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Research Article

Some Identities on the q-Bernoulli Numbers and Polynomials with Weight 0

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Recently, Kim (2011) has introduced the q-Bernoulli numbers with weight α. In this paper, we consider the q-Bernoulli numbers and polynomials with weight $\alpha = 0$ and give p-adic q-integral representation of Bernstein polynomials associated with q-Bernoulli numbers and polynomials with weight 0. From these integral representation on $\mathbb{Z}_p$, we derive some interesting identities on the q-Bernoulli numbers and polynomials with weight 0.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Let $|\cdot|_p$ be a p-adic norm with $|x|_p = p^{-r}$, where $x = p^r s/t$ and $(p,s) = (p,t) = (s,t) = 1$, $r \in \mathbb{Q}$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$, and $[x]_q = (1-q^x)/(1-q)$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$\begin{aligned} I_q(f) &= \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_q\left(x + p^N\mathbb{Z}_p\right) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \end{aligned} \tag{1.1}$$

(see [1–5]). For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. From (1.1), we note that

$$q^n I_q(f_n) - I_q(f) = (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} q^l f'(l), \tag{1.2}$$

where $f'(l) = df(x)/dx|_{x=l}$, (see [3, 6, 7]). In the special case, $n = 1$, we get

$$q\int_{\mathbb{Z}_p} f(x+1)d\mu_q(x) - \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = (q-1)f(0) + \frac{q-1}{\log q}f'(0). \tag{1.3}$$

Throughout this paper, we assume that $\alpha \in \mathbb{Q}$.

The q-Bernoulli numbers with weight α are defined by Kim [8] as follows:

$$\widetilde{\beta}_{0,q}^{(\alpha)} = 1, \qquad q\left(q^{\alpha}\widetilde{\beta}_{q}^{(\alpha)} + 1\right)^n - \widetilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.4}$$

with the usual convention about replacing $(\widetilde{\beta}_q^{(\alpha)})^n$ with $\widetilde{\beta}_{n,q}^{(\alpha)}$. From (1.4), we can derive the following equation:

$$\begin{aligned} \widetilde{\beta}_{n,q}^{(\alpha)} &= \frac{1}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{\alpha l+1}{[\alpha l+1]_q} \\ &= -\frac{n\alpha}{[\alpha]_q}\sum_{m=0}^{\infty}q^{m\alpha+m}[m]_{q^{\alpha}}^{n-1} + (1-q)\sum_{m=0}^{\infty}q^m[m]_{q^{\alpha}}^n. \end{aligned} \tag{1.5}$$

By (1.1), (1.4), and (1.5), we get

$$\widetilde{\beta}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p}[x]_{q^{\alpha}}^n d\mu_q(x) = \frac{1}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{\alpha l+1}{[\alpha l+1]_q}. \tag{1.6}$$

The q-Bernoulli polynomials with weight α are defined by

$$\widetilde{\beta}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p}[x+y]_{q^{\alpha}}^n d\mu_q(y) = \sum_{l=0}^{n}\binom{n}{l}q^{\alpha l x}[x]_{q^{\alpha}}^{n-l}\widetilde{\beta}_{l,q}^{(\alpha)}. \tag{1.7}$$

By (1.6) and (1.7), we easily see that

$$\widetilde{\beta}_{n,q}^{(\alpha)}(x) = \frac{1}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{\alpha l+1}{[\alpha l+1]_q}. \tag{1.8}$$

Let $C(\mathbb{Z}_p)$ be the set of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the p-adic analogue of Bernstein operator of order n for f is given by

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^{n}f\left(\frac{k}{n}\right)B_{k,n}(x) = \sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}, \tag{1.9}$$

where $n, k \in \mathbb{Z}_+$ (see [1, 9, 10]). For $n, k \in \mathbb{Z}_+$, the p-adic Bernstein polynomials of degree n are defined by $B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ for $x \in \mathbb{Z}_p$, (see [1, 10, 11]).

In this paper, we consider Bernstein polynomials to express the p-adic q-integral on $\mathbb{Z}_p$ and investigate some interesting identities of Bernstein polynomials associated with the q-Bernoulli numbers and polynomials with weight 0 by using the expression of p-adic q-integral on $\mathbb{Z}_p$ of these polynomials.

2. q-Bernoulli Numbers with Weight 0 and Bernstein Polynomials

In the special case, $\alpha = 0$, the q-Bernoulli numbers with weight 0 will be denoted by $\widetilde{\beta}_{n,q}^{(0)} = \widetilde{\beta}_{n,q}$. From (1.4), (1.5), and (1.6), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \widetilde{\beta}_{n,q} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} x^n d\mu_q(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{xt} d\mu_q(x) \\ &= \left(\frac{q-1}{\log q}\right)\left(\frac{t+\log q}{qe^t - 1}\right). \end{aligned} \tag{2.1}$$

