrt{1 - s^{p-q}}} \right)} d\theta \]

\[ \leq C_{1,\delta}. \quad (3.15) \]

Moreover, put \( s = \epsilon^{1/(q+1)} t \). Then for \( 0 < \epsilon \ll 1 \),

\[ I \leq \frac{2}{\sqrt{1 - \delta^{p-q}}} \int_0^\delta \frac{t^{p+1}}{(t^{q+1} + 1)^{3/2}} \epsilon^{1/(q+1)} dt \]

\[ \leq 2 \frac{\delta^{p+1}}{\sqrt{1 - \delta^{p-q}}} \epsilon^{(2p-3q+1)/(2(q+1))} = o(\epsilon^{(q-1)/(2(q+1))}). \quad (3.16) \]

By (3.14), (3.15), and (3.16), we have

\[ |L_1(\epsilon)| = o(\epsilon^{-(q-1)/(2(q+1))}). \quad (3.17) \]

By this, (3.10), and (3.11), we obtain (3.9). \( \square \)

Now, we study the asymptotic behavior of \( \epsilon(\lambda) \) as \( \lambda \to \infty \).

**Lemma 3.3.** As \( \lambda \to \infty \),

\[ \epsilon(\lambda) = K_1 (1 + o(1)) \lambda^{-2(p+1)(q+1)/(q-1)2(p-q+3)). \quad (3.18) \]
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Proof. By (1.2) and (2.12), we have

\[ \nu(\lambda) = \lambda^{(p-1)/(2(p-q))} \mu(\lambda)^{(1-q)/(2(p-q))} \]
\[ = \lambda^{(p-1)/(2(p-q))} \left( C_1 \lambda^{(p+1)/(2(p-q+3))} \right)^{1/(2(p-q))} \left( 1 + o(1) \right) \]  \hspace{1cm} (3.19)
\[ = C_1^{(1-q)/(2(p-q))} \left( 1 + o(1) \right) \lambda^{(p+1)/(2(p-q+3))}. \]

On the other hand, by Lemmas 3.1 and 3.2 and Taylor expansion, we have

\[ \nu(\lambda) = \sqrt{2(q+1)} \left( \frac{p+1}{q+1} \right)^{-(q-1)/(2(p-q))} \left( 1 + \epsilon(\lambda) \right)^{-(q-1)/(2(p-q))} L(\epsilon(\lambda)) \]
\[ = \sqrt{2(q+1)} \left( \frac{p+1}{q+1} \right)^{-(q-1)/(2(p-q))} \left( 1 - \frac{q-1}{2(p-q)} \epsilon(\lambda) + o(\epsilon(\lambda)) \right) \]
\[ \times \left( \frac{\Gamma(1/(q+1)) \Gamma((q-1)/2(q+1))}{(q+1)\sqrt{\pi}} \epsilon(\lambda)^{-(q-1)/(2(q+1))} \right) \]
\[ + o(\epsilon(\lambda)^{-(q-1)/(2(q+1))}) \]  \hspace{1cm} (3.20)
\[ \times \epsilon(\lambda)^{-(q-1)/(2(q+1))} \left( 1 + o(1) \right). \]

By this and (3.19), we obtain (3.18).  \hspace{1cm} \Box

4. Asymptotic behavior of \( \|z_\lambda\|_{p+1} \)

In this section, we calculate \( \|z_\lambda\|_{p+1} \). Note that \( z_\lambda(t) = z_\lambda(-t) \) for \( t \in I_\nu(\lambda) \). Then by (3.7) and putting \( s = z_\lambda(t) \), we have

\[ \|z_\lambda\|_{p+1} = 2 \int_0^{\nu(\lambda)/2} \|z_\lambda(t)^{p+1}\| dt \]
\[ = 2 \int_0^{\nu(\lambda)/2} \|z_\lambda(t)^{p+1}\|^{(q-1)/2} \frac{-z_\lambda'(t)}{\|w_\lambda\|_{\infty}^{(q-1)/2} \sqrt{2/(q+1)m(\epsilon(\lambda), z_\lambda(t))}} dt \]  \hspace{1cm} (4.1)
\[ = \sqrt{2(q+1)} \frac{\|w_\lambda\|_{\infty}^{(q-1)/2}}{J(\epsilon(\lambda))}, \]

where

\[ J(\epsilon) := \int_0^1 \frac{s^{p+1}}{m(\epsilon, s)} ds \quad (\epsilon > 0). \]  \hspace{1cm} (4.2)
Therefore, we study the precise asymptotics of $J(\epsilon)$ as $\epsilon \to 0$. Put $s = \sin^{2/(p-q)} \theta$. Then as $\epsilon \to 0$,

