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Research Article

Positive Solutions for a Higher-Order Nonlinear Neutral Delay Differential Equation

Zeqing Liu,[1] Ming Jia,[1] Jeong Sheok Ume,[2] and Shin Min Kang[3]

[1] *Department of Mathematics, Liaoning Normal University, Liaoning, Dalian 116029, China*
[2] *Department of Applied Mathematics, Changwon National University, Changwon 641-773, Republic of Korea*
[3] *Department of Mathematics and the Research Institute of Natural Science, Gyeongsang National University, Jinju 660-701, Republic of Korea*

Correspondence should be addressed to Jeong Sheok Ume, jsume@changwon.ac.kr

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This paper deals with the higher-order nonlinear neutral delay differential equation $(d^n/dt^n)[x(t) + \sum_{i=1}^{m} p_i(t)x(T_i(t))] + (d^{n-1}/dt^{n-1})f(t, x(\alpha_1(t)), \ldots, x(\alpha_k(t))) + h(t, x(\beta_1(t)), \ldots, x(\beta_k(t))) = g(t)$, $t \geq t_o$, where $n, m, k \in \mathbb{N}, p_i, \tau_i, \beta_j, g \in C([t_o, +\infty), \mathbb{R}), \alpha_j \in C^{n-1}([t_o, +\infty), \mathbb{R}), f \in C^{n-1}([t_o, +\infty) \times \mathbb{R}^k, \mathbb{R}), h \in C([t_o, +\infty) \times \mathbb{R}^k, \mathbb{R})$, and $\lim_{t\to+\infty}\tau_i(t) = \lim_{t\to+\infty}\alpha_j(t) = \lim_{t\to+\infty}\beta_j(t) = +\infty, i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, k\}$. By making use of the Leray-Schauder nonlinear alterative theorem, we establish the existence of uncountably many bounded positive solutions for the above equation. Our results improve and generalize some corresponding results in the field. Three examples are given which illustrate the advantages of the results presented in this paper.

1. Introduction and Preliminaries

This paper is concerned with the higher-order nonlinear neutral delay differential equation:

$$\frac{d^n}{dt^n}\left[x(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t))\right] + \frac{d^{n-1}}{dt^{n-1}} f(t, x(\alpha_1(t)), \ldots, x(\alpha_k(t))) \\ + h(t, x(\beta_1(t)), \ldots, x(\beta_k(t))) = g(t), \quad t \geq t_0, \tag{1.1}$$

where $n, m, k \in \mathbb{N}$, $p_i, \tau_i, \beta_j, g \in C([t_0, +\infty), \mathbb{R})$, $\alpha_j \in C^{n-1}([t_0, +\infty), \mathbb{R})$, $f \in C^{n-1}([t_0, +\infty) \times \mathbb{R}^k, \mathbb{R})$, $h \in C([t_0, +\infty) \times \mathbb{R}^k, \mathbb{R})$, and

$$\lim_{t\to+\infty}\tau_i(t) = \lim_{t\to+\infty}\alpha_j(t) = \lim_{t\to+\infty}\beta_j(t) = +\infty, \quad i \in \{1, 2, \ldots, m\}, \quad j \in \{1, 2, \ldots, k\}. \tag{1.2}$$

Theory of neutral delay differential equations has undergone a rapid development in the last over thirty years. We refer the readers to [1–8] and the references therein for a wealth of reference materials on the subject. The authors [1–8] and others discussed the oscillation, nonoscillation, and existence of a nonoscillatoy solution for some special cases of (1.1) under various conditions. By using the Banach fixed point theorem, Zhang et al. [4] and Kulenović and Hadžiomerspahić [1] studied, respectively, the existence of a nonoscillatory solution for the first-order neutral delay differential equation:

$$\frac{d}{dt}\left[x(t)+p(t)x(t-\tau)\right]+P(t)x(t-\sigma)-Q(t)x(t-\delta)=0, \quad t\geq t_0, \tag{1.3}$$

where $\tau>0$, $\sigma,\delta\in\mathbb{R}^+$, $P,Q\in C([t_0,+\infty),\mathbb{R}^+)$, and $p\in C([t_0,+\infty),\mathbb{R})$, and the second-order neutral delay differential equation with positive and negative coefficients:

$$\frac{d^2}{dt^2}\left[x(t)+px(t-\tau)\right]+P(t)x(t-\sigma)-Q(t)x(t-\delta)=0, \quad t\geq t_0, \tag{1.4}$$

where $p\in\mathbb{R}\setminus\{\pm 1\}$, $\sigma,\delta\in\mathbb{R}^+$ and $P,Q\in C([t_0,+\infty),\mathbb{R}^+)$. Zhang et al. [6] considered the second-order nonlinear neutral differential equation with positive and negative terms:

$$\frac{d^2}{dt^2}\left[x(t)-px(\tau(t))\right]+f_1(t,x(\sigma_1(t)))-f_2(t,x(\sigma_2(t)))=0, \quad t\geq t_0 \tag{1.5}$$

and its corresponding equation with forced term:

$$\frac{d^2}{dt^2}\left[x(t)-px(\tau(t))\right]+f_1(t,x(\sigma_1(t)))-f_2(t,x(\sigma_2(t)))=g(t), \quad t\geq t_0, \tag{1.6}$$

where $t\geq t_0$, $p,\tau,\sigma_i\in C([t_0,\infty),\mathbb{R})$, $f_i\in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, and $\lim_{t\to+\infty}\tau(t)=\lim_{t\to+\infty}\sigma_i(t)=+\infty$ for $i\in\{1,2\}$. Lin [2] investigated sufficient conditions of oscillation and nonoscillation for the second-order nonlinear neutral differential equation:

$$\frac{d^2}{dt^2}\left[x(t)-p(t)x(t-\tau)\right]+q(t)f(x(t-\sigma))=0, \quad t\geq 0, \tag{1.7}$$

where $\tau>0,\sigma>0$, $p,q\in C(\mathbb{R}^+,\mathbb{R}^+)$, $f\in C(\mathbb{R},\mathbb{R})$ with $xf(x)>0$ for all $x\neq 0$. Liu and Huang [3] used the coincidence degree theory to establish the existence and uniqueness of T-periodic solutions for the second-order neutral functional differential equation of the form

