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Research Article

A Periodic Problem of a Semilinear Pseudoparabolic Equation

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A class of periodic problems of pseudoparabolic type equations with nonlinear periodic sources are investigated. A rather complete classification of the exponent p is given, in terms of the existence and nonexistence of nontrivial and nonnegative periodic solutions.

1. Introduction

The purpose of this paper is to give a complete classification of the exponent p, in terms of the existence and nonexistence of nontrivial nonnegative classical periodic solutions for the pseudoparabolic equation with nonlinear periodic sources

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \Delta u + \alpha(x,t)u^p, \quad (x,t) \in \Omega \times \mathbb{R}, \tag{1.1}$$

subject to the homogeneous boundary value condition and periodic condition

$$\begin{aligned} u(x,t) &= 0, \quad x \in \partial\Omega,\ t \in \mathbb{R}, \\ u(x,t) &= u(x,t+\omega), \quad x \in \Omega,\ t \in \mathbb{R}, \end{aligned} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $k > 0$ and $p \geq 0$ are all constants, and $\alpha(x,t)$ is an appropriately smooth and positive function which is periodic in time with periodicity $\omega > 0$.

Pseudo-parabolic equations are characterized by the occurrence of a time derivative appearing in the highest-order term [1] and arise in applications from radiation with time delay [2], dynamic capillary pressure in unsaturated flow [3], and heat conduction involving two temperatures [4], and so forth. They can also be used as a regularization of ill-posed transport problems, especially as a quasi-continuous approximation to discrete models for population dynamics [5]. Actually, comparing with another regularized method, the Cahn-Hilliard equations, pseudo-parabolic equations are more incorporated with the out-of-equilibrium viscoelastic relaxation effects according to experimental results [6]. Furthermore, pseudo-parabolic equations are closely related to the well-know, BBM equations [7] which are advocated as a refinement of KdV equations.

Since the last century, pseudo-parabolic equations have been studied in different aspects, such as the integral representations of solutions [8], long-time behavior of solutions [9], Riemann problem and Riemann-Hilbert problem [10], and nonlocal boundary value problems [11]. However, as far as we know, the researches on periodic problems for pseudo-parabolic equations are far from those of parabolic equations [12–18]. Among the earliest works for periodic parabolic equations, Seidman's work [18] caused much attention, in which one can find the existence of nontrivial periodic solutions, for the case $k = 0$ and $p = 0$ of (1.1), namely,

$$\frac{\partial u}{\partial t} = \Delta u + \alpha(x,t), \tag{1.3}$$

where the function $\alpha(x,t)$ is periodic in t. From then on, many authors dealt with semilinear equations of the form

$$\frac{\partial u}{\partial t} = \Delta u + \alpha(x,t)u^p, \quad p > 0. \tag{1.4}$$

It were Beltramo and Hess [12] who first considered the case $p = 1$ of (1.4) and showed that only for some special $\alpha(x,t)$ can the equation have nontrivial periodic solutions. It seems that the exponent $p = 1$ of the source is a singular value. Indeed, this interesting phenomenon was verified by Esteban [13, 14]. Her results imply that, for p in a neighborhood of 1 except for $p = 1$, nontrivial periodic solutions exist definitely for any $\alpha(x,t) > 0$. Her results also imply the existence of positive periodic solutions when $p > 1$ with $N \leq 2$, or $1 < p < N/(N-2)$ with $N > 2$, for any positive $\alpha(x,t)$. At the same time, she also indicated that if $p \geq (N+2)/(N-2)$ with $N > 2$, then the equation might have no positive periodic solution. In fact, at least for star-shaped domains, there is definitely no such solution. So, this is another interesting phenomenon, and it is imaginable that $p = (N+2)/(N-2)$ should be a critical value. In fact, until 2004, this guess was solved by Quittner [17] who proved the existence of positive periodic solutions for the case $1 < p < (N+2)/(N-2)$ with $N > 2$, although there are still some restrictions on the structure of $\alpha(x,t)$.

Looking back to periodic problems of pseudo-parabolic equations, to our knowledge, most works are devoted to space periodic problems. For instance, the existence and uniqueness for regular solutions of the well-known BBM equation with $\partial\Delta u/\partial t$ were proved by the differential-difference method in [19]. The existence and blowup of solutions to the initial and periodic boundary value problem for the Camassa-Holm equation were considered in [20]. In [21], Kaikina et al. considered the periodic boundary value problem for the following pseudo-parabolic equation:

$$\frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t} = \alpha\Delta u - \lambda|u|^{p-1}u, \tag{1.5}$$

where $\alpha > 0$, $\lambda > 0$, and $p > 1$. Their proof revealed that if the initial data is small enough, then there exists a unique solution. Once removing the assumption that the initial data is small, then one should add $N \leq 4$ with $p > 1$ or $N \geq 5$ with $1 < p \leq N/(N-4)$ to assure the existence of a unique solution. Further, from their results, one can also find that the solutions of (1.5) exhibit power-law decay in time or dichotomous large-time behavior which unlike the usual exponentially decay in time arose in periodic problems.

For time periodic problems of pseudo-parabolic equations, according to our survey, expect the early works of Matahashi and Tsutsumi and the recent research of Li et al., there are no other investigations. In [22, 23], Matahashi and Tsutsumi have established the existence theorems of time periodic solutions for the linear case

$$\frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t} = \Delta u + f(x,t) \tag{1.6}$$

and the semilinear case

$$\frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t} = \Delta u - |u|^{p-1}u + f(x,t) \tag{1.7}$$

for $1 < p < 1 + 4/N$ with $N = 2,3,4$ or $0 < p < 3$ with $N = 1$, respectively. As for one-dimensional case with $p > 1$ for (1.1) and (1.2), we refer to the joint work with two authors of this paper for the existence of nontrivial and nonnegative periodic solutions; see [24].

In this paper, we consider the time periodic problem (1.1) and (1.2) when $N \geq 1$ and $p \geq 0$. Certainly, some researches focus on the source which has the general form $f(u)$, but here we are quite interested in the special source u^p (which was also studied by many authors, see [9] e.g.) and the existence and nonexistence of nontrivial nonnegative classical periodic solutions in different intervals divided by p. It will be shown that, as an important aspect of good viscosity approximation to the corresponding periodic problem of the semilinear heat equation, there still exist two critical values $p_0 = 1$ and $p_c = (N+2)/(N-2)$ for the exponent p. Precisely speaking, we have the following conclusions

(i) There exist at least one positive classical periodic solution in the case $0 \leq p < 1$

(ii) When $1 < p < (N+2)/(N-2)$ for $N > 2$, or $1 < p < \infty$ for $N \leq 2$ with convex domain Ω, there exist at least one nontrivial nonnegative classical periodic solution

(iii) When $p \geq (N+2)/(N-2)$ for $N > 2$ with star-shaped domain Ω and $\alpha(x,t)$ is independent of t, there is no nontrivial and nonnegative periodic solution

(iv) For the singular case $p = 1$, only for some special $\alpha(x,t)$ can the problem have positive classical periodic solutions.