It is easy to show that

$$\begin{aligned} \frac{t+\log q}{qe^t-1} &= \frac{t}{q-1}\left(\frac{1-q^{-1}}{e^t-q^{-1}}\right) + \frac{\log q}{q-1}\left(\frac{1-q^{-1}}{e^t-q^{-1}}\right) \\ &= \frac{t}{q-1}\sum_{n=0}^{\infty} H_n\left(q^{-1}\right)\frac{t^n}{n!} + \frac{\log q}{q-1}\sum_{n=0}^{\infty} H_n\left(q^{-1}\right)\frac{t^n}{n!} \\ &= \frac{1}{q-1}\sum_{n=1}^{\infty} nH_{n-1}\left(q^{-1}\right)\frac{t^n}{n!} + \frac{\log q}{q-1}\sum_{n=0}^{\infty} H_n\left(q^{-1}\right)\frac{t^n}{n!}, \end{aligned} \tag{2.2}$$

where $H_n(q^{-1})$ are the nth Frobenius-Euler numbers.

By (2.1) and (2.2), we get

$$\widetilde{\beta}_{n,q} = \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{n}{\log q} H_{n-1}(q^{-1}) + H_n(q^{-1}) & \text{if } n > 0. \end{cases} \tag{2.3}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, we have*

$$\widetilde{\beta}_{n,q} = \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{n}{\log q} H_{n-1}(q^{-1}) + H_n(q^{-1}) & \text{if } n > 0, \end{cases} \tag{2.4}$$

where $H_n(q^{-1})$ are the nth Frobenius-Euler numbers.

From (1.5), (1.6), and (1.7), we have

$$\widetilde{\beta}_{0,q} = 1, \quad q\left(\widetilde{\beta}_q + 1\right)^n - \widetilde{\beta}_{n,q} = \begin{cases} \dfrac{q-1}{\log q} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{2.5}$$

with the usual convention about replacing $(\widetilde{\beta}_q)^n$ with $\widetilde{\beta}_{n,q}$. By (1.7), the nth q-Bernoulli polynomials with weight 0 are given by

$$\widetilde{\beta}_{n,q}(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_q(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \widetilde{\beta}_{l,q}. \tag{2.6}$$

From (2.6), we can derive the following function equation:

$$\left(\frac{q-1}{\log q}\right)\left(\frac{t+\log q}{qe^t - 1}\right)e^{xt} = \sum_{n=0}^{\infty} \widetilde{\beta}_{n,q}(x)\frac{t^n}{n!}. \tag{2.7}$$

Thus, by (2.7), we get that

$$\widetilde{\beta}_{n,q^{-1}}(1-x) = (-1)^n \widetilde{\beta}_{n,q}(x), \quad \text{for } n \in \mathbb{Z}_+. \tag{2.8}$$

By the definition of p-adic q-integral on $\mathbb{Z}_p$, we see that

$$\int_{\mathbb{Z}_p} (1-x)^n d\mu_q(x) = (-1)^n \int_{\mathbb{Z}_p} (x-1)^n d\mu_q(x) = (-1)^n \widetilde{\beta}_{n,q}(-1). \tag{2.9}$$

Therefore, by (2.8) and (2.9), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, we have*

$$(-1)^n \widetilde{\beta}_{n,q}(x) = \widetilde{\beta}_{n,q^{-1}}(1-x). \tag{2.10}$$

In particular, $x = -1$, we get

$$\int_{\mathbb{Z}_p} (1-y)^n d\mu_q(y) = (-1)^n \widetilde{\beta}_{n,q}(-1) = \widetilde{\beta}_{n,q^{-1}}(2). \tag{2.11}$$

From (2.5), we can derive the following equation:

$$q^2 \widetilde{\beta}_{n,q}(2) = q^2 + nq\frac{q-1}{\log q} - q + \widetilde{\beta}_{n,q}, \quad \text{if } \ n > 1. \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\widetilde{\beta}_{n,q}(2) = 1 + \frac{n}{q}\left(\frac{q-1}{\log q}\right) - \frac{1}{q} + \frac{1}{q^2}\widetilde{\beta}_{n,q}. \tag{2.13}$$

Taking the p-adic q-integral on $\mathbb{Z}_p$ for one Bernstein polynomials in (1.9), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_q(x) &= \binom{n}{k}\int_{\mathbb{Z}_p} x^k(1-x)^{n-k}d\mu_q(x)\\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\int_{\mathbb{Z}_p} x^{k+l}d\mu_q(x)\\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\widetilde{\beta}_{k+l,q}.\end{aligned} \tag{2.14}$$

From the symmetry of Bernstein polynomials, we note that

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_q(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x)d\mu_q(x)\\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p}(1-x)^{n-l}d\mu_q(x).\end{aligned} \tag{2.15}$$

Let $n > k+1$. Then, by Theorem 2.3 and (2.15), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_q(x) &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\left(1 - \frac{n-l}{q^{-1}}\left(\frac{q^{-1}-1}{\log q}\right) - q + q^2\widetilde{\beta}_{n-l,q^{-1}}\right)\\ &= \begin{cases} 1 + n\left(\dfrac{q-1}{\log q}\right) - q + q^2\widetilde{\beta}_{n,q^{-1}} & \text{if } k = 0,\\ n\left(\dfrac{1-q}{\log q}\right) + nq^2\widetilde{\beta}_{n,q^{-1}} + nq^2\widetilde{\beta}_{n-1,q^{-1}} & \text{if } k = 1,\\ \dbinom{n}{k}q^2\displaystyle\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\widetilde{\beta}_{n-l,q^{-1}} & \text{if } k > 1.\end{cases}\end{aligned} \tag{2.16}$$

By comparing the coefficients on the both sides of (2.14) and (2.16), we obtain the following theorem.