$$\frac{d^2}{dt^2}[x(t)+Bx(t-\delta)]+C\frac{dx(t)}{dt}+g(x(t-\tau(t)))=p(t), \quad t\geq 0, \tag{1.8}$$

where $\tau, p, g : \mathbb{R} \to \mathbb{R}$ are continuous functions, B, δ, C are constants, τ and p are T-periodic, $C \neq 0, |B| \neq 1$, and $T > 0$. Zhou and Zhang [8] extended the results in [1] to the higher-order neutral functional differential equation with positive and negative coefficients:

$$\frac{d^n}{dt^n}\left[x(t) + px(t-\tau)\right] + (-1)^{n+1}\left[P(t)x(t-\sigma) - Q(t)x(t-\delta)\right] = 0, \quad t \geq t_0, \tag{1.9}$$

where $p \in \mathbb{R} \setminus \{\pm 1\}$, $\tau, \sigma, \delta \in \mathbb{R}^+$ and $P, Q \in C([t_0, +\infty), \mathbb{R}^+)$. Zhou et al. [7] used the Krasnoselskii fixed point theorem and the Schauder fixed point theorem to prove the existence results of a nonoscillatory solution for the forced higher-order nonlinear neutral functional differential equation:

$$\frac{d^n}{dt^n}\left[x(t) + p(t)x(t-\tau)\right] + \sum_{i=1}^{m} q_i(t) f(x(t-\sigma_i)) = g(t), \quad t \geq t_0, \tag{1.10}$$

where $\tau, \sigma_i \in \mathbb{R}^+$, $p, q_i, g \in C([t_0, +\infty), \mathbb{R})$ for $i \in \{1, 2, \ldots, m\}$ and $f \in C(\mathbb{R}, \mathbb{R})$. Zhang et al. [5] obtained some sufficient conditions for the oscillation of all solutions of the even order nonlinear neutral differential equations with variable coefficients:

$$\frac{d^n}{dt^n}\left[x(t) + p(t)x(\tau(t))\right] + q(t) f(x(\sigma(t))) = 0, \quad t \geq t_0, \tag{1.11}$$

where n is an even number, $p, q, \sigma, \tau \in C([t_0, +\infty), \mathbb{R}^+)$ with $0 \leq p(t) < 1$, for all $t \geq t_0$, $\lim_{t \to +\infty} \tau(t) = \lim_{t \to +\infty}, \sigma_i(t) = +\infty$ and $f \in C([t_0, +\infty), \mathbb{R})$.

The purpose of this paper is to investigate the solvability of (1.1). By constructing appropriate mappings and using the Laray-Schauder nonlinear alternative theorem, we establish a few sufficient conditions which ensure the existence of uncountably many bounded positive solutions for (1.1). Our results improve and generalize some corresponding results in [1, 2, 4, 6–8]. Three examples are given to illustrate the advantages of the results presented in this paper.

Throughout this paper, we assume that $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ denote the sets of all real numbers, nonnegative numbers, and positive integers, respectively, and

$$\nu = \inf\left\{\tau_i(t), \alpha_j(t), \beta_j(t) : t \in [t_0, +\infty), \quad i \in \{1, 2, \ldots, m\},\ j \in \{1, 2, \ldots, k\}\right\}. \tag{1.12}$$

Let $\mathrm{CB}([\nu, +\infty), \mathbb{R})$ stand for the Banach space of all continuous and bounded functions in $[\nu, +\infty)$ with norm $\|x\| = \sup_{t \geq \nu} |x(t)|$ for all $x \in \mathrm{CB}([\nu, +\infty), \mathbb{R})$ and

$$\begin{gathered} E(N) = \{x \in \mathrm{CB}([\nu, +\infty)\ , \mathbb{R}) : \ x(t) \geq N \text{ for } t \geq \nu\}, \\ U(M) = \{x \in E(N) : \|x\| < M\}, \end{gathered} \tag{1.13}$$

where $M, N \in \mathbb{R}^+$ with $M > N > 0$. Clearly, $E(N)$ is a nonempty closed convex subset of $\mathrm{CB}([\nu, +\infty), \mathbb{R})$ and $U(M)$ is an open subset of $E(N)$.

By a solution of (1.1), we mean a function $x \in C([\nu, +\infty), \mathbb{R})$ with some $T \geq t_0 + |\nu|$ such that $x(t) + \sum_{i=1}^{m} p_i(t) x(\tau_i(t))$ is n times continuously differentiable in $[T, +\infty)$ and

$f(t, x(\alpha_1(t)), \dots, x(\alpha_k(t)))$ is $n-1$ times continuously differentiable in $[T, +\infty)$ and (1.1) holds for $t \geq T$.

Lemma 1.1 (the Leray-Schauder nonlinear alterative theorem [9]). *Let E be a closed convex subset of a Banach space X and let U be an open subset of E with $p^\star \in U$. Also, $G : \overline{U} \to E$ is a continuous, condensing mapping with $G(\overline{U})$ bounded, where $\overline{U}$ denotes the closure of U Then,*

(A_1) *G has a fixed point in $\overline{U}$, or*

(A_2) *there are $x \in \partial U$ and $\lambda \in (0,1)$ with $x = (1-\lambda)p^\star + \lambda G x$.*

2. Main Results

Now, we apply the Leray-Schauder nonlinear alterative theorem to investigate the existence of uncountably many bounded positive solutions of (1.1) under certain conditions.