From the existing investigations, we can see that, not only for space periodic problem but also for time periodic problem of pseudo-parabolic equations, the results are still far from complete. Specially, notice that pseudo-parabolic equations can be used to describe models which are sensitive to time periodic factors (e.g., seasons), such as aggregating populations [5, 25], and there are some numerical results and analysis of stabilities of solutions [26–28] which indicate that time periodic solutions should exist, so it is reasonable to study the periodic problem (1.1) and (1.2). Our results reveal that the exponents p_0 and p_c are consistent with the corresponding semilinear heat equation [12–14, 17]. This fact exactly indicates that the viscous effect of the third-order term is not strong enough to change the exponents. However, due to the existence of the third-order term $k(\partial\Delta u/\partial t)$, the proof is more complicated than the proof for the case $k = 0$. Actually, the viscous term $k(\partial\Delta u/\partial t)$ seems to have its own effect [29], and our future work will be with a particular focus on this. Moreover, comparing with the previous works of pseudo-parabolic equations, our conclusions not only coincide with those of [22] but also contain the results of [24].

The content of this paper is as follows. We describe, in Section 2, some preliminary notations and results for our problem. Section 3 concerns with the case $0 \le p < 1$, and the existence of positive classical periodic solutions is established. Subsequently, in Section 4, we discuss the case $p > 1$, in which we will investigate the existence and nonexistence of nontrivial nonnegative classical periodic solutions. The singular case $p = 1$ will be discussed in Section 5.

2. Preliminaries

In this section, we will recall some standard definitions and notations needed in our investigation. Specially, we will prove that if the weak solution under consideration belongs to L^∞, then it is just the classical solution.

Let $\tau \in \mathbb{R}$ be fixed, and set

$$\begin{aligned} &Q = \Omega \times (0,T), \qquad Q_\omega = \Omega \times (\tau, \tau+\omega), \\ &S = \inf_Q \alpha(x,t), \qquad L = \sup_Q \alpha(x,t). \end{aligned} \tag{2.1}$$

In order to prove the existence of periodic solutions, we only need to consider the following problem:

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \Delta u + \alpha(x,t)|u|^p, \quad (x,t) \in Q_\omega, \tag{2.2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \ t \in (\tau, \tau+\omega), \tag{2.3}$$

$$u(x,\tau) = u(x,\tau+\omega), \quad x \in \Omega. \tag{2.4}$$

Though the final existence results in this paper are established for the classical solutions, but due to the proof procedure, we first need to consider solutions in the distribution sense. Denote by E, E_0 the reasonable weak solutions space, namely,

$$\begin{aligned} &E = \left\{ u \in L^{p+1}(\Omega); \frac{\partial u}{\partial t} \in L^2(Q_\omega), \frac{\partial \nabla u}{\partial t} \in L^2(Q_\omega), \nabla u \in L^2(Q_\omega) \right\}, \\ &E_0 = \left\{ u \in E; u(x,t) = 0 \text{ for any } x \in \partial\Omega \right\}. \end{aligned} \tag{2.5}$$

Definition 2.1. A function $u \in E$ is called to be a weak ω-periodic upper solution of the problem (2.2)–(2.4) provided that for, any nonnegative function $\varphi \in C_0^1(\overline{Q}_\omega)$, there holds

$$\begin{aligned}
\iint_{Q_\omega} \frac{\partial u}{\partial t}\varphi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_\omega} k\frac{\partial \nabla u}{\partial t}\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \geq -\iint_{Q_\omega} \nabla u \nabla\varphi \,\mathrm{d}x\ \mathrm{d}t& \\
+ \iint_{Q_\omega} \alpha(x,t)|u|^p\varphi \,\mathrm{d}x\,\mathrm{d}t,& \quad (x,t) \in Q_\omega, \\
u(x,t) \geq 0, \quad x \in \partial\Omega,\ t \in (\tau, \tau+\omega),& \\
u(x,\tau) \geq u(x,\tau+\omega), \quad x \in \Omega.&
\end{aligned} \tag{2.6}$$

Replacing "≥" by "≤" in the above inequalities, it follows the definition of a weak lower solution. Furthermore, if u is a weak upper solution as well as a weak lower solution, then we call it a weak solution of (2.2)–(2.4).

In what follows, we show that the weak solution defined above is classical if it belongs to $L^\infty(Q_\omega)$. Meanwhile, the classical solution u is nonnegative which implies that we can throw off the symbol of absolute value of $|u|$.

Theorem 2.2. *If the weak solution u in Definition 2.1 also belongs to $L^\infty(Q_\omega)$, then there holds $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_\omega)$ and $\partial u/\partial t \in C^{2+\alpha,\alpha/2}(\overline{Q}_\omega)$, namely, u is just the classical solution. Furthermore, u is nonnegative.*

Proof. We lift the regularity of the weak solution u step by step, via using the following abstract setting of pseudo-parabolic equation:

$$\frac{\partial u}{\partial t} + \frac{1}{k}u = (I - k\Delta)^{-1}\left(\frac{1}{k}u + \alpha(x,t)u^p\right), \tag{2.7}$$

which can be derived by Fourier transform [9] or by reducing the pseudo-parabolic equation to a system of second-order equations [30], namely,

$$\begin{aligned}
v &= (I - k\Delta)u, \\
\frac{\partial v}{\partial t} + \frac{1}{k}v &= \frac{1}{k}u + \alpha(x,t)u^p.
\end{aligned} \tag{2.8}$$

Since the regularity of the weak solution u is not sufficient at the beginning, we start our proof from the abstract form in a weaker sense. From Definition 2.1, if u in E_0 is a weak solution, then, after a small deformation, it also satisfies

$$\begin{aligned}
\iint_{Q_\omega}\left(\frac{\partial u}{\partial t}\varphi + k\frac{\partial \nabla u}{\partial t}\nabla\varphi\right)\mathrm{d}x\,\mathrm{d}t = -\frac{1}{k}\iint_{Q_\omega}(u\varphi + k\nabla u\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t& \\
+ \iint_{Q_\omega}\left(\frac{1}{k}u + \alpha(x,t)|u|^p\right)\varphi\,\mathrm{d}x\,\mathrm{d}t.&
\end{aligned} \tag{2.9}$$

As what Showalter et al. have done in [1, 31], by using the Lax-Milgram theorem on bounded positive-definite bilinear forms in Hilbert space, we obtain the corresponding Friedrichs extensions of $I - k\Delta$, denoted by M_0, with domain $D(M_0)$ dense in $\overset{\circ}{W}{}_2^{1,0}(Q_\omega)$, satisfying the identity

$$\iint_{Q_\omega} (\phi\varphi + k\nabla\phi\nabla\varphi)\mathrm{d}x\,\mathrm{d}t = \left(M_0\phi, \varphi\right)_{L^2(Q_\omega)}, \tag{2.10}$$

whenever $\phi \in D(M_0)$ and $\varphi \in \overset{\circ}{W}{}_2^{1,0}(Q_\omega)$. The range of M_0 is all of $L^2(Q_\omega)$, and M_0 has an inverse which is a bounded mapping of $L^2(Q_\omega)$ into $\overset{\circ}{W}{}_2^{1,0}(Q_\omega)$. Then the weak solution u in E_0 is just the weak solution of the following equation:

$$\frac{\partial u}{\partial t} = -\frac{1}{k}u + M_0^{-1}\left(\frac{1}{k}u + \alpha(x,t)|u|^p\right). \tag{2.11}$$