$$J(\epsilon) \to J(0) = \int_0^1 \frac{s^{(2p-q+1)/2}}{\sqrt{1-s^{p-q}}} ds$$

$$= \frac{2}{p-q} \int_0^{\pi/2} \sin^{(p+3)/(p-q)} \theta d\theta$$

$$= \frac{\sqrt{\pi}}{p-q} \frac{q+3}{p-3} \frac{\Gamma((q+3)/2(p-q))}{\Gamma((p+3)/2(p-q))} = J_0.$$  (4.3)

We use here the formulas

$$\int_0^{\pi/2} \sin^r \theta d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma((r+1)/2)}{\Gamma(r/2+1)} (r > -1),$$  (4.4)

Therefore, put

$$J_1(\epsilon) := J(\epsilon) - J_0 := -\epsilon J_2(\epsilon),$$

$$J_2(\epsilon) := \int_0^1 \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon,s) m(0,s)(m(\epsilon,s) + m(0,s))} ds.$$  (4.5)

Now, we study the asymptotic behavior of $J_2(\epsilon)$ as $\epsilon \to 0$.

**Lemma 4.1.** (1) If $p > (3q-1)/2$, then $J_2(\epsilon) \to K_2$ as $\epsilon \to 0$.

(2) If $p < (3q-1)/2$, then as $\epsilon \to 0$,

$$J_2(\epsilon) = K_3 (1 + o(1)) \epsilon^{(2p-3q+1)/(2q+1)}.$$  (4.6)

(3) If $p = (3q-1)/2$, then as $\epsilon \to 0$,

$$J_2(\epsilon) = -\frac{1}{2(q+1)} (1 + o(1)) \log \epsilon.$$  (4.7)

**Proof.** (1) Since $p > (3q-1)/2$, we have $(2p-3q-1)/2 > -1$. Therefore, by Lebesgue's convergence theorem, as $\epsilon \to 0$,

$$J_2(\epsilon) \to \frac{1}{2} \int_0^1 \frac{s^{(2p-3q-1)/2}(1-s^{p+1})}{(1-s^{p-q})^{3/2}} ds = K_2.$$  (4.8)

(2) We have the following two steps.
Step 1. Assume that \( p < (3q - 1)/2 \). We introduce \( J_3(\epsilon) \) to approximate \( J_2(\epsilon) \):

\[
J_3(\epsilon) := \int_0^1 \frac{s^{(2p-q+1)/2}}{s^{q+1} + \epsilon (s^{q+1}/2 + \sqrt{s^{q+1}} + \epsilon)} ds
= J_4(\epsilon, \delta) + J_5(\epsilon, \delta)
::= \int_0^\delta \frac{s^{(2p-q+1)/2}}{s^{q+1} + \epsilon (s^{q+1}/2 + \sqrt{s^{q+1}} + \epsilon)} ds
+ \int_\delta^1 \frac{s^{(2p-q+1)/2}}{s^{q+1} + \epsilon (s^{q+1}/2 + \sqrt{s^{q+1}} + \epsilon)} ds,
\]

where \( 0 < \delta \ll 1 \) is a fixed small constant. We study the asymptotic behaviors of \( J_3, J_4, \) and \( J_5 \) as \( \epsilon \to 0 \). Note that \( 0 < (2p - 3q + 2) / (q + 1) < 1 \) since \( p < (3q - 1)/2 \). Then put \( s = \epsilon^{1/(q+1)} \tan^{2/(q+1)} \theta \) and \( y = \tan(\theta/2) \) to obtain

\[
J_3(\epsilon) = \frac{2}{q+1} e^{(2p-3q+1)/(2(q+1))} \int_0^{\tan^{-1}(1/\sqrt{\epsilon})} \frac{\tan^{(2p-2q+2)/(q+1)}}{1 + \sin \theta} d\theta
\times \int_0^{\tan^{-1}(1/\sqrt{\epsilon})} \frac{\tan^{(2p-2q+2)/(q+1)}}{1 + y^{2(p+2)/(q+1)}} (1 - y^{(2p-2q+2)/(q+1)}) dy
= \frac{2^{2(p+2)/(q+1)}}{q+1} (1 + o(1)) e^{(2p-3q+1)/(2(q+1))}
\times \int_0^1 \frac{y^{(2p-2q+2)/(q+1)}}{(1 + y)^{2(p+2)/(q+1)}} (1 - y^{(2p-2q+2)/(q+1)}) dy
= K_3 (1 + o(1)) e^{(2p-3q+1)/(2(q+1))}.
\]