Theorem 2.1. *Assume that there exist constants M, N, p_0, t_1 and functions $F, H \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying*

$$|f(t, u_1, \dots, u_k)| \leq F(t), \quad \forall (t, u_1, \dots, u_k) \in [t_0, +\infty) \times [N, M]^k, \tag{2.1}$$

$$|h(t, v_1, \dots, v_k)| \leq H(t), \quad \forall (t, v_1, \dots, v_k) \in [t_0, +\infty) \times [N, M]^k, \tag{2.2}$$

$$\max\left\{\int_{t_0}^{+\infty} F(s)\,ds, \int_{t_0}^{+\infty} s^{n-1} \max\{|g(s)|, H(s)\}\,ds\right\} < +\infty, \tag{2.3}$$

$$0 < N < (1-2p_0)M, \qquad \sum_{i=1}^{m} |p_i(t)| \leq p_0 < \frac{1}{2}, \quad \forall t \geq t_1 \geq t_0. \tag{2.4}$$

Then, (1.1) *has uncountably many bounded positive solutions in $\overline{U(M)}$.*

Proof. Let $L \in (p_0 M + N, (1-p_0)M)$. It follows from (2.3) and (2.4) that there exists a constant $T > 1 + |t_0| + |t_1| + |\nu|$ satisfying

$$\int_{T}^{+\infty} F(s)\,ds + \int_{T}^{+\infty} s^{n-1}\left[|g(s)| + H(s)\right] ds < \min\left\{L - p_0 M - N, (1-p_0)M - L, \frac{M-N}{2}\right\}. \tag{2.5}$$

Choose $\epsilon_0 \in (0, \min\{L - p_0 M - N, (1-p_0)M - L, (M - N/2)\})$ with

$$\int_{T}^{+\infty} F(s)\,ds + \int_{T}^{+\infty} s^{n-1}\left[|g(s)| + H(s)\right] ds < \min\left\{L - p_0 M - N, (1-p_0)M - L, \frac{M-N}{2}\right\} - \epsilon_0. \tag{2.6}$$

Put $p^\star = M - \epsilon_0$. Clearly, $p^\star \in U(M)$. Define two mappings $A_L, B_L : \overline{U(M)} \to CB([\nu, +\infty), \mathbb{R})$ by

$$(A_L x)(t) = \begin{cases} L - \sum_{i=1}^{m} p_i(t) x(\tau_i(t)) + \dfrac{(-1)^n}{(n-1)!} \displaystyle\int_t^{+\infty} (s-t)^{n-1} g(s)\, ds, & t \geq T, \\ (A_L x)(T), & \nu \leq t < T, \end{cases} \tag{2.7}$$

$$(B_L x)(t) = \begin{cases} \displaystyle\int_t^{+\infty} f(s, x(\alpha_1(s)), \dots, x(\alpha_k(s)))\, ds \\ \quad + \dfrac{(-1)^{n-1}}{(n-1)!} \displaystyle\int_t^{+\infty} (s-t)^{n-1} h(s, x(\beta_1(s)), \dots, x(\beta_k(s)))\, ds, & t \geq T, \\ (B_L x)(T), & \nu \leq t < T \end{cases} \tag{2.8}$$

for all $x \in \overline{U(M)}$. It is clear to see that $A_L x$ and $B_L x$ are continuous for each $x \in \overline{U(M)}$. Let $D_L = A_L + B_L$. In view of (2.1), (2.2), and (2.4)–(2.8), we get that

$$\begin{aligned} &(A_L x)(t) + (B_L x)(t) \\ &\quad = L - \sum_{i=1}^{m} p_i(t) x(\tau_i(t)) + \frac{(-1)^n}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} g(s)\, ds \\ &\qquad + \int_t^{+\infty} f(s, x(\alpha_1(s)), \dots, x(\alpha_k(s)))\, ds \\ &\qquad + \frac{(-1)^{n-1}}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} h(s, x(\beta_1(s)), \dots, x(\beta_k(s)))\, ds \\ &\quad \geq L - p_0 M - \int_T^{+\infty} F(s)\, ds - \int_T^{+\infty} s^{n-1} \left[|g(s)| + H(s) \right] ds \\ &\quad \geq L - p_0 M - \min\left\{ L - p_0 M - N, (1-p_0)M - L, \frac{M-N}{2} \right\} + \varepsilon_0 \\ &\quad > N, \quad \forall (t, x) \in [T, +\infty) \times \overline{U(M)}, \end{aligned} \tag{2.9}$$

which gives that $D_L : \overline{U(M)} \to E(N)$.

Now, we show that $B_L : \overline{U(M)} \to \mathrm{CB}([\nu, +\infty), \mathbb{R})$ is continuous and compact. Let $\{x_m\}_{m \in \mathbb{N}} \subseteq \overline{U(M)}$ be an arbitrary sequence and $x \in C([\nu, +\infty), \mathbb{R})$ with

$$\|x_m - x\| \longrightarrow \quad \text{as } m \longrightarrow \infty. \tag{2.10}$$

Since $\overline{U(M)}$ is closed, it follows that $x \in \overline{U(M)}$. For any $(s, m) \in [T, +\infty) \times \mathbb{N}$, put

$$\begin{aligned} F_m(s) &= \left| f(s, x_m(\alpha_1(s)), \dots, x_m(\alpha_k(s))) - f(s, x(\alpha_1(s)), \dots, x(\alpha_k(s))) \right|, \\ H_m(s) &= \left| h(s, x_m(\beta_1(s)), \dots, x_m(\beta_k(s))) - h(s, x(\beta_1(s)), \dots, x(\beta_k(s))) \right|. \end{aligned} \tag{2.11}$$

It follows from (2.1), (2.2), and (2.11) that

$$|F_m(s)| \le 2F(s), \quad |H_m(s)| \le 2H(s), \quad \forall (s,m) \in [T,+\infty) \times \mathbb{N}, \tag{2.12}$$

which together with (2.8)–(2.11), the continuity of f, h, α_j, β_j for $j \in \{1,2,\ldots,k\}$, and the Lebesgue dominated convergence theorem yields that