We can also relate the extended operator M_0 to the operator M_1 which are just the extension of $I - k\Delta$ to the domain $\overset{\circ}{W}{}_2^{1,0}(Q_\omega) \cap W_2^{2,0}(Q_\omega)$ in the sense of generalized derivatives. M_1 has a continuous inverse operator from $L^2(Q_\omega)$ to $\overset{\circ}{W}{}_2^{1,0}(Q_\omega) \cap W_2^{2,0}(Q_\omega)$. Thus, from

$$\frac{\partial u}{\partial t} = -\frac{1}{k}u + M_1^{-1}\left(\frac{1}{k}u + \alpha(x,t)|u|^p\right) \tag{2.12}$$

and $u \in L^\infty(Q_\omega)$, there hold

$$\frac{\partial u}{\partial t} \in L^2(Q_\omega), \qquad \frac{\partial u}{\partial t} + \frac{1}{k}u \in \overset{\circ}{W}{}_2^{1,0}(Q_\omega) \cap W_2^{2,0}(Q_\omega). \tag{2.13}$$

Multiplying $e^{t/k}$ on both sides of (2.12), we get

$$\frac{\partial}{\partial t}\left(e^{t/k}u\right) = e^{t/k}M_1^{-1}\left(\frac{1}{k}u + \alpha(x,t)|u|^p\right). \tag{2.14}$$

For any $t \in [0,\omega]$, integrating the above equation in $[t, t+\omega]$ and using the periodicity of u yield

$$u(x,t) = \left(e^{(t+\omega)/k} - e^{t/k}\right)^{-1}\int_t^{t+\omega} e^{s/k}M_1^{-1}\left(\frac{1}{k}u(x,s) + \alpha(x,t)|u|^p(x,s)\right)\mathrm{d}s, \tag{2.15}$$

which with (2.13) imply that

$$u \in \overset{\circ}{W}{}_2^{1,0}(Q_\omega) \cap W_2^{2,1}(Q_\omega), \qquad \frac{\partial u}{\partial t} \in \overset{\circ}{W}{}_2^{1,0}(Q_\omega) \cap W_2^{2,0}(Q_\omega). \tag{2.16}$$

Thus, the weak solution u is just the strong solution which satisfies (2.2) almost everywhere in Q_ω. Furthermore, following the discussion in [32, 33], we can introduce a linear operator $M_r : W_r^{2,0}(Q_\omega) \cap \{\phi(x,t) = 0 \text{ for any } x \in \Omega\} \to L^r(Q_\omega)$ by $M_r\phi := (I - k\Delta)\phi$, whose inverse $M_r^{-1} : L^r(Q_\omega) \to W_r^{2,0}(Q_\omega) \cap \{\phi(x,t) = 0 \text{ for any } x \in \Omega\}$ is continuous. Then, we have

$$\frac{\partial u}{\partial t} = -\frac{1}{k}u + M_r^{-1}\left(\frac{1}{k}u + \alpha(x,t)|u|^p\right). \tag{2.17}$$

Similar to the above discussion, we can deduce that

$$u \in W_r^{2,1}(Q_\omega), \quad \frac{\partial u}{\partial t} \in W_r^{2,0}(Q_\omega), \quad \text{for any } 1 \le r < \infty. \tag{2.18}$$

From the Isotropic Embedding Theorem [34], we know that

$$u \in C^{\alpha,\alpha/2}\left(\overline{Q}_\omega\right), \quad 0 < \alpha < 2 - \frac{N+2}{r} \text{ with } r > \frac{N+2}{2}. \tag{2.19}$$

As in [35, 36], $(I - k\Delta)^{-1}$ is bounded from $C^{\alpha,\alpha/2}(\overline{Q}_\omega)$ to $C^{2+\alpha,\alpha/2}(\overline{Q}_\omega)$, then u satisfies

$$\frac{\partial u}{\partial t} = -\frac{1}{k}u + (I - k\Delta)^{-1}\left(\frac{1}{k}u + \alpha(x,t)|u|^p\right). \tag{2.20}$$

In the same way, we have

$$u \in C^{2+\alpha,1+\alpha/2}\left(\overline{Q}_\omega\right), \quad \frac{\partial u}{\partial t} \in C^{2+\alpha,\alpha/2}\left(\overline{Q}_\omega\right), \tag{2.21}$$

which implies that u is the classical solution.

Once u is the classical solution, we conclude that $u \ge 0$. Inspired by the method in [24, 37], we suppose, to the contrary, that there exists a pair of points $(x_0, t_0) \in \Omega \times (0, \omega)$ such that

$$u(x_0, t_0) < 0. \tag{2.22}$$

Since u is continuous, then there exists a domain Ω_0 such that $u(x, t_0) < 0$ in Ω_0 and $u(x, t_0) = 0$ on $\partial\Omega_0$. Multiplying (2.2) by φ, which is the principle eigenfunction of $-\Delta$ in Ω_0 with homogeneous Dirichlet boundary condition, and integrating on Ω_0, we get

$$(1 + k\lambda_0)\int_{\Omega_0} \frac{\partial u}{\partial t}\varphi \, dx + \lambda_0 \int_{\Omega_0} u\varphi \, dx = \int_{\Omega_0} \alpha(x,t)|u|^p\varphi \, dx, \tag{2.23}$$

where λ_0 is the first eigenvalue. Integrating the above inequality from 0 to ω and using the periodicity of u, we have

$$\lambda_0 \int_0^\omega \int_{\Omega_0} u\varphi \, dx \, dt > 0. \tag{2.24}$$

By the mean value theorem, there exists a point $t^* \in (0,\omega)$ such that

$$\int_{\Omega_0} u(x,t^*)\varphi \,\mathrm{d}x > 0. \tag{2.25}$$

Actually (2.23) is equivalent to

$$\int_{\Omega_0} \frac{\partial e^{t\lambda_0/(1+k\lambda_0)}u}{\partial t}\varphi \,\mathrm{d}x = \frac{1}{1+k\lambda_0}\int_{\Omega_0} e^{t\lambda_0/(1+k\lambda_0)}\alpha(x,t)|u|^p\varphi \,\mathrm{d}x. \tag{2.26}$$

Integrating the above inequality from t^* to ω implies that

$$\int_{\Omega_0} e^{\omega\lambda_0/(1+k\lambda_0)}u(x,\omega)\varphi \,\mathrm{d}x > 0. \tag{2.27}$$

Recalling the periodicity of u, we see that

$$\int_{\Omega_0} u(x,0)\varphi \,\mathrm{d}x > 0. \tag{2.28}$$

Then integrating (2.26) over $(0,t)$ implies that

$$\int_{\Omega_0} e^{t\lambda_0/(1+k\lambda_0)}u(x,t)\varphi \,\mathrm{d}x > 0, \quad t \in (0,\omega) \tag{2.29}$$

which contradicts with $u(x,t_0) < 0$ in Ω_0. □

3. The Case $0 \le p < 1$

In this section, we consider the case $0 \le p < 1$, in which we will show that there exists at least one positive periodic solution.