Similarly, we obtain

\[
J_4(\epsilon, \delta) = K_3 (1 + o(1)) e^{(2p-3q+1)/(2(q+1))},
J_5(\epsilon, \delta) \leq \frac{1}{\delta^{q+1}}.
\]

Since \( p < (3q - 1)/2 \), this along with (4.10) implies that \( J_3(\epsilon)/J_4(\epsilon, \delta) \to 1 \) as \( \epsilon \to 0 \) for a fixed \( \delta \).

Step 2. We show that as \( \epsilon \to 0 \),

\[
\frac{J_2(\epsilon)}{J_3(\epsilon)} \to 1.
\]
Let an arbitrary $0 < \delta \ll 1$ be fixed. Put

$$J_2(\epsilon) = J_6(\epsilon, \delta) + J_7(\epsilon, \delta)$$

$$= \int_{\delta}^{1} \frac{s^{p+1} (1 - s^{p+1})}{m(\epsilon, s)m(0, s)(m(\epsilon, s) + m(0, s))} \, ds$$

(4.13)

$$+ \int_{\delta}^{1} \frac{s^{p+1} (1 - s^{p+1})}{m(\epsilon, s)m(0, s)(m(\epsilon, s) + m(0, s))} \, ds.$$ 

Then for $0 < \epsilon \ll 1$,

$$|J_7(\epsilon, \delta)| \leq C_{2,\delta} \int_{\delta}^{1} \frac{1 - s^{p+1}}{(1 - s^{p-q})^{3/2}} \, ds \leq C_{3,\delta}. \quad (4.14)$$

Moreover, by (3.13), we obtain

$$(1 - \delta^{p+1}) \int_{0}^{\delta} \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1} + \epsilon (s^{q+1}/2 + s^{q+1} + \epsilon)}} \, ds$$

$$\leq J_6(\epsilon, \delta) \leq \frac{1}{(1 - \delta^{p-q})^{3/2}} \int_{0}^{\delta} \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1} + \epsilon (s^{q+1}/2 + s^{q+1} + \epsilon)}} \, ds. \quad (4.15)$$

This implies

$$(1 - \delta^{p+1}) J_4(\epsilon, \delta) \leq J_6(\epsilon, \delta) \leq \frac{1}{(1 - \delta^{p-q})^{3/2}} J_4(\epsilon, \delta). \quad (4.16)$$

By (4.11), (4.14), and (4.16), we see that $J_7(\epsilon, \delta) = o(J_6(\epsilon, \delta))$ as $\epsilon \to 0$ for a fixed $\delta$ since $p < (3q - 1)/2$. Then by (4.10), (4.11), (4.13), (4.14), and (4.16),

$$(1 - \delta^{p+1}) \leq \liminf_{\epsilon \to 0} \frac{J_6(\epsilon, \delta)}{J_4(\epsilon, \delta)} = \liminf_{\epsilon \to 0} \frac{J_2(\epsilon)}{J_3(\epsilon)} \leq \limsup_{\epsilon \to 0} \frac{J_2(\epsilon)}{J_3(\epsilon)} = \limsup_{\epsilon \to 0} \frac{J_6(\epsilon, \delta)}{J_4(\epsilon, \delta)} \leq \frac{1}{(1 - \delta^{p-q})^{3/2}}. \quad (4.17)$$

By letting $\delta \to 0$, we obtain (4.12). Then by (4.10) and (4.12), we obtain (4.6).

(3) If $p = (3q - 1)/2$, then by the asymptotic formula

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + O\left(\frac{1}{x^3}\right) \quad (x \gg 1), \quad (4.18)$$
Taylor expansion of \( \tan x \) at \( x = \pi/4 \), and (4.10), we obtain

\[
J_3(\epsilon) = \frac{8}{q+1} \int_0^{\tan((1/2)(\tan^{-1}(1/\sqrt{\epsilon})))} \frac{y}{(1+y)^3(1-y)} \, dy \\
= \frac{1}{q+1} \int_0^{\tan((1/2)(\tan^{-1}(1/\sqrt{\epsilon})))} \left( \frac{1}{1-y} + \frac{1}{1+y} + \frac{2}{(1+y)^2} - \frac{4}{(1+y)^3} \right) \, dy \\
= \frac{1}{q+1} \left( -\log \left| 1 - \tan \left( \frac{1}{2} \tan^{-1} \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \right| + \log 2 - \frac{1}{2} + o(1) \right) \\
= \frac{1}{q+1} \left( -\frac{1}{2} (1 + o(1)) \log \epsilon + \log 2 - \frac{1}{2} + o(1) \right).
\]