$$\begin{aligned}
&|(B_L x_m)(t) - (B_L x)(t)| \\
&\quad \le \int_t^{+\infty} \left| f(s, x_m(\alpha_1(s)),\ldots,x_m(\alpha_k(s))) - f(s, x(\alpha_1(s)),\ldots,x(\alpha_k(s)))\right| ds \\
&\qquad + \frac{1}{(n-1)!} \\
&\qquad \times \int_t^{+\infty} (s-t)^{n-1} \left| h\left(s, x_m\left(\beta_1(s)\right),\ldots,x_m\left(\beta_k(s)\right)\right) - h\left(s, x\left(\beta_1(s)\right),\ldots,x\left(\beta_k(s)\right)\right)\right| ds \\
&\quad \le \int_T^{+\infty} F_m(s)ds + \frac{1}{(n-1)!}\int_T^{+\infty} s^{n-1} H_m(s)ds, \quad \forall t \ge T, \\
&\limsup_{m\to\infty} \|B_L x_m - B_L x\| \le \limsup_{m\to\infty}\left(\int_T^{+\infty} F_m(s)ds + \frac{1}{(n-1)!}\int_T^{+\infty} s^{n-1} H_m(s)ds\right) = 0,
\end{aligned} \tag{2.13}$$

which means that B_L is continuous in $\overline{U(M)}$. It follow from (2.1), (2.2), (2.6), and (2.8) that

$$\begin{aligned}
\|B_L x\| &= \sup_{t\ge\nu} |(B_L x)(t)| \\
&\le \int_T^{+\infty} F(s)ds + \frac{1}{(n-1)!}\int_T^{+\infty} s^{n-1} H(s)ds \\
&< \min\left\{L - p_0 M - N, (1-p_0)M - L, \frac{M-N}{2}\right\} - \epsilon_0 \\
&< M, \quad \forall x \in \overline{U(M)},
\end{aligned} \tag{2.14}$$

which yields that $B_L(\overline{U(M)})$ is uniformly bounded in $[\nu, +\infty)$.

Let ε be an arbitrary positive number. Equation (2.3) ensures that there exists $T^* > T$ satisfying

$$\int_{T^*}^{+\infty} F(s)ds + \int_{T^*}^{+\infty} s^{n-1} H(s)ds < \frac{\varepsilon}{2}. \tag{2.15}$$

Set

$$\delta = \frac{\varepsilon}{1 + 4\left[Q + M + Q(T^* - T)^{n-1}\right]}, \qquad Q = \max\{F(t), H(t) : t \in [T, T^*]\}. \tag{2.16}$$

For any $x \in \overline{U(M)}$ and $t_1, t_2 \in [\nu, +\infty)$ with $|t_1 - t_2| < \delta$, we consider the following three cases.

Case 1 ($T^* \le t_1 < t_2$). In view of (2.1), (2.2), (2.8), and (2.15), we deduce that

$$
\begin{aligned}
&|(B_L x)(t_2) - (B_L x)(t_1)| \\
&\quad = \Bigg| \int_{t_2}^{+\infty} f(s, x(\alpha_1(s)), \ldots, x(\alpha_k(s)))ds \\
&\qquad + \frac{(-1)^{n-1}}{(n-1)!} \int_{t_2}^{+\infty} (s-t_2)^{n-1} h(s, x(\beta_1(s)), \ldots, x(\beta_k(s)))ds \\
&\qquad - \int_{t_1}^{+\infty} f(s, x(\alpha_1(s)), \ldots, x(\alpha_k(s)))ds \\
&\qquad - \frac{(-1)^{n-1}}{(n-1)!} \int_{t_1}^{+\infty} (s-t_1)^{n-1} h(s, x(\beta_1(s)), \ldots, x(\beta_k(s)))ds \Bigg| \\
&\quad \le 2\int_{T^*}^{+\infty} F(s)ds + \frac{2}{(n-1)!} \int_{T^*}^{+\infty} s^{n-1} H(s)ds \\
&\quad < \varepsilon.
\end{aligned}
\tag{2.17}
$$

Case 2 ($T \le t_1 < t_2 \le T^*$). Suppose that $n = 1$. It follows from (2.1), (2.2), (2.6), and (2.8) that

$$
\begin{aligned}
&|(B_L x)(t_2) - (B_L x)(t_1)| \\
&\quad = \Bigg| \int_{t_2}^{+\infty} f(s, x(\alpha_1(s)), \ldots, x(\alpha_k(s)))ds + \int_{t_2}^{+\infty} h(s, x(\beta_1(s)), \ldots, x(\beta_k(s)))ds \\
&\qquad - \int_{t_1}^{+\infty} f(s, x(\alpha_1(s)), \ldots, x(\alpha_k(s)))ds - \int_{t_1}^{+\infty} h(s, x(\beta_1(s)), \ldots, x(\beta_k(s)))ds \Bigg| \\
&\quad \le \int_{t_1}^{t_2} (F(s) + H(s))ds \\
&\quad \le 2Q|t_1 - t_2| \\
&\quad < \varepsilon.
\end{aligned}
\tag{2.18}
$$

Suppose that $n \in N \setminus \{1\}$. It follows from the mean value theorem that, for each $s \in (t_2, +\infty)$, there exists $\zeta \in (s - t_2, s - t_1)$ satisfying

$$
\left| (s-t_2)^{n-1} - (s-t_1)^{n-1} \right| = (n-1)\zeta^{n-2}|t_1 - t_2| \le (n-1)s^{n-1}|t_1 - t_2|, \tag{2.19}
$$

which together with (2.1), (2.2), (2.6), and (2.8) yields that

$$\begin{aligned}
&|(B_Lx)(t_2)-(B_Lx)(t_1)|\\
&\quad=\Bigg|\int_{t_2}^{+\infty}f(s,x(\alpha_1(s)),\dots,x(\alpha_k(s)))ds\\
&\qquad+\frac{(-1)^{n-1}}{(n-1)!}\int_{t_2}^{+\infty}(s-t_2)^{n-1}h(s,x(\beta_1(s)),\dots,x(\beta_k(s)))ds\\
&\qquad-\int_{t_1}^{+\infty}f(s,x(\alpha_1(s)),\dots,x(\alpha_k(s)))ds\\
&\qquad-\frac{(-1)^{n-1}}{(n-1)!}\int_{t_1}^{+\infty}(s-t_1)^{n-1}h(s,x(\beta_1(s)),\dots,x(\beta_k(s)))ds\Bigg|\\
&\quad\le\int_{t_1}^{t_2}F(s)ds+\frac{1}{(n-1)!}\left(\int_{t_2}^{+\infty}\left|(s-t_2)^{n-1}-(s-t_1)^{n-1}\right|H(s)ds+\int_{t_1}^{t_2}(s-t_1)^{n-1}H(s)ds\right)\\
&\quad\le Q|t_1-t_2|+\frac{(n-1)|t_1-t_2|}{(n-1)!}\int_{t_2}^{+\infty}s^{n-1}H(s)ds+(T^*-T)^{n-1}Q|t_1-t_2|\\
&\quad\le\left[Q+M+(T^*-T)^{n-1}Q\right]|t_1-t_2|\\
&\quad<\varepsilon.
\end{aligned}\tag{2.20}$$