Theorem 3.1. *Assume that* $0 \le p < 1$*. Then the problem* (1.1) *and* (1.2) *admits at least one positive periodic classical solution* u*.*

Proof. We prove the theorem by constructing monotone sequence. Just as what we have done in Section 2, we may as well consider the problem (2.2)–(2.4). First, we construct a coupled upper and lower classical solution of (2.2)–(2.4). Choose $\widetilde{R}$ to be appropriately large such that $\Omega \subset B_{\widetilde{R}/2}$. Let λ_1, $\widetilde{\lambda}_1$ be the first principle eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary value conditions on Ω and $B_{\widetilde{R}}$ of $\mathbb{R}^n$, respectively. Furthermore, we let ϕ and φ be the corresponding principle eigenfunctions normalized by $\|\phi\|_{L^\infty(\Omega)} = 1$ and $\|\psi\|_{L^\infty(B_{\widetilde{R}})} = 1$. Precisely speaking, ϕ and ψ satisfy

$$\begin{aligned}
&-\Delta\phi(x) = \lambda_1\phi, \quad x \in \Omega,\\
&\phi(x)\big|_{\partial\Omega} = 0,\\
&-\Delta\psi(x) = \widetilde{\lambda}_1\psi, \quad x \in B_{\widetilde{R}},\\
&\phi(x)\big|_{\partial B_{\widetilde{R}}} = 0.
\end{aligned} \tag{3.1}$$

It is well known (see e.g., [38]) that $\phi(x) > 0$ for $x \in \Omega$ and $\psi > 0$ for $x \in B_{\widetilde{R}}$. Therefore, there exists a constant $\gamma > 0$ such that $\psi(x) > \gamma$ for $x \in \Omega$. Set

$$\Phi = \kappa_1\phi(x), \qquad \Psi = \kappa_2\psi(x), \tag{3.2}$$

where κ_1 and κ_2 are constants which are to be determined later.

Actually, if we choose $\kappa_1 = (S/\lambda_1)^{1/(1-p)}$, then a simple calculation yields that

$$-\Delta\Phi(x) = -\kappa_1\Delta\phi = \kappa_1\lambda_1\phi \le \left(\frac{S}{\lambda_1}\right)^{1/(1-p)}\lambda_1\phi = S\left(\frac{S}{\lambda_1}\right)^{p/(1-p)}\phi^p \le \alpha(x,t)\Phi^p. \tag{3.3}$$

Then Φ is a lower positive classical periodic solution of (2.2)–(2.4). Moreover, Ψ is an upper positive classical periodic solution of (2.2)–(2.4) if and only if

$$\widetilde{\lambda}_1\Psi^{1-p} \ge \alpha(\mathrm{x},t), \tag{3.4}$$

which is ensured by

$$\kappa_2 \ge \max\left\{\frac{1}{\gamma}\left(\frac{L}{\widetilde{\lambda}_1}\right)^{1/(1-p)}, \frac{\kappa_1}{\gamma}\right\}. \tag{3.5}$$

Clearly we also have $\Psi(x) \ge \Phi(x)$.

Set $\widehat{u} = \Phi$ and $\widetilde{u} = \Psi$ be the coupled bounded lower and upper classical periodic solutions of (2.2)–(2.4). We get a function sequences $\{u_m\}_{m=0}^{\infty}$ via the following iteration process

$$\frac{\partial u_m}{\partial t} - k\frac{\partial\Delta u_m}{\partial t} = \Delta u_m + \alpha(x,t)|u_{m-1}|^p, \quad (x,t) \in Q_\omega, \tag{3.6}$$

$$u_m(x,t) = 0, \quad x \in \partial\Omega,\ t \in (\tau, \tau+\omega), \tag{3.7}$$

$$u_m(x,\tau) = u_{m-1}(x,\tau+\omega), \quad x \in \Omega \tag{3.8}$$

for $m = 1, 2, \ldots$, where $u_0 = \hat{u}$. The existence and uniqueness of classical solutions for the above problem can be proved by the method in [36] and Theorem 2.2, so u_m is well defined. Then we have that the above sequence is monotone and bounded, that is,

$$\hat{u} = u_0 \leq u_1 \leq \cdots \leq u_m \leq u_{m+1} \leq \cdots \leq \tilde{u}. \tag{3.9}$$

Since $u_0 = \hat{u}$ is the lower solution, we get

$$\begin{gathered}
\frac{\partial(u_1 - u_0)}{\partial t} - k\frac{\partial \Delta(u_1 - u_0)}{\partial t} \geq \Delta(u_1 - u_0), \quad (x,t) \in Q_\omega, \\
(u_1 - u_0)(x,t) = 0, \quad x \in \partial\Omega,\ t \in (\tau, \tau+\omega), \\
(u_1 - u_0)(x,\tau) \geq 0, \quad x \in \Omega.
\end{gathered} \tag{3.10}$$

By using the comparison principle of pseudo-parabolic equation [35, 39], we have that $u_1 \geq u_0$, and

$$\begin{gathered}
\frac{\partial u_1}{\partial t} - k\frac{\partial \Delta u_1}{\partial t} = \Delta u_1 + \alpha |u_0|^p \leq \Delta u_1 + \alpha |u_1|^p, \quad (x,t) \in Q_\omega, \\
u_1(x,t) = 0, \quad x \in \partial\Omega,\ t \in (\tau, \tau+\omega), \\
u_1(x,\tau) = u_0(x, \tau+\omega) \leq u_1(x, \tau+\omega), \quad x \in \Omega,
\end{gathered} \tag{3.11}$$

which means that u_1 is a lower periodic solution. Furthermore, for $\tilde{u} - u_1$, there also holds

$$\begin{gathered}
\frac{\partial(\tilde{u} - u_1)}{\partial t} - k\frac{\partial \Delta(\tilde{u} - u_1)}{\partial t} \geq \Delta(\tilde{u} - u_1), \quad (x,t) \in Q_\omega, \\
(\tilde{u} - u_1)(x,t) \geq 0, \quad x \in \partial\Omega,\ t \in (\tau, \tau+\omega), \\
(\tilde{u} - u_1)(x,\tau) \geq \tilde{u}(x, \tau+\omega) - \hat{u}(x, \tau+\omega) \geq 0, \quad x \in \Omega,
\end{gathered} \tag{3.12}$$

which indicates that $\tilde{u} \geq u_1$ by the comparison principle. Repeating the above procedures, there holds (3.9). Due to that u_m is monotone of m, then there exists a function u such that $u_m \to u$ in Q_ω, and $u(x,t) = u(x, t+\omega)$.

Multiplying both sides of (3.6) by u_m, integrating the result over Q_ω, and recalling (3.7), (3.8) yields

$$\iint_{Q_\omega} |\nabla u_m|^2 \, \mathrm{d}x\, \mathrm{d}t = \iint_{Q_\omega} \alpha(x,t) u_m |u_{m-1}|^p \, \mathrm{d}x\, \mathrm{d}t \leq \omega |\Omega| L \|\tilde{u}\|_\infty^{p+1}. \tag{3.13}$$

Next, multiplying both sides of (3.7) by $\partial u_m / \partial t$, we have

$$\iint_{Q_\omega} \left|\frac{\partial u_m}{\partial t}\right|^2 \mathrm{d}x\, \mathrm{d}t + k \iint_{Q_\omega} \left|\frac{\partial \nabla u_m}{\partial t}\right|^2 \mathrm{d}x\, \mathrm{d}t = \iint_{Q_\omega} \alpha(x,t) \frac{\partial u_m}{\partial t} |u_{m-1}|^p \mathrm{d}x\, \mathrm{d}t. \tag{3.14}$$

Using the Young inequality to the above equality yields

$$\iint_{Q_\omega} \left|\frac{\partial u_m}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{Q_\omega} \left|\frac{\partial \nabla u_m}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t \le \omega|\Omega|L^2\|\tilde{u}\|_\infty^{2p}. \tag{3.15}$$

Hence, when $n \to \infty$, it follows that, for any $r > 0$,

$$\begin{aligned} u_m &\longrightarrow u, \quad \text{in } L^r(Q_\omega), \\ \nabla u_m &\rightharpoonup \nabla u, \quad \text{in } L^2(Q_\omega), \\ \frac{\partial u_m}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t}, \quad \text{in } L^2(Q_\omega), \\ \frac{\partial \nabla u_m}{\partial t} &\rightharpoonup \frac{\partial \nabla u}{\partial t}, \quad \text{in } L^2(Q_\omega), \end{aligned} \tag{3.16}$$

which imply that $u \in E_0$ is the periodic solution of (2.2)–(2.4).