Theorem 2.4. *For $n, k \in \mathbb{Z}_+$ with $n > k+1$, we have*

$$\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\widetilde{\beta}_{1+l,q} = \frac{1-q}{\log q} + q^2\widetilde{\beta}_{n,q^{-1}} + q^2\widetilde{\beta}_{n-1,q^{-1}},$$
$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\widetilde{\beta}_{k+l,q} = q^2\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\widetilde{\beta}_{n-l,q^{-1}}, \quad \text{if } k>1. \tag{2.17}$$

In particular, when $k = 0$, we have

$$\sum_{l=0}^{n}\binom{n}{l}(-1)^l\widetilde{\beta}_{l,q} = 1 + n\ \frac{q-1}{\log q} - q + q^2\widetilde{\beta}_{n,q^{-1}}. \tag{2.18}$$

Let $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k + 1$. Then we see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)d\mu_q(x)\\
&= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p} x^{2k}(1-x)^{n+m-2k}d\mu_q(x)\\
&= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\int_{\mathbb{Z}_p}(1-x)^{n+m-l}d\mu_q(x)\\
&= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(1-(n+m-l)\left(\frac{1-q}{\log q}\right) - q + q^2\widetilde{\beta}_{n+m-l,q^{-1}}\right)\\
&= \begin{cases} 1 + (n+m)\left(\dfrac{q-1}{\log q}\right) - q + q^2\widetilde{\beta}_{n+m,q^{-1}} & \text{if } k=0,\\ \dbinom{n}{k}\dbinom{m}{k}q^2\displaystyle\sum_{l=0}^{2k}\dbinom{2k}{l}(-1)^{l+2k}\widetilde{\beta}_{n+m-l,q^{-1}} & \text{if } k>0. \end{cases}
\end{aligned} \tag{2.19}$$

For $m, n, k \in \mathbb{Z}_+$, we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)d\mu_q(x) &= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p} x^{2k}(1-x)^{n+m-2k}d\mu_q(x)\\
&= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^l\int_{\mathbb{Z}_p} x^{2k+l}d\mu_q(x)\\
&= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^l\widetilde{\beta}_{l+2k,q}.
\end{aligned} \tag{2.20}$$

By comparing the coefficients on the both sides of (2.19) and (2.20), we obtain the following theorem.

Theorem 2.5. *For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k + 1$, we have*

$$\sum_{l=0}^{n+m} \binom{n+m}{l} (-1)^l \widetilde{\beta}_{l,q} = 1 + (n+m)\left(\frac{q-1}{\log q}\right) - q + q^2 \widetilde{\beta}_{n+m,q^{-1}}. \tag{2.21}$$

In particular, when $k \neq 0$, we have

$$\sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \widetilde{\beta}_{l+2k,q} = q^2 \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \widetilde{\beta}_{n+m-l,q^{-1}}. \tag{2.22}$$

For $s \in \mathbb{N}$, let $k, n_1, \ldots, n_s \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk + 1$. By the same method above, we get

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}(x) \right) d\mu_q(x) = \begin{cases} 1 + \left(\sum_{i=1}^{s} n_i \right) \left(\frac{q-1}{\log q} \right) - q + q^2 \widetilde{\beta}_{n_1+n_2+\cdots+n_s,q^{-1}} & \text{if } k = 0, \\ \left(\prod_{i=1}^{s} \binom{n_i}{k} \right) q^2 \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \widetilde{\beta}_{n_1+n_2+\cdots+n_s-l,q^{-1}} & \text{if } k > 0. \end{cases} \tag{2.23}$$

From the binomial theorem, we note that

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}(x) \right) d\mu_q(x) = \left(\prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l \widetilde{\beta}_{l+sk,q}. \tag{2.24}$$

By comparing the coefficients on the both sides of (2.23) and (2.24), we obtain the following theorem.

Theorem 2.6. *For $s \in \mathbb{N}$, let $k, n_1, \ldots, n_s \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk + 1$. Then, we have*

$$\sum_{l=0}^{n_1+\cdots+n_s} \binom{n_1+\cdots+n_s}{l} (-1)^l \widetilde{\beta}_{l,q} = 1 + \left(\sum_{i=1}^{s} n_i \right) \left(\frac{q-1}{\log q} \right) - q + q^2 \widetilde{\beta}_{n_1+\cdots+n_s,q^{-1}}. \tag{2.25}$$

In particular, when $k \neq 0$, we have

$$\sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l \widetilde{\beta}_{l+sk,q} = q^2 \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \widetilde{\beta}_{n_1+\cdots+n_s-l,q^{-1}}. \tag{2.26}$$

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