Case 3 ($\nu\le t_1<t_2\le T$). Equation (2.8) gives that

$$|(B_Lx)(t_2)-(B_Lx)(t_1)|=|(B_Lx)(T)-(B_Lx)(T)|=0.\tag{2.21}$$

Thus, $B_L(\overline{U(M)})$ is equicontinuous in $[\nu,+\infty)$. Hence, $B_L(\overline{U(M)})$ is a relatively compact subset of $C([\nu,+\infty),\mathbb{R})$. That is, B_L is a compact mapping.

Note that for any $x,y\in\overline{U(M)}$ and $t\ge T$

$$\begin{aligned}
&\left|(A_Lx)(t)-(A_Ly)(t)\right|\\
&\quad=\Bigg|L-\sum_{i=1}^{m}p_i(t)x(\tau_i(t))+\frac{(-1)^n}{(n-1)!}\int_t^{+\infty}(s-t)^{n-1}g(s)ds\\
&\qquad-L+\sum_{i=1}^{m}p_i(t)y(\tau_i(t))-\frac{(-1)^n}{(n-1)!}\int_t^{+\infty}(s-t)^{n-1}g(s)ds\Bigg|\\
&\quad\le\sum_{i=1}^{m}\left|p_i(t)\right|\left\|x-y\right\|\\
&\quad\le p_0\left\|x-y\right\|,
\end{aligned}\tag{2.22}$$

which implies that

$$\left\|A_Lx - A_Ly\right\| \leq p_0\left\|x - y\right\|, \quad \forall x, y \in \overline{U(M)}, \tag{2.23}$$

which together with (2.4) gives that A_L is a contraction mapping. It follows that $D_L : \overline{U(M)} \to E(N)$ is a continuous and condensing mapping. Let $x_0(t) = N$ for all $t \in [\nu, +\infty)$. Notice that $x_0 \in \overline{U(M)}$. Thus, (2.4)–(2.7), (2.14), and (2.23) yield that

$$\begin{aligned}
\|D_Lx\| &\leq \|A_Lx\| + \|B_Lx\| \\
&\leq \|A_Lx - A_Lx_0\| + \|A_Lx_0\| + M \\
&\leq p_0\|x - x_0\| + M + \sup_{t\geq T}\left|L - \sum_{i=1}^{m} p_i(t)x_0(\tau_i(t)) + \frac{(-1)^n}{(n-1)!}\int_t^{+\infty}(s-t)^{n-1}g(s)ds\right| \\
&\leq p_0(M + N) + M + L + p_0N + \int_T^{+\infty} s^{n-1}\left|g(s)\right|ds \\
&\leq (2 + p_0)M + 2p_0N + L, \quad \forall x \in \overline{U(M)},
\end{aligned} \tag{2.24}$$

that is, $D_L(\overline{U(M)})$ is uniformly bounded in $[\gamma, +\infty)$.

Put

$$\begin{aligned}
S_1 &= \{x \in \mathrm{CB}([\nu, +\infty), \mathbb{R}) : N \leq x(t) \leq M, \quad \forall t \geq \nu, \ \|x\| = M\}, \\
S_2 &= \{x \in \mathrm{CB}([\nu, +\infty), \mathbb{R}) : N \leq x(t) \leq M, \\
&\qquad \forall t \geq \nu \text{ and there exists } t^* \geq \nu \text{ satisfying } x(t^*) = N\}.
\end{aligned} \tag{2.25}$$

It is easy to verify that $\partial U(M) = S_1 \cup S_2$.

Next, we show that (A_2) in Lemma 1.1 does not hold. Otherwise, there exist $x \in \partial U(M)$ and $\lambda \in (0, 1)$ satisfying $x = (1 - \lambda)p^\star + \lambda D_Lx$. We have to discuss the following possible cases.

Case 1. Let $x \in S_1$. By means of (2.1), (2.2), and (2.4)–(2.8), we get that, for $t \geq T$,

$$\begin{aligned}
x(t) &= (1 - \lambda)p^\star + \lambda[(A_Lx)(t) + (B_Lx)(t)] \\
&\leq (1 - \lambda)(M - \epsilon_0) \\
&\quad + \lambda\left[L + \sum_{i=1}^{m}\left|p_i(t)\right|x(\tau_i(t)) + \frac{1}{(n-1)!}\int_t^{+\infty} s^{n-1}\left|g(s)\right|ds\right. \\
&\quad + \int_t^{+\infty}\left|f(s, x(\alpha_1(s)), \dots, x(\alpha_k(s)))\right|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(n-1)!}\int_t^{+\infty} s^{n-1}\left|h(s,x(\beta_1(s)),\dots,x(\beta_k(s)))\right|ds\Bigg]\\
&\le (1-\lambda)(M-\epsilon_0)+\lambda\left[L+p_0M+\int_t^{+\infty}F(s)ds+\frac{1}{(n-1)!}\int_t^{+\infty}s^{n-1}\left[\left|g(s)\right|+H(s)\right]ds\right]\\
&< (1-\lambda)(M-\epsilon_0)+\lambda\left[L+p_0M+\min\left\{L-p_0M-N,(1-p_0)M-L,\frac{M-N}{2}\right\}-\epsilon_0\right]\\
&\le M-\epsilon_0,
\end{aligned}
\tag{2.26}
$$

which implies that

$$
M=\|x\|=\sup_{t\ge\nu}|x(t)|\le M-\epsilon_0<M,
\tag{2.27}
$$

which is a contradiction.