Furthermore, the above weak periodic solution u we find is just the positive classical periodic solution of the problem (1.1) and (1.2). From (3.9) and the convergent procedures, we have that

$$0 < \Phi(x) \le u \le \Psi(x) \le \kappa_2, \quad x \in \Omega, \tag{3.17}$$

namely, that $u \in L^\infty(Q_\omega)$. Thus, from Theorem 2.2, u is the positive classical periodic solution of (1.1) and (1.2). □

4. The Case $p > 1$

In what follows, we pay our attention to the case $p > 1$, in which we will determine an exponent p_c, such that $1 < p < p_c$ and $p \ge p_c$ are corresponding to the existence and nonexistence of nontrivial and nonnegative periodic solutions, respectively. To prove the existence of periodic solutions, we need the following lemma, which can be found in [40].

Lemma 4.1. *Let $\mathbb{R}^+ := [0, +\infty)$, and let $(E, \|\cdot\|)$ be a real Banach-space. Let $G : \mathbb{R}^+ \times E \to E$ be continuous and map-bounded subsets on relatively compact subsets. Suppose moreover that G satisfies*

(a) $G(0,0) = 0$,

(b) *there exist $R > 0$ such that*

  (i) *$u \in E$, $\|u\| \le R$ and $u = G(0,u)$ implies $u = 0$,*

  (ii) *$\deg(\mathbf{id} - G(0,\cdot), B(0,R), 0) = 0$.*

Let J denote the set of solutions to the problem

$$u = G(l, u) \tag{4.1}$$

in $\mathbb{R}^+ \times E$. *Let* $\mathcal{E}$ *denote the component (closed connected subset maximal with respect to inclusion) of* J *to which* $(0,0)$ *belongs. Then if*

$$\mathcal{E} \cap (\{0\} \times E) = \{(0,0)\}, \tag{4.2}$$

then $\mathcal{E}$ *is unbounded in* $\mathbb{R}^+ \times E$.

Define an operator G by

$$\begin{gathered} G\Big(\mathbb{R}^+, L_\omega^\infty\Big((\tau,\tau+\omega), L^{\tilde{p}}(\Omega)\Big)\Big) \longrightarrow L_\omega^\infty\Big((\tau,\tau+\omega), L^{\tilde{p}}(\Omega)\Big), \\ G(l,v) = u, \end{gathered} \tag{4.3}$$

where $(2N/(N+2))p < 1+p \leq \tilde{p} < 2N/(N-2)$ is a constant. Let u be a solution of the following problem

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \Delta \mathrm{u} + \alpha(x,t)(|v|+l)^p, \quad (x,t) \in Q_\omega, \tag{4.4}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (\tau,\tau+\omega), \tag{4.5}$$

$$u(x,\tau) = u(x,\tau+\omega), \quad x \in \Omega. \tag{4.6}$$

We aim to apply Lemma 4.1 to get the existence of nontrivial weak periodic solutions and then by lifting the regularity of the weak solutions (Theorem 2.2) to get the existence of classical solutions. For these purposes, firstly, we need to verify the compactness and continuity of the operator G.

Lemma 4.2. *When* $1 < p < (N+2)/(N-2)$ *with* $N > 2$ *or* $1 < p < \infty$ *with* $N \leq 2$, *the operator* G *is completely continuous.*

Proof. To verify the compactness of the operator G, we first need to make some a priori estimates. Multiplying (4.4) by u and integrating over Ω yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 \mathrm{d}x + \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x \leq L\int_\Omega u(|v|+l)^p \mathrm{d}x. \tag{4.7}$$

Integrating the above inequality from τ to $\tau+\omega$ and combining with the Hölder inequality and the Isotropic Embedding Theorem [34], we conclude that

$$\begin{aligned} \iint_{Q_\omega} |\nabla u|^2 \mathrm{d}x\,\mathrm{d}t &\leq L\left(\iint_{Q_\omega} |u|^{\tilde{p}/(\tilde{p}-p)} \mathrm{d}x\,\mathrm{d}t\right)^{(\tilde{p}-p)/\tilde{p}} \left(\iint_{Q_\omega} (|v|+l)^{\tilde{p}} \mathrm{d}x\,\mathrm{d}t\right)^{p/\tilde{p}} \\ &\leq C\left(\iint_{Q_\omega} |\nabla u|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \left(\iint_{Q_\omega} (|v|+l)^{\tilde{p}} \mathrm{d}x\,\mathrm{d}t\right)^{p/\tilde{p}}. \end{aligned} \tag{4.8}$$

Thus, we have

$$\iint_{Q_\omega} |u|^2 \mathrm{d}x\,\mathrm{d}t \le C_1, \qquad \iint_{Q_\omega} |\nabla u|^2 \mathrm{d}x\,\mathrm{d}t \le C_1, \tag{4.9}$$

where C_1 depends only on l, p, $\tilde{p}$, $\|\alpha\|_{L^\infty(Q_\omega)}$, Q_ω, and $\sup_{t\in(\tau,\tau+\omega)}\|v\|_{L^{\tilde{p}}(\Omega)}$. By the mean value theorem, we see that there exists a point $\hat{t} \in (\tau, \tau+\omega)$ such that

$$\int_\Omega \left(\left|u\left(x,\hat{t}\right)\right|^2 + k\left|\nabla u\left(x,\hat{t}\right)\right|^2\right)\mathrm{d}x \le C_1\omega. \tag{4.10}$$

Integrating (4.7) from $\hat{t}$ to t gives

$$\int_\Omega \left(|u|^2 + k|\nabla u|^2\right)\mathrm{d}x \le C_2, \quad t \in \left[\hat{t}, \tau+\omega\right]. \tag{4.11}$$

Noticing the periodicity of u, we arrive at

$$\int_\Omega \left(|u(x,\tau)|^2 + k|\nabla u(x,\tau)|^2\right)\mathrm{d}x \le C_2, \tag{4.12}$$

from which it is easy to obtain that

$$\sup_{t\in[\tau,\tau+\omega]} \int_\Omega \left(|u|^2 + k|\nabla u|^2\right)(x,t)\mathrm{d}x \le C_2, \tag{4.13}$$

where C_2 depends only on $l, p, \tilde{p}, k$, $\|\alpha\|_{L^\infty(Q_\omega)}$, Q_ω, and $\sup_{t\in(\tau,\tau+\omega)}\|v\|_{L^{\tilde{p}(\Omega)}}$. Using the Isotropic Embedding Theorem [34] yields

$$\int_\Omega |u|^r \mathrm{d}x \le C_3, \tag{4.14}$$

where

$$r = \begin{cases} 1 \le r \le \dfrac{2N}{N-2}, & N > 2, \\ 1 \le r < +\infty, & N \le 2, \end{cases} \tag{4.15}$$

and C_3 depends on l, p, $\tilde{p}$, k, N, $\|\alpha\|_{L^\infty(Q_\omega)}$, Q_ω, and $\sup_{t\in(\tau,\tau+\omega)}\|v\|_{L^{\tilde{p}(\Omega)}}$. Multiplying (4.4) by $\partial u/\partial t$ and integrating over Ω yield

$$\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 \mathrm{d}x + k\int_\Omega \left|\frac{\partial \nabla u}{\partial t}\right|^2 \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \mathrm{d}x = \int_\Omega \frac{\partial u}{\partial t}\alpha(x,t)(|v|+l)^p \mathrm{d}x. \tag{4.16}$$

