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Research Article

Ternary Weighted Function and Beurling Ternary Banach Algebra $l_1^\omega(S)$

Mehdi Dehghanian and Mohammad Sadegh Modarres Mosadegh

Department of Mathematics, Yazd University, P.O. Box 89195-741, Yazd, Iran

Correspondence should be addressed to Mehdi Dehghanian, mdehghanian.math@gmail.com

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Let S be a ternary semigroup. In this paper, we introduce our notation and prove some elementary properties of a ternary weight function ω on S. Also, we make ternary weighted algebra $l_1^\omega(S)$ and show that $l_1^\omega(S)$ is a ternary Banach algebra.

1. Introduction

The notion of an n-ary group was introduced by Dörnte [1] (inspired by E. Nöther) and is a natural generalization of the notion of a group and a ternary group considered by Certaine [2] and Kasner [3].

In the first section, which have preliminary character, we review some basic definitions and properties related to ternary groups and semigroups (cf. also Belousov [4] and Rusakov [5]).

Dudek [6], Feĭzullaev [7], Kim and Fred [8], and Lyapin [9] have also studied the properties of the ternary semigroups.

The present paper may be described as an introduction to harmonic analysis on ternary semigroups. In Section 2, we introduce our notation and prove some elementary properties of a ternary weight function. In Section 3, we make ternary weighted algebra $l_1^\omega(S)$ and show that $l_1^\omega(S)$ is a ternary Banach algebra.

Definition 1.1. A nonempty set G with one ternary operation $[] : G \times G \times G \to G$ is called a ternary groupoid and denoted by $(G, [])$.

We say that $(G, [])$ is a ternary semigroup if the operation $[]$ is associative, that is, if

$$[[xyz]uv] = [x[yzu]v] = [xy[zuv]] \tag{1.1}$$

hold for all $x, y, z, u, v \in G$.

Definition 1.2. A ternary semigroup $(G, [])$ is a ternary group if for all $a, b, c \in G$, there are $x, y, z \in G$ such that

$$[xab] = [ayb] = [abz] = c. \tag{1.2}$$

One can prove (post [10]) that elements x, y, and z are uniquely determined. Moreover, according to the suggestion of post [10], one can prove (cf. Dudek et al. [11]) that, in the above definition, under the assumption of the associativity, it suffices only to postulate the existence of a solution of $[ayb] = c$, or equivalently, of $[xab] = [abz] = c$.

In a ternary group, the equation $[xxz] = x$ has a unique solution which is denoted by $z = \overline{x}$ and called the skew element to x (cf. Dörnte, [1]). As a consequence of results obtained in Dörnte [1], we have the following theorem.

Theorem 1.3. *In any ternary group $(G, [])$ for all $x, y, z \in G$, the following identities take place:*

$$\begin{gathered}
[xx\overline{x}] = [x\overline{x}x] = [\overline{x}xx] = x, \\
[yx\overline{x}] = [y\overline{x}x] = [x\overline{x}y] = [\overline{x}xy] = y, \\
\overline{[xyz]} = [\overline{z}\,\overline{y}\,\overline{x}], \\
\overline{\overline{x}} = x.
\end{gathered} \tag{1.3}$$

Other properties of skew elements are described in Dudek [12] and I. Dudek and W. A. Dudek [13].

Definition 1.4 (see [14]). Let $(G, [])$ be a ternary group, $^{-1}$ it's inverse operation, and G be equipped with a topology O. Then, we say that $(G, [], O)$ is a topological ternary group if and only if

(i) ternary operation $[]$ is continuous in O, and

(ii) the 2-operation $^{-1}$ is continuous in O.

Let G be a ternary group and A any subset of G. We denote by $\overline{A}$ denote set of all $\overline{x}$ (skew element) such that $x \in A$, that is,

$$\overline{A} = \{\overline{x} : x \in A\}. \tag{1.4}$$

Definition 1.5. A ternary Banach algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \to [xyz]$ of A^3 into A, which is associative in the sense that $[[xyz]uv] = [xy[zuv]] = [x[yzu]v]$, and satisfy $\|[xyz]\| \le \|x\|\|y\|\|z\|$.

Let A be a ternary Banach algebra and $A_1,\ A_2$, and A_3 subsets of A. We define

$$[A_1A_2A_3] = \{[a_1a_2a_3] : a_1 \in A,\ a_2 \in A_2,\ a_3 \in A_3\}. \tag{1.5}$$

Let A and B be a ternary Banach algebra. A linear mapping $\phi : A \to B$ is called to be a ternary homomorphism if $\phi[xyz] = [\phi(x)\phi(y)\phi(z)]$.