Case 2. Let $x\in S_2$. It follows from (2.1), (2.2), and (2.4)–(2.8) that

$$
\begin{aligned}
N&=x(t^*)\\
&=(1-\lambda)p^\star+\lambda\left[(A_Lx)(t^*)+(B_Lx)(t^*)\right]\\
&=(1-\lambda)(M-\epsilon_0)+\lambda\left[(A_Lx)(\max\{t^*,T\}\ )+(B_Lx)(\max\{t^*,T\})\right]\\
&\ge(1-\lambda)(M-\epsilon_0)+\lambda\Bigg[L-\sum_{i=1}^m\left|p_i(\max\{t^*,T\})\right|x(\tau_i(\max\{t^*,T\}))\\
&\quad-\frac{1}{(n-1)!}\int_{\max\{t^*,T\}}^{+\infty}s^{n-1}\left|g(s)\right|ds-\int_{\max\{t^*,T\}}^{+\infty}\left|f(s,x(\alpha_1(s)),\dots,x(\alpha_k(s)))\right|ds\\
&\quad-\frac{1}{(n-1)!}\int_{\max\{t^*,T\}}^{+\infty}s^{n-1}\left|h(s,x(\beta_1(s)),\dots,x(\beta_k(s)))\right|ds\Bigg]\\
&\ge(1-\lambda)(M-\epsilon_0)\\
&\quad+\lambda\left[L-p_0M-\int_{\max\{t^*,T\}}^{+\infty}F(s)ds-\frac{1}{(n-1)!}\int_{\max\{t^*,T\}}^{+\infty}s^{n-1}\left[\left|g(s)\right|+H(s)\right]ds\right]\\
&\ge(1-\lambda)(M-\epsilon_0)+\lambda\left[L-p_0M-\min\left\{L-p_0M-N,(1-p_0)M-L,\frac{M-N}{2}\right\}+\epsilon_0\right]\\
&\ge(1-\lambda)(M-\epsilon_0)+\lambda(N+\epsilon_0)\\
&\ge\min\{M-\epsilon_0,N+\epsilon_0\}\\
&=N+\epsilon_0,
\end{aligned}
\tag{2.28}
$$

which is absurd.

Thus, Lemma 1.1 ensures that D_L has a fixed point $x \in \overline{U(M)}$; that is,

$$\begin{aligned} x(t) = L &- \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) + \frac{(-1)^n}{(n-1)!}\int_t^{+\infty}(s-t)^{n-1}g(s)ds \\ &+ \int_t^{+\infty} f(s, x(\alpha_1(s)), \dots, x(\alpha_k(s)))ds \\ &+ \frac{(-1)^{n-1}}{(n-1)!}\int_t^{+\infty}(s-t)^{n-1}h(s, x(\beta_1(s)), \dots, x(\beta_k(s)))ds, \quad \forall t \geq T, \end{aligned} \tag{2.29}$$

which yields that

$$\begin{aligned} &\frac{d^n}{dt^n}\left[x(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t))\right] + \frac{d^{n-1}}{dt^{n-1}} f(t, x(\alpha_1(t)), \dots, x(\alpha_k(t))) \\ &\quad + h(t, x(\beta_1(t)), \dots, x(\beta_k(t))) = g(t), \quad \forall t \geq T, \end{aligned} \tag{2.30}$$

which means that $x \in \overline{U(M)}$ is a bounded positive solution of (1.1).

Let $L_1, L_2 \in (p_0 M + N, (1-p_0)M)$ with $L_1 \neq L_2$. Similarly, we can prove that, for each $r \in \{1, 2\}$, there exist a constant $T_r > 1 + |t_0| + |t_1| + |\nu|$ and two mappings $A_{L_r}, B_{L_r} : \overline{U(M)} \to \mathrm{CB}([\nu, +\infty), \mathbb{R})$ satisfying (2.6)–(2.8), where T, L, A_L, and B_L are replaced by T_r, L_r, A_{L_r}, and B_{L_r}, respectively, and $A_{L_r} + B_{L_r}$ has a fixed point $z_r \in \overline{U(M)}$, which is a bounded positive solution of (1.1) in $\overline{U(M)}$. In order to prove that (1.1) possesses uncountably many bounded positive solutions in $\overline{U(M)}$, we need only to prove that $z_1 \neq z_2$. By means of (2.1)–(2.3), we know that there exists $T_3 > \max\{T_1, T_2\}$ satisfying

$$\int_{T_3}^{+\infty} F(s)ds + \int_{T_3}^{+\infty} s^{n-1}H(s)ds < \frac{|L_1 - L_2|}{4}. \tag{2.31}$$

It follows from (2.1), (2.2), (2.4), (2.7), (2.8), and (2.31) that for $t \geq T_3$

$$\begin{aligned} &|z_1(t) - z_2(t)| \\ &\quad = \Bigg| L_1 - L_2 - \sum_{i=1}^{m} p_i(t) z_1(\tau_i(t)) + \sum_{i=1}^{m} p_i(t) z_2(\tau_i(t)) \\ &\qquad + \int_t^{+\infty} f(s, z_1(\alpha_1(s)), \dots, z_1(\alpha_k(s)))ds - \int_t^{+\infty} f(s, z_2(\alpha_1(s)), \dots, z_2(\alpha_k(s)))ds \\ &\qquad + \frac{(-1)^{n-1}}{(n-1)!}\int_t^{+\infty}(s-t)^{n-1} h(s, z_1(\beta_1(s)), \dots, z_1(\beta_k(s)))ds \\ &\qquad - \frac{(-1)^{n-1}}{(n-1)!}\int_t^{+\infty}(s-t)^{n-1} h(s, z_2(\beta_1(s)), \dots, z_2(\beta_k(s)))ds \Bigg| \\ &\quad \geq |L_1 - L_2| - p_0\|z_1 - z_2\| - 2\int_{T_3}^{+\infty} F(s)ds - 2\int_{T_3}^{+\infty} s^{n-1}H(s)ds \\ &\quad \geq |L_1 - L_2| - p_0\|z_1 - z_2\| - \frac{|L_1 - L_2|}{2}, \end{aligned} \tag{2.32}$$

which implies that

$$\|z_1 - z_2\| \geq \frac{|L_1 - L_2|}{2(1+p_0)} > 0, \tag{2.33}$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 2.2. *Assume that there exist constants M, N, p_0, t_1 and functions $F, H \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (2.1)–(2.3) *and*

$$N < (1+p_0)M, \ \max\{p_i(t) : 1 \leq i \leq m\} \leq 0, \qquad \sum_{i=1}^{m} p_i(t) \geq p_0 > -1, \quad \forall t \geq t_1 \geq t_0. \tag{2.34}$$