(4.19)

By this and the same arguments as those in the proof of (2), we obtain (4.7).

\[ \square \]

5. Proof of Theorem 2.1

By summing up Lemmas 3.1, 3.2, 3.3, and 4.1, we now prove Theorem 2.1. By (2.15), (3.1), (4.1), and (4.5), we have

\[
\mu(\lambda) \frac{(2p-q+3)/(2(p-q))}{(2p-q+3)/(2(p-q))} \left\| W_{\lambda} \right\|_{\infty}^{(2p-q+3)/2} J(\epsilon(\lambda)) \\
= \frac{\sqrt{2(q+1)}}{y^{p+1}} \lambda^{(p+3)/(2(p-q))} \left( \frac{p+1}{q+1} \right)^{(2p-q+3)/(2(p-q))} \left( 1 + \epsilon(\lambda) \right)^{(2p-q+3)/(2(p-q))} (f_0 - \epsilon(\lambda)J_2(\epsilon(\lambda))). \tag{5.1}
\]

Moreover, it is easy to check that

\[
\left( \frac{\sqrt{2(q+1)}}{y^{p+1}} \right)^{(2p-q)/(2(p-q))} \frac{p+1}{q+1} J_0^{(2p-q)/(2(p-q))} = C_1. \tag{5.2}
\]

By this, (5.1), and Taylor expansion, we obtain

\[
\mu(\lambda) = \left( \frac{\sqrt{2(q+1)}}{y^{p+1}} \right)^{(2p-q)/(2(p-q)+3)} \frac{p+1}{q+1} J_0^{(2p-q)/(2(p-q)+3)} \lambda^{(p+3)/(2(p-q)+3)} \\
\times \left( 1 + \epsilon(\lambda) \right) \left( 1 - \frac{2(p-q)}{(2p-q+3)J_0} \epsilon(\lambda)J_2(\epsilon(\lambda)) + o(\epsilon(\lambda)J_2(\epsilon(\lambda))) \right)
\]
\[ \mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \times \left( 1 + \epsilon(\lambda) - \frac{2(p-q)}{(2p-q+3)J_0} (1 + o(1)) \epsilon(\lambda) J_2(\epsilon(\lambda)) \right). \] (5.3)

There are three cases to consider.

**Case 5.1.** Assume that \( p > (3q-1)/2 \). Then by Lemmas 3.3 and 4.1(1), we have

\[ \mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \times \left\{ 1 + \left( 1 - \frac{2(p-q)K_2}{(2p-q+3)J_0} \right) \epsilon(\lambda) + o(\epsilon(\lambda)) \right\} \]

\[ = C_1 \lambda^{(p+3)/(2p-q+3)} \times \left\{ 1 + \left( 1 - \frac{2(p-q)K_2}{(2p-q+3)J_0} \right) K_1 (1 + o(1)) \lambda^{-2(p+1)(q+1)/(2p-q+3)(q-1)} \right\}. \] (5.4)

This implies (2.6).

**Case 5.2.** Assume that \( p < (3q-1)/2 \). Then by Lemma 3.3, (4.6), and (5.3), we have

\[ \mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \times \left\{ 1 - \frac{2(p-q)K_2}{(2p-q+3)J_0} K_3 K_1^{(2p-q+3)/(2(q+1))} (1 + o(1)) \lambda^{-2(p+1)(q+1)/(2p-q+3)(q-1)} \right\}. \] (5.5)

This implies (2.8).

**Case 5.3.** Assume that \( p = (3q-1)/2 \). Then by Lemma 3.3, we have

\[ \log \epsilon(\lambda) = -\frac{2(p+1)(q+1)}{(q-1)(2p-q+3)} (1 + o(1)) \log \lambda. \] (5.6)

This along with (4.7) and (5.3) implies

\[ \mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \times \left\{ 1 - \frac{2(p-q)(p+1)}{(q-1)(2p-q+3)^2J_0} \right\} K_1 (1 + o(1)) \lambda^{-2(p+1)(q+1)/(2p-q+3)(q-1)} \log \lambda \] (5.7)

This implies (2.10). Thus, the proof is complete.
Two-parameter eigenvalue problems

References


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