Definition 1.6. Let S be a ternary semigroup, and let $l_1(S)$ denote the set of mappings f of S into $\mathbb{C}$ such that

$$\sum_{s\in S}|f(s)| < \infty \tag{1.6}$$

with pointwise addition and scalar multiplication and with the norm

$$\|f\|_1 = \sum_{s\in S}|f(s)|. \tag{1.7}$$

Let S be a ternary semigroup; for $f, g, h \in l_1(S)$, we define

$$[f,g,h]_*(x) = \sum_{x=[rst]} f(r)g(s)h(t) \quad (x \in S), \tag{1.8}$$

and it is called ternary convolution product on $l_1(S)$.

Theorem 1.7 (see [15]). *Let S be a ternary semigroup, then ternary convolution product on $l_1(S)$ is associative.*

Theorem 1.8 (see [15]). *Let S be a ternary semigroup, then $(l_1(S), [, ,]_*)$ under the usual norm is a ternary Banach algebra.*

2. Ternary Weight Function on Ternary Semigroup and Ternary Group

Definition 2.1. A ternary weight on ternary semigroup $(S, [])$ is a positive real function $\omega : S \to \mathbb{R}^+$ such that

$$\omega([rst]) \le \omega(r)\omega(s)\omega(t) \quad \forall r, s, t \in S. \tag{2.1}$$

Remark 2.2. If ω_1 and ω_2 are two ternary weight function, then $\omega_1\omega_2$ is a ternary weight function.

Example 2.3. Let $(\mathbb{Z}, [])$ with $[xyz] = x + y + z$ be a ternary group. For $\alpha > 0$, define $\omega_\alpha = (1 + |n|)^\alpha$. Then, ω_α is a weight ternary function.

Theorem 2.4. *Let K be a compact subset of a topological ternary group G, ω a ternary weight function on G, and the interior of $\{x : \omega(x) < n\}$ is nonempty for some $n \in \mathbb{N}$. Then, there exists $a, b \in \mathbb{R}$ such that*

$$0 < a \leq \omega(x) \leq b \tag{2.2}$$

for all $x \in K$.

Proof. First we establish the existence of b. To that end, for $n \in \mathbb{N}$, let

$$U_n := \{x \in G : \omega(x) < n\}. \tag{2.3}$$

Clearly $\cup_{n=1}^{\infty} U_n = G$. Choose $n \in \mathbb{N}$ such that $(U_n)^\circ \neq \emptyset$(interior U_n). Fix $g \in (U_n)^\circ$, and let $V = (U_n)^\circ$. Then V is an open neighborhood of the g, and, hence, by compactness of K, there exist $y_1, y_2, \ldots, y_m \in K \cup \overline{K}$ such that

$$K \cup \overline{K} \subseteq [V\overline{g}y_1] \cup [V\overline{g}y_2] \cup \cdots \cup [V\overline{g}y_m]. \tag{2.4}$$

Now, define $b > 0$ by

$$b = n\omega(\overline{g}) \max\{\omega(y_i) : 1 \leq i \leq m\}. \tag{2.5}$$

If $x \in K \cup \overline{K}$, then $x = [v\overline{g}y_i]$ for some $v \in V$ and $i \in \{1, 2, \ldots, m\}$, and, hence,

$$\omega(x) = \omega([v\overline{g}y_i]) \leq \omega(v)\omega(\overline{g})\omega(y_i) \leq n\omega(\overline{g})\omega(y_i) \leq b. \tag{2.6}$$

Thus, $\omega(x) \leq b$, for all $x \in K \cup \overline{K}$.

Next let

$$a = \inf\{\omega(x) : x \in K\}, \tag{2.7}$$

and suppose that $a = 0$. Then, there exists a sequence $(x_n)_n$ in K such that $\omega(x_n) \to 0$. Since

$$1 \leq \omega(x_n)\omega(\overline{x}_n), \tag{2.8}$$

we must have $\omega(\overline{x}_n) \to \infty$, which contradicts boundedness of ω on the compact set $\overline{K}$. □

Proposition 2.5. *Let ω be a ternary weight function on a ternary semigroup S such that $\{x \in S : \omega(x) < \epsilon\}$ is finite for some $\epsilon > 0$. Then $\omega(x) \geq 1$, for all $x \in S$.*