Integrating the above equality from τ to $\tau+\omega$ and using the Hölder inequality and the Isotropic Embedding Theorem [34], we get

$$
\begin{aligned}
&\iint_{Q_\omega}\left(\left|\frac{\partial u}{\partial t}\right|^2+k\left|\frac{\partial \nabla u}{\partial t}\right|^2\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad\le L\left(\iint_{Q_\omega}\left|\frac{\partial u}{\partial t}\right|^{\tilde{p}/(\tilde{p}-p)}\mathrm{d}x\,\mathrm{d}t\right)^{(\tilde{p}-p)/\tilde{p}}\left(\iint_{Q_\omega}(|v|+l)^{\tilde{p}}\mathrm{d}x\,\mathrm{d}t\right)^{p/\tilde{p}}\\
&\quad\le C\left(\iint_{Q_\omega}\left|\frac{\partial \nabla u}{\partial t}\right|^2\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\iint_{Q_\omega}(|v|+l)^{\tilde{p}}\mathrm{d}x\,\mathrm{d}t\right)^{p/\tilde{p}}.
\end{aligned}
\tag{4.17}
$$

Then one has

$$
\iint_{Q_\omega}\left(\left|\frac{\partial u}{\partial t}\right|^2+k\left|\frac{\partial \nabla u}{\partial t}\right|^2\right)\mathrm{d}x\,\mathrm{d}t\le C_4,
\tag{4.18}
$$

where C_4 depends on l, p, $\tilde{p}$, $\|\alpha\|_{L^\infty(Q_\omega)}$, Q_ω, and $\sup_{t\in(\tau,\tau+\omega)}\|v\|_{L^{\tilde{p}}(\Omega)}$. Moreover, by means of (4.13), (4.14), and (4.18), we obtain the compactness of the operator G, while, for the continuity of G, it is easy to obtain just by a simple and cumbersome real analysis process, so we omit it. The proof is complete. □

By using the above lemmas, we obtain the following results.

Theorem 4.3. *Assume $\alpha(x,t)\in C^1(\mathbb{R}^N,\mathbb{R})$. If Ω is a convex domain and*

$$
1<p\begin{cases}<\dfrac{N+2}{N-2}, & \text{for } N>2,\\ <+\infty, & \text{for } N\le 2,\end{cases}
\tag{4.19}
$$

then the problem (1.1) *and* (1.2) *admits at least one nontrivial nonnegative classical periodic solution.*

Proof. We will complete the proof by using Lemma 4.1. Recalling the definition of the operator G and Lemma 4.2, we see that the operator G is completely continuous. In what follows, we first need to check the condition (a) in Lemma 4.1. Let $u=G(0,0)$ that is, u is a solution of the following problem:

$$
\frac{\partial u}{\partial t}-k\frac{\partial \Delta u}{\partial t}=\Delta u,\qquad (x,t)\in Q_\omega,
\tag{4.20}
$$

$$
u(x,t)=0,\quad (x,t)\in\partial\Omega\times(\tau,\tau+\omega),
\tag{4.21}
$$

$$
u(x,\tau)=u(x,\tau+\omega),\quad x\in\Omega.
\tag{4.22}
$$

Clearly, the above problem admits only zero solution. In fact, multiplying the first equation of (4.20) by u and integrating over Q_ω yield

$$\iint_{Q_\omega} |\nabla u|^2 \mathrm{d}x\,\mathrm{d}t = 0. \tag{4.23}$$

Recalling the Poincaré inequality, we see that

$$\iint_{Q_\omega} u^2 \mathrm{d}x\,\mathrm{d}t \le C \iint_{Q_\omega} |\nabla u|^2 \mathrm{d}x\,\mathrm{d}t = 0, \tag{4.24}$$

which implies that $u = 0$ a.e. in Q_ω.

Secondly, we will show that there exists an $R > 0$ such that if $u = G(0, u)$ and

$$\sup_t \|u(\cdot, t)\|_{L^{\tilde{p}}(\Omega)} < R, \tag{4.25}$$

then $u \equiv 0$. Taking $l = 0$, replacing v by u in (4.4), and then multiplying the equation by u on both sides and integrating over Ω yield

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 \mathrm{d}x + \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x \\ &\quad \le L\int_\Omega |u|^{p+1}\mathrm{d}x \le L|\Omega|^{(\tilde{p}-p-1)/\tilde{p}}\left(\int_\Omega |u|^{\tilde{p}}\mathrm{d}x\right)^{(p+1)/\tilde{p}}. \end{aligned} \tag{4.26}$$

By virtue of the Isotropic Embedding Theorem, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 \mathrm{d}x + \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \mathrm{d}x + \mu\left(\int_\Omega |u|^{\tilde{p}}\mathrm{d}x\right)^{2/\tilde{p}} \le L|\Omega|^{(\tilde{p}-p-1)/\tilde{p}}\left(\int_\Omega |u|^{\tilde{p}}\mathrm{d}x\right)^{(p+1)/\tilde{p}}, \tag{4.27}$$

where μ is the constant in the Isotropic Embedding Theorem; that is,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 \mathrm{d}x + \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \mathrm{d}x \\ &\quad \le \left(L|\Omega|^{(\tilde{p}-p-1)/\tilde{p}} - \mu\left(\int_\Omega |u|^{\tilde{p}}\mathrm{d}x\right)^{(1-p)/\tilde{p}}\right)\left(\int_\Omega |u|^{\tilde{p}}\mathrm{d}x\right)^{(p+1)/\tilde{p}}. \end{aligned} \tag{4.28}$$

Thus, if

$$\sup_t \int_\Omega |u|^{\tilde{p}} \le R_0, \tag{4.29}$$

where

$$R_0 = \left(\frac{\mu}{2L}\right)^{\tilde{p}/(p-1)} |\Omega|^{-(\tilde{p}-p-1)/(p-1)}, \tag{4.30}$$

then we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 \mathrm{d}x + \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \mathrm{d}x \le -L|\Omega|^{(\tilde{p}-p-1)/\tilde{p}}\int_\Omega |u|^{p+1}\mathrm{d}x, \tag{4.31}$$

which means $u = 0$ a.e. in Q_ω.