Then, (1.1) *has uncountably many bounded positive solutions in $\overline{U(M)}$.*

Proof. Let $L \in (N, (1+p_0)M)$. It follows from (2.3) and (2.34) that there exists a constant $T > 1 + |t_0| + |t_1| + |\nu|$ satisfying

$$\int_T^{+\infty} F(s)ds + \int_T^{+\infty} s^{n-1}\left[\left|g(s)\right| + H(s)\right]ds < \min\left\{L - N, (1+p_0)M - L, \frac{M-N}{2}\right\}. \tag{2.35}$$

Take $\epsilon_0 \in (0, \min\{L - N, \ (1+p_0)M - L, \ (M - N/2)\})$ such that

$$\int_T^{+\infty} F(s)ds + \int_T^{+\infty} s^{n-1}\left[\left|g(s)\right| + H(s)\right]ds \leq \min\left\{L - N, (1+p_0)M - L, \frac{M-N}{2}\right\} - \epsilon_0. \tag{2.36}$$

The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

Theorem 2.3. *Assume that there exist constants M, N, p_0, t_1 and functions $F, H \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (2.1)–(2.3) *and*

$$N < (1-p_0)M, \ \min\{p_i(t) : 1 \leq i \leq m\} \geq 0, \qquad \sum_{i=1}^{m} p_i(t) \leq p_0 < 1, \quad \forall t \geq t_1 \geq t_0. \tag{2.37}$$

Then, (1.1) *has uncountably many bounded positive solutions in $\overline{U(M)}$.*

Proof. Let $L \in (p_0 M + N, M)$. It follows from (2.3) and (2.37) that there exists a constant $T > 1 + |t_0| + |t_1| + |\nu|$ satisfying

$$\int_T^{+\infty} F(s)ds + \int_T^{+\infty} s^{n-1}\left[\left|g(s)\right| + H(s)\right]ds < \min\left\{L - p_0 M - N, M - L, \frac{M-N}{2}\right\}. \tag{2.38}$$

Choose $\epsilon_0 \in (0,\ \min\{L - p_0 M - N,\ M - L,\ (M - N/2)\})$ such that

$$\int_T^{+\infty} F(s)ds + \int_T^{+\infty} s^{n-1}\left[|g(s)| + H(s)\right]ds \le \min\left\{L - p_0 M - N, M - L, \frac{M-N}{2}\right\} - \epsilon_0. \quad (2.39)$$

The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

Remark 2.4. Theorems 2.1–2.3 extend, improve, and unify the theorem in [1], Theorem 2.2 in [2], Theorem 1 in [4], Theorems 2.1 and 2.3 in [6], Theorems 1 and 3 in [7], and Theorems 1 and 3 in [8].

3. Examples and Applications

Now, we construct three nontrivial examples to show the superiority and applications of Theorems 2.1–2.3, respectively.

Example 3.1. Consider the higher-order nonlinear neutral delay differential equation:

$$\begin{aligned}
&\frac{d^n}{dt^n}\left[x(t) - \frac{t^2\cos t}{1+6t^2}x(t+2) + \frac{(-1)^n t\sin(1-t^3)}{1+4t}x\left(t^3 - t\right)\right] \\
&\quad + \frac{d^{n-1}}{dt^{n-1}}\left[\frac{1+t^2x^4(t-1) - tx^2(t^2+2)}{1+t^5} + \frac{tx^2(t^2+2)\sin^2(t - t^3x^2(t-2))}{1+x^2(t-2)+t^3}\right] \\
&\quad + \frac{t^3 + (1/t)}{1+t^{n+4}}\cos^3\left(x\left(t^2\ln t\right)\right) + \frac{\ln(1+x^2(2^t)) + t^2\sin(x(t^2))}{1+t^{n+3}+t^2} \\
&= \frac{1-t^3}{t^{n+4}\ln(1+t^3)}, \quad t \ge 3.
\end{aligned} \quad (3.1)$$

Let $t_0 = t_1 = 3$, $p_0 = 5/12$, $m = 2$, $k = 3$, $M = 36$, $N = 3$, $\nu = 1$,

$$\begin{aligned}
&p_1(t) = -\frac{t^2\cos t}{1+6t^2}, \quad p_2(t) = \frac{(-1)^n t\sin(1-t^3)}{1+4t}, \quad \tau_1(t) = t+2, \quad \tau_2(t) = t^3 - t, \\
&\alpha_1(t) = t-1, \quad \alpha_2(t) = t^2+2, \quad \alpha_3(t) = t-2, \quad \beta_1(t) = t^2\ln t, \quad \beta_2(t) = 2^t, \\
&\beta_3(t) = t^2, \quad f(t,u,v,w) = \frac{1+t^2u^4 - tv^2}{1+t^5} + \frac{tv^2\sin^2(t - t^3w^2)}{1+w^2+t^3}, \\
&h(t,u,v,w) = \frac{t^3+(1/t)}{1+t^{n+4}}\cos^3 u + \frac{\ln(1+v^2) + t^2\sin w}{1+t^{n+3}+t^2}, \\
&F(t) = \frac{1+t^2M^4 + tM^2}{1+t^5} + \frac{tM^2}{1+N^2+t^3}, \quad H(t) = \frac{t^3+(1/t)}{1+t^{n+4}} + \frac{\ln(1+M^2)+t^2}{1+t^{n+3}+t^2}, \\
&g(t) = \frac{1-t^3}{t^{n+4}\ln(1+t^3)}, \quad \forall (t,u,v,w) \in [t_0, +\infty) \times \mathbb{R}^3.
\end{aligned} \quad (3.2)$$

It is clear that (2.1)–(2.4) hold. Consequently, Theorem 2.1 ensures that (3.1) has uncountably many bounded positive solutions in $\overline{U(M)}$. But Theorem in [1], Theorems 2.1 and 2.3 in [6], Theorems 1 and 3 in [7], and Theorems 1 and 3 in [8] are null for (3.1).