Proof. Suppose that $\omega(x) < 1$ for some $x \in S$. Choose $p \in \mathbb{N}$ such that $\omega(x^{3^p}) < \epsilon$. Then

$$\omega\left(x^{3^p+3^n+1}\right) \leq \omega\left(x^{3^p}\right)\omega\left(x^{3^n}\right)\omega(x) < \omega\left(x^{3^p}\right)\omega\left(x^{3^n}\right) \leq \omega\left(x^{3^p}\right)\omega(x)^{3^n} < \epsilon. \tag{2.9}$$

Thus, $\{x^{3^p+3^n} : n \in \mathbb{N}\}$ is an infinite subset of $\{y \in S : \omega(y) < \epsilon\}$ contrary to the hypothesis. Thus, the result follows. □

Corollary 2.6. *Let ω be a ternary weight function on a compact topological ternary group G, and the interior of $\{x : \omega(x) < n\}$ is nonempty for some $n \in \mathbb{N}$. Then, $\omega(x) \geq 1$ for all $x \in G$.*

Proof. This follows trivially from Theorem 2.4 and Proposition 2.5. □

Let $C(S)$ be the set of all complex-valued continuous functions on S, $C_b(S)$ the space of all bounded functions in $C(S)$ under the supremum norm $\|\cdot\|_S$ and ω a continuous ternary weight function on S. We define the space of ternary weighted continuous functions $C(S,\omega)$ by

$$C(S,\omega) = \{f \in C(S) : f\omega \in C_b(S)\} \tag{2.10}$$

with the norm given by

$$\|f\|_\omega = \|f\omega\|_S. \tag{2.11}$$

Define

$$\begin{gathered}\Omega(x,y) := \sup\left\{\frac{\omega(z)}{\omega([xyz])} : z \in S\right\}, \\ {}_{x,y}f\ (z) = f([xyz])\end{gathered} \tag{2.12}$$

for all $x, y, z \in S$.

Lemma 2.7. *Let ω be a continuous ternary weight function on a ternary semigroup S. Then, ${}_{x,y}f \in C(S,\omega)$, for all $f \in C(S,\omega)$ and $x \in S$ if and only if $\Omega(x,y)$ is finite.*

Proof. To establish the necessary condition, we note that $\omega^{-1} \in C(S,\omega)$ and so if $x \in S$, we have ${}_{x,y}\omega^{-1} \in C(S,\omega)$. Hence,

$$\infty > \left\| {}_{x,y}\omega^{-1} \right\| = \sup\left\{\frac{\omega(z)}{\omega([xyz])} : z \in S\right\} = \Omega(x,y). \tag{2.13}$$

Conversely, suppose that $\Omega(x,y)$ is finite, for all $x \in S$, and let

$$\begin{aligned}\left|{}_{x,y}f(z)\omega(z)\right| = \left|f([xyz])\omega(z)\right| &= \left|f([xyz])\omega([xyz])\right|\frac{\omega(z)}{\omega([xyz])} \\ &\leq \|f\|_\omega \Omega(x,y)\end{aligned} \tag{2.14}$$

and so $\|{}_{x,y}f\|_\omega < \infty$. Hence, ${}_{x,y}f \in C(S,\omega)$. □

Corollary 2.8. *Let ω be a continuous ternary weight function on a compact topological ternary group G, and the interior of $\{x : \omega(x) < n\}$ is nonempty for some $n \in \mathbb{N}$. Then, ${}_{x,y}f \in C(S,\omega)$ for all $f \in C(S,\omega)$ and $x \in G$.*

Proof. For all x and y in G, we have

$$\frac{\omega(z)}{\omega([xyz])} = \frac{\omega([\overline{y}\,\overline{x}[xyz]])}{\omega([xyz])} \leq \frac{\omega(\overline{y})\omega(\overline{x})\omega([xyz])}{\omega([xyz])} = \omega(\overline{y})\omega(\overline{x}), \tag{2.15}$$

and so $\Omega(x,y) \leq \omega(\overline{x})\omega(\overline{y})$. By Lemma 2.7, our result follows. □

3. Ternary Beurling Algebra $l_1^\omega(S)$

Let S be a ternary semigroup. In [15] introduce ternary Banach algebra $l_1(S)$. Now, we make ternary Beurling algebra $l_1^\omega(S)$ and show some elementary properties.