Next, we check the condition (b) in Lemma 4.1, namely, there exists an $R < R_0$, such that

$$\deg(\mathbf{id} - G(0,\cdot), B(0,R), 0) = 1. \tag{4.32}$$

Consider the following problem:

$$\begin{gathered}
\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \Delta u + \sigma\alpha(x,t)|v|^p, \quad (x,t) \in Q_\omega, \\
u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (\tau, \tau+\omega), \\
u(x,\tau) = u(x,\tau+\omega), \quad x \in \Omega,
\end{gathered} \tag{4.33}$$

where $\sigma \in [0,1]$. Construct a homotopic mapping

$$\begin{gathered}
T : [0,1] \times L^\infty_\omega\left((\tau,\tau+\omega), L^{\tilde{p}}(\Omega)\right) \longrightarrow L^\infty_\omega\left((\tau,\tau+\omega), L^{\tilde{p}}(\Omega)\right), \\
T(\sigma, v) = u.
\end{gathered} \tag{4.34}$$

Similar to Lemma 4.2, we see that T is completely continuous. Assume that

$$\sup_{t\in(\tau,\tau+\omega)} \|v(\cdot,t)\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} \le R, \tag{4.35}$$

where $R \leq R_0$ is to be determined. Multiplying the first equation of the above problem by u and integrating over Ω yield

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|u|^2\mathrm{d}x + \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|\nabla u|^2\mathrm{d}x + \int_{\Omega}|\nabla u|^2\mathrm{d}x \\
&\quad \leq L\left(\int_{\Omega}|u|^{\tilde{p}/(\tilde{p}-p)}\mathrm{d}x\right)^{(\tilde{p}-p)/\tilde{p}}\left(\int_{\Omega}|v|^{\tilde{p}}\mathrm{d}x\right)^{p/\tilde{p}} \\
&\quad \leq L\left(\int_{\Omega}|\nabla u|^2\mathrm{d}x\right)^{1/2}\left(\int_{\Omega}|v|^{\tilde{p}}\mathrm{d}x\right)^{p/\tilde{p}} \\
&\quad \leq C\left(\int_{\Omega}|\nabla u|^2\mathrm{d}x\right)^{1/2} R^{p/\tilde{p}}.
\end{aligned} \tag{4.36}$$

Integrating from τ to $\tau+\omega$ gives

$$\iint_{Q_\omega}|\nabla u|^2\mathrm{d}x \leq CR^{2p/\tilde{p}}. \tag{4.37}$$

Then, we further have

$$\iint_{Q_\omega}|u|^2\mathrm{d}x \leq CR^{2p/\tilde{p}}. \tag{4.38}$$

By means of the integral mean value theorem, we see that there exists a $t_\sigma \in [\tau, \tau+\omega]$ such that

$$\int_{\Omega}\left(|u(x,t_\sigma)|^2 + |\nabla u(x,t_\sigma)|^2\right)\mathrm{d}x \leq CR^{2p/\tilde{p}}. \tag{4.39}$$

By the periodicity of u and a similar process as Lemma 4.2, we obtain that

$$\sup_{\in(\tau,\tau+\omega)}\int_{\Omega}\left(|u|^2 + k|\nabla u|^2\right)\mathrm{d}x \leq CR^{2p/\tilde{p}}. \tag{4.40}$$

Using the Isotropic Embedding Theorem gives

$$\sup_{t\in(\tau,\tau+\omega)}\int_{\Omega}|u(x,t)|^{\tilde{p}}\mathrm{d}x \leq CR^{p} < R. \tag{4.41}$$

If R with $R \leq R_0$ is appropriately small, such that $p > 1$, therefore,

$$\deg(\mathrm{id} - T(1,\cdot), B(0,R), 0) = \deg(\mathrm{id} - T(0,\cdot), B(0,R), 0) = 1, \tag{4.42}$$

which means that

$$\deg(\mathrm{id} - G(0,\cdot), B(0,R), 0) = 1. \tag{4.43}$$

To show that the problem (1.1) and (1.2) admits at least one nontrivial periodic solution, it remains to check the boundedness of the set $\mathcal{E}$ in Lemma 4.1. Otherwise, the set of solutions to the problem $u = G(l,u)$ is unbounded. Therefore, there exist l_n, u_n such that $u_n = G(l_n, u_n)$ and

$$l_n + \sup_{t\in(\tau,\tau+\omega)} \|u_n(\cdot,t)\|_{L^\infty(\Omega)} \longrightarrow \infty, \tag{4.44}$$

which implies that

$$l_n + \|u_n\|_{L^\infty(Q_\omega)} \longrightarrow \infty. \tag{4.45}$$

If this were true, then we would have

$$\frac{l_n}{\|u_n\|_{L^\infty(Q_\omega)}} \longrightarrow 0. \tag{4.46}$$

Suppose the contrary, and note that if l_n is bounded, then $\|u_n\|_{L^\infty(Q_\omega)} \to \infty$, which means (4.46). Thus, without loss of generality, we may assume that $0 < l_n \to +\infty$. Making change of variable

$$v = \frac{u_n}{l_n}, \tag{4.47}$$

we have

$$\frac{\partial v_n}{\partial t} - k\frac{\partial \Delta v_n}{\partial t} - \Delta v_n = \alpha(x,t) l_n^{p-1}(|v_n| + 1)^p. \tag{4.48}$$

If $\|v_n\|_{L^\infty(Q_\omega)}$ are bounded uniformly, that is, there is a constant $C > 0$ such that $\|v_n\|_{L^\infty(Q_\omega)} < C$, then, for any $\varphi \in C_T^1(Q_\omega)$ with $\varphi|_{\partial\Omega} = 0$, we have

$$\begin{aligned} &\iint_{Q_\omega} \frac{\partial v_n}{\partial t}\varphi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_\omega} k\frac{\partial \nabla v_n}{\partial t}\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_\omega} \nabla v_n \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \iint_{Q_\omega} \alpha(x,t) l_n^{p-1}(|v_n| + 1)^p \varphi \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.49}$$

Noticing the density of $C_T^1(Q_\omega)$ in $L_T^2((\tau,\tau+\omega), W^{1,2}(\Omega))$, then it is sound to take $\varphi = v_n$ thus, we have

$$\iint_{Q_\omega} |\nabla v_n|^2 \mathrm{d}x\,\mathrm{d}t \le CLl_n^{p-1}. \tag{4.50}$$

In addition, for any $0 \le \varphi(x) \in C_0^1(\Omega)$, we also have

$$\begin{aligned} l_n^{p-1}\iint_{Q_\omega} \alpha(x,t)\varphi\,\mathrm{d}x\,\mathrm{d}t &\le l_n^{p-1}\iint_{Q_\omega} \alpha(x,t)(|v_n|+1)^p\varphi\,\mathrm{d}x\,\mathrm{d}t \\ &\le \left(\iint_{Q_\omega} |\nabla v_n|^2\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\iint_{Q_\omega} |\nabla\varphi|^2\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\ &\le \left(\int\int_{Q_\omega} |\nabla\varphi|^2\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(CLl_n^{p-1}\right)^{1/2}, \end{aligned} \tag{4.51}$$

that is,

$$l_n^{(p-1)/2}\iint_{Q_\omega} \alpha(x,t)\varphi\,\mathrm{d}x\,\mathrm{d}t \le \left(\iint_{Q_\omega} |\nabla\varphi|^2\mathrm{d}x\,\mathrm{d}t\right)^{1/2}. \tag{4.52}$$