Example 3.2. Consider the higher-order nonlinear neutral delay differential equation:

$$\begin{aligned}
&\frac{d^n}{dt^n}\left[x(t)-\frac{t^2}{1+3t^2}x\left(t+t^2\right)-\frac{2t^2}{1+7t^2}x\left(t^2-4t\right)\right] \\
&\quad+\frac{d^{n-1}}{dt^{n-1}}\left[\frac{2+t^4x^2(t-(1/t))}{1+t^6}+\frac{t^2x^2(t-(1/t))}{(1+t^4)(1+tx^2(2t^2-t))}\right] \\
&\quad+\frac{t^3-x^2\left(t^2-t\right)-t^2x^3(t\ln(1+t))}{(1+t^{n+4})\left[2+\sin^2\left(t^3x(t^2-t)x^4(t\ln(1+t))\right)\right]} \\
&=\frac{1+\sqrt{t}}{1+t^{n+4}}\left(1-t+t^2\ln(1+2|t|)\right),\quad t\geq 1.
\end{aligned}\tag{3.3}$$

Let $t_0=t_1=1$, $p_0=-13/21$, $m=2$, $k=2$, $M=400$, $N=100$, $\nu=-4$,

$$\begin{gathered}
p_1(t)=-\frac{t^2}{1+3t^2},\qquad p_2(t)=-\frac{2t^2}{1+7t^2},\qquad \tau_1(t)=t+t^2,\qquad \tau_2(t)=t^2-4t,\\
\alpha_1(t)=t-\frac{1}{t},\qquad \alpha_2(t)=2t^2-t,\qquad \beta_1(t)=t^2-t,\qquad \beta_2(t)=t\ln(1+t),\\
f(t,u,v)=\frac{2+t^4u^2}{1+t^6}+\frac{t^2u^2}{(1+t^4)(1+tv^2)},\qquad h(t,u,v)=\frac{t^3-u^2-t^2v^3}{(1+t^{n+4})\left(2+\sin^2\left(t^3uv^4\right)\right)}\\
F(t)=\frac{2+t^4M^2}{1+t^6}+\frac{t^2M^2}{(1+t^4)(1+tN^2)},\qquad H(t)=\frac{t^3+M^2+t^2M^3}{2+2t^{n+4}},\\
g(t)=\frac{1+\sqrt{t}}{1+t^{n+4}}\left(1-t+t^2\ln(1+2|t|)\right),\quad \forall(t,u,v)\in[t_0,+\infty)\times\mathbb{R}^2.
\end{gathered}\tag{3.4}$$

It is easy to verify that (2.1)–(2.3) and (2.34) hold. Consequently, Theorem 2.2 guarantees that (3.3) has uncountably many bounded positive solutions in $\overline{U(M)}$. But Theorem in [1], Theorem 1 in [4], Theorem 2.3 in [6], Theorem 3 in [7], and Theorem 3 in [8] are useless for (3.3).

Example 3.3. Consider the higher-order nonlinear neutral delay differential equation:

$$\frac{d^n}{dt^n}\left[x(t)+\frac{t^4}{1+t^2+4t^4}x\left(t^4+1\right)+\frac{2\ln\left(1+t^2\right)}{1+3\ln(1+t^2)}x\left(2+\ln^2 t\right)\right]$$
$$+\frac{d^{n-1}}{dt^{n-1}}\left[\frac{t+x^3\left(t^3+2t^2\right)-(-1)^n t^2\sin\left(x\left(t^2\right)\right)}{1+t^5+t^2x^4(t^2)}\right]$$
$$+\frac{t^3-x^2(t-2)+t^2x^5(t-2)}{(1+t^{n+5})(1+tx^2(t^2\ln(1+t)))}=\frac{t-(-1)^n\ln 1+\sqrt{1+t^2}}{t^{n+(5/2)}+\sqrt{1+\cos^2 t}},\quad t\geq 2. \tag{3.5}$$

Let $t_0=t_1=2$, $p_0=11/12$, $m=2$, $k=2$, $M=24$, $N=1$, $\nu=0$,

$$p_1(t)=\frac{t^4}{1+t^2+4t^4},\qquad p_2(t)=\frac{2\ln\left(1+t^2\right)}{1+3\ln(1+t^2)},\qquad \tau_1(t)=t^4+1,\qquad \tau_2(t)=2+\ln^2 t,$$
$$\alpha_1(t)=t^3+2t^2,\qquad \alpha_2(t)=t^2,\qquad \beta_1(t)=t-2,\qquad \beta_2(t)=t^2\ln(1+t),$$
$$f(t,u,v)=\frac{t+u^3-(-1)^n t^2\sin v}{1+t^5+t^2v^4},\qquad h(t,u,v)=\frac{t^3-u^2+t^2u^5}{(1+t^{n+5})(1+tv^2)},$$
$$F(t)=\frac{t+M^3+t^2}{1+t^5+t^2N^4},\qquad H(t)=\frac{t^3+M^2+t^2M^5}{(1+t^{n+5})(1+tN^2)},$$
$$g(t)=\frac{t-(-1)^n\ln\left(1+\sqrt{1+t^2}\right)}{t^{n+(5/2)}+\sqrt{1+\cos^2 t}},\quad \forall(t,u,v)\in[t_0,+\infty)\times\mathbb{R}^2. \tag{3.6}$$

Obviously, (2.1)–(2.3) and (2.37) hold. It follows from Theorem 2.3 that (3.5) has uncountably many bounded positive solutions in $\overline{U(M)}$. But Theorem in [1], Theorem 2.2 in [2], Theorem 2.1 in [4], Theorem 2.1 in [6], Theorem 1 in [7], and Theorem 1 in [8] are inapplicable for (3.5).

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