Definition 3.1. Let S be a ternary semigroup, let ω be a ternary weight on S, and let $l_1^\omega(S)$ denote the set of mappings f of S into $\mathbb{C}$ such that

$$\sum_{s\in S} |f(s)|\omega(s) < \infty, \tag{3.1}$$

with pointwise addition and scalar multiplication, with ternary convolution

$$[f,g,h]_*(x) = \sum_{x=[rst]} f(r)g(s)h(t) \quad (x \in S), \tag{3.2}$$

and with the norm

$$\|f\|_{1,\omega} = \|f\omega\|_1 = \sum_{s\in S} |f(s)|\omega(s). \tag{3.3}$$

Theorem 3.2. *Let S be a ternary semigroup and ω be a ternary weight function on S. Then, $(l_1^\omega(S), [,,]_*, \|\cdot\|_{1,\omega})$ is a ternary Banach algebra.*

Proof. By Theorem 1.7, we need only to check that

$$\|[f,g,h]_*\|_{1,\omega} \leq \|f\|_{1,\omega}\|g\|_{1,\omega}\|h\|_{1,\omega}, \tag{3.4}$$

for all $f, g, h \in l_1^{\omega}(S)$. This justifies the last inequality of the following calculation:

$$
\begin{aligned}
\|[f,g,h]_*\|_{1,\omega} &= \sum_{x\in S}\left|[f,g,h]_*(x)\omega(x)\right| \\
&= \sum_{x\in S}\left|\sum_{x=[rst]} f(r)g(s)h(t)\omega([rst])\right| \\
&\leq \sum_{x\in S}\sum_{x=[rst]}\left|f(r)\omega(r)\right|\left|g(s)\omega(s)\right|\left|h(t)\omega(t)\right| \\
&\leq \sum_{r}\left|f(r)\omega(r)\right|\sum_{s}\left|g(s)\omega(s)\right|\sum_{t}\left|h(t)\omega(t)\right| \\
&= \|f\|_{1,\omega}\|g\|_{1,\omega}\|h\|_{1,\omega}.
\end{aligned}
\tag{3.5}
$$

□

The ternary algebra $l_1^{\omega}(S)$ is called the ternary Beurling algebra on S associated with the ternary weight ω.

If $\omega(s) = 1, (s \in S)$, we obtained $l_1^{\omega}(S) = l_1(S)$.

If $\omega(s) \geq 1$ for all $s \in S$, then $l_1^{\omega}(S)$ is a ternary subalgebra of $l_1(S)$, and if $\omega(s) \leq 1$ for all $s \in S$, then $l_1(S)$ is a ternary subalgebra of $l_1^{\omega}(S)$.

Proposition 3.3. *Let ω and ω' are ternary weights on S and $\phi : l_1^{\omega}(S) \to l_1^{\omega'}(S)$ is a continuous nonzero ternary homomorphism. If $[l_1^{\omega}(S), f, g]_*$ is norm dense in $l_1^{\omega}(S)$. Then, the norm closure of $[l_1^{\omega'}(S), \phi(f), \phi(g)]_*$ contains the norm closure of $[l_1^{\omega'}(S), l_1^{\omega'}(S), \phi(h)]_*$ for all h in $l_1^{\omega}(S)$.*

Proof. Since $[l_1^{\omega}(S), f, g]_*$ is norm dense in $l_1^{\omega}(S)$, we can find a sequence $\{\lambda_n\}$ in $l_1^{\omega}(S)$ for which $\lim([\lambda_n, f, g]_*) = h$, with the limit taken in the norm topology. By the continuity of ϕ, this implies that

$$\lim\left(\left[\phi(\lambda_n), \phi(f), \phi(g)\right]_*\right) = \phi(h). \tag{3.6}$$

Hence, $\phi(h)$, and, therefore, the norm closure of $[l_1^{\omega'}(S), l_1^{\omega'}(S), \phi(h)]_*$ as well, belongs to the norm closure of $[l_1^{\omega'}(S), \phi(f), \phi(g)]_*$. This completes the proof of the proposition. □

Proposition 3.4. *Let ω and ω' are ternary weights on S, and $\phi : l_1^{\omega}(S) \to l_1^{\omega'}(S)$ is a continuous nonzero ternary homomorphism. If $[l_1^{\omega}(S), l_1^{\omega}(S), f]_*$ is norm dense in $l_1^{\omega}(S)$. Then, the norm closure of $[l_1^{\omega'}(S), l_1^{\omega'}(S), \phi(f)]_*$ contains the norm closure of $[l_1^{\omega'}(S), l_1^{\omega'}(S), \phi(g)]_*$ for all g in $l_1^{\omega}(S)$.*

Proof. Since $[l_1^{\omega}(S), l_1^{\omega}(S), f]_*$ is norm dense in $l_1^{\omega}(S)$, we can find sequences $\{h_n\}$ and $\{\lambda_n\}$ in $l_1^{\omega}(S)$ for which $\lim([h_n, \lambda_n, f]_*) = g$, with the limit taken in the norm topology. By the continuity of ϕ, this implies that

$$\lim\left(\left[\phi(h_n), \phi(\lambda_n), \phi(f)\right]_*\right) = \phi(g). \tag{3.7}$$

Hence, $\phi(g)$, and, therefore, the norm closure of $[l_1^{\omega'}(S), l_1^{\omega'}(S), \phi(g)]_*$ as well, belongs to the norm closure of $[l_1^{\omega'}(S), l_1^{\omega'}(S), \phi(f)]_*$. This completes the proof of the proposition. □

Definition 3.5. Let G be a ternary group, let ω be a ternary weight on G, and let $l_1^{\omega}(G)$ denote the set of mappings f of G into $\mathbb{C}$ such that

$$\sum_{s\in G} |f(s)|\omega(s) < \infty \tag{3.8}$$

with pointwise addition and scalar multiplication, with ternary convolution

$$[f, g, h]_*(x) = \sum_{s,t\in G} f\left(\left[x\bar{s}\bar{t}\right]\right)g(t)h(s) \quad (x \in G) \tag{3.9}$$

and with the norm

$$\|f\|_{1,\omega} = \|f\omega\|_1 = \sum_{s\in G} |f(s)|\omega(s). \tag{3.10}$$

It is clear that $l_1^{\omega}(G)$ is a ternary Banach algebra.

Theorem 3.6. *Let G be a topological ternary group, ω a ternary weight function on G such that interior $\{x : \omega(x) < n\}$ not empty for some $n \in \mathbb{N}$. Then,*

(i) *every compactly supported function in $l_1(G)$ belongs to $l_1^{\omega}(G)$ and*

(ii) *For every $x, y \in G$ and $f \in l_1^{\omega}(G)$, ${}_{\bar{y},\bar{x}}f \in l_1^{\omega}(G)$ and $\|{}_{\bar{y},\bar{x}}f\|_{1,\omega} \le \omega(x)\omega(y)\|f\|_{1,\omega}$.*

Proof. (i) It is immediately since ω is bounded on compact subsets of G by Theorem 2.4.

(ii) It follows simply from submultiplicativity of ω

$$\begin{aligned}
\|{}_{\bar{y},\bar{x}}f\|_{1,\omega} &= \sum_{z\in G} |f([\bar{y}\,\bar{x}z])|\omega(z) \\
&= \sum_{z\in G} |f([\bar{y}\,\bar{x}z])|\omega([\bar{y}\,\bar{x}z])\frac{\omega(z)}{\omega([\bar{y}\,\bar{x}z])} \\
&\le \omega(x)\omega(y)\sum_{z\in G} |f([\bar{y}\,\bar{x}z])|\omega([\bar{y}\,\bar{x}z]) \\
&= \omega(x)\omega(y)\|f\|_{1,\omega}.
\end{aligned} \tag{3.11}$$

□

Let G be a ternary group and ω be a ternary weight function, define

$$l_p^{\omega}(G) = \{f : f\omega \in l_p(G)\} \tag{3.12}$$

such that

$$\|f\|_{p,\omega} = \|f\omega\|_p = \left(\sum_{x\in G} |f(x)|^p \omega(x)^p\right)^{1/p}. \tag{3.13}$$

For $p \geq 1$, we have $l_1^{\omega}(G) \subset l_p^{\omega}(G)$ and $\|f\|_{p,\omega} \leq \|f\|_{1,\omega}$.

Theorem 3.7. *Let G be a discrete uncountable ternary group. Then, there exists set $A \subset G$ such that $A = \overline{A}$ and $l_p(A) \subset l_p^{\omega}(G)$.*

Proof. Since G is uncountable, for some $C > 0$, the set $A = \{x : \max(\omega(x), \omega(\overline{x})) \leq C\}$ is also uncountable. Note that $A = \overline{A}$. Now, if $f \in l_p(A)$, then

$$\|f\|_{p,\omega}^p = \sum_{x\in A} |f(x)\omega(x)|^p \leq C^p \sum_{x\in A} \|f\|_p^p \tag{3.14}$$

so that $l_p(A) \subset l_p^{\omega}(G)$. □

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