Clearly, it is a contradiction since $l_n \to \infty$. Therefore, (4.46) holds, which also implies that $\|u_n\|_{L^\infty(Q_\omega)} \to \infty$. Let $\rho_n = \|u_n\|_{L^\infty(Q_\omega)} = u_n(x_n,t_n) \to \infty$. By the convexity of Ω, we see that there exists a $\delta_0 > 0$ such that $\operatorname{dist}(x_n,\partial\Omega) \ge \delta_0$; see, for example, [41, 42]. Then, there exists a subsequence, and for simplicity, we still denote it by x_n such that $x_n \to x_0$, $t_n \to t_0$ with $\operatorname{dis}(x_n,\partial\Omega) \ge \delta_0$. Let

$$\omega_{nj}(y,s) = \rho_{\mathrm{n}}^{-1}u_n\left(\rho_n^{-(p-1)/2}y + x_0, t_j + js\right), \qquad \tilde{\alpha}_{nj} = \alpha\left(\rho_n^{-(p-1)/2}y + x_0, t_j + js\right), \tag{4.53}$$

and let

$$\Omega_n = \left\{y; y = \rho_n^{(p-1)/2}(x-x_0)\ \text{for}\ x \in \Omega\right\}, \qquad Q_{nj} = \Omega_n \times \left(\frac{\tau - t_j}{j}, \frac{\tau+\omega-t_j}{j}\right). \tag{4.54}$$

Then, ω_{nj} with $\|\omega_{nj}\|_{L^\infty(Q_{n\omega})} = 1$ on Q_{nj} satisfies

$$\rho_n^{1-p}\frac{\partial\omega_{nj}}{\partial s} - k\frac{\partial\Delta\omega_{nj}}{\partial s} - j\Delta\omega_{nj} = j\tilde{\alpha}(y,s)\left(\left|\omega_{nj}\right| + l_n\rho_n^{-1}\right)^p. \tag{4.55}$$

Similar to the proof of Theorem 2.2, we can deduce that $\omega_{nj} \ge 0$ which admits throwing off the symbol of absolute value of $|\omega_{nj}|$. Therefore, for any $\phi(y,s) \in C_T^1(Q_{nj})$ with $\phi = 0$ on $\partial\Omega_n$, we have

$$\begin{aligned} &\iint_{Q_{nj}} \rho_n^{1-p}\frac{\partial\omega_{nj}}{\partial s}\phi\,\mathrm{d}y\,\mathrm{d}s + \iint_{Q_{nj}} k\frac{\partial\nabla\omega_{nj}}{\partial s}\nabla\phi\,\mathrm{d}y\,\mathrm{d}s + j\iint_{Q_{nj}} \nabla\omega_{nj}\nabla\phi\,\mathrm{d}y\,\mathrm{d}s \\ &\quad = j\iint_{Q_{nj}} \tilde{\alpha}(y,s)\left(\omega_{nj} + l_n\rho_n^{-1}\right)^p\phi\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{4.56}$$

Taking $\phi = \omega_{nj}$, we have

$$j\iint_{Q_{nj}} \left|\nabla\omega_{nj}\right|^2 \mathrm{d}y\,\mathrm{d}s = j\iint_{Q_{nj}} \tilde{\alpha}(y,s)\left(\omega_{nj} + l_n\rho_n^{-1}\right)^p \omega_{nj}\mathrm{d}y\,\mathrm{d}s \le C|\Omega_n|, \tag{4.57}$$

which means that there exists $\sigma_j \in [(\tau - t_j)/j, (\tau + \omega - t_j)/j)$ such that

$$\int_{\Omega} \left|\nabla\omega_{nj}(y,\sigma_j)\right|^2 \mathrm{d}y \le C|\Omega_n|. \tag{4.58}$$

For any $s > \sigma_j$, taking $\phi = \chi(\sigma_j, s)(\partial\omega_{nj}/\partial s)$ in (4.56) yields

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_n} \left|\nabla\omega_{nj}(y,s)\right|^2 \mathrm{d}y \le \frac{1}{2}\int_{\Omega_n} &\left|\nabla\omega_{nj}(y,\sigma_j)\right|^2 \mathrm{d}y \\
&+ \frac{1}{p+1}\int_{\Omega_n} \tilde{\alpha}_{nj}(y,\sigma_j)\left(\omega_{nj}(y,\sigma_j) + l_n\rho_n^{-1}\right)^{p+1} \mathrm{d}y \\
&- \frac{1}{p+1}\int_{\Omega_n} \tilde{\alpha}_{nj}(y,s)\left(\omega_{nj}(y,s) + l_n\rho_n^{-1}\right)^{p+1} \mathrm{d}y \\
&- \frac{j}{p+1}\int_{\sigma_j}^{s}\int_{\Omega_n} \frac{\partial\tilde{\alpha}_{nj}}{\partial s}(y,s)\left(\omega_{nj} + l_n\rho_n^{-1}\right)^{p+1} \mathrm{d}y\,\mathrm{d}s \\
\le C_5|\Omega_n|,&
\end{aligned} \tag{4.59}$$

where C_5 is a constant independent of j, n, Ω_n. By the periodicity of ω_{nj}, we further have

$$\int_{\Omega_n} \left|\nabla\omega_{nj}\left(y, \frac{\tau - t_j}{j}\right)\right|^2 \mathrm{d}y \le C_5|\Omega_n|. \tag{4.60}$$

Repeating the process above, we finally obtain that for any $s \in ((\tau - t_j)/j, \sigma_j)$,

$$\int_{\Omega_n} \left|\nabla\omega_{nj}(y,s)\right|^2 \mathrm{d}y \le C_6|\Omega_n|. \tag{4.61}$$

Summing up, we finally obtain that

$$\sup_s \int_{\Omega_n} \left|\nabla\omega_{nj}(y,s)\right|^2 \mathrm{d}y \le \widehat{C}|\Omega_n|. \tag{4.62}$$

In addition, we note that, for any $\varphi \in C_0^1(\Omega_n)$, we have

$$j\iint_{Q_{nj}} \nabla\omega_{nj}\nabla\varphi\,\mathrm{d}y\,\mathrm{d}s = j\iint_{Q_{nj}} \tilde{\alpha}_{nj}(y,s)\left(\omega_{nj} + l_n\rho_n^{-1}\right)^p \varphi\,\mathrm{d}y\,\mathrm{d}s. \tag{4.63}$$

By Lebesgue differential theorem, there exists $s_j \in ((\tau - t_j)/j, (\tau + \omega - t_j)/j)$ such that

$$\int_{\Omega_n} \nabla \omega_{nj}(y, s_j) \nabla \varphi \, dy = \int_{\Omega_n} \tilde{\alpha}_{nj}(y, s_j) \left(\omega_{nj}(y, s_j) + l_n \rho_n^{-1} \right)^p \varphi \, dy. \tag{4.64}$$

Then, there exists a function $\omega_n \in W^{1,2}(\Omega_n)$ with $\|\omega_n\|_{L^\infty} = 1$ such that as $j \to \infty$ (passing to a subsequence if necessary)

$$\begin{aligned} &\nabla \omega_{nj} \rightharpoonup \nabla \omega_n \quad \text{in } L^2(\Omega_n); \qquad \omega_{nj} \longrightarrow \omega_n \quad \text{in } L^r(\Omega_n) \quad \text{for } 1 \le r \le \frac{2N}{N-2}, \\ &\tilde{\alpha}_{nj}(y, s_j) \longrightarrow \tilde{\alpha}_n(y) \quad \text{uniformly}, \qquad \tilde{\alpha}_n(y) \in C^\beta(\Omega_n) \quad \text{for some } 0 < \beta < 1, \end{aligned} \tag{4.65}$$

we obtain that

$$\int_{\Omega_n} \nabla \omega_n \nabla \varphi \, dy = \int_{\Omega_n} \tilde{\alpha}(y) \left(\omega_n + l_n \rho_n^{-1} \right)^p dy. \tag{4.66}$$

Take $\varphi = \omega_n \eta^4(x)$, where

$$\eta = \begin{cases} 1, & x \in B_R(0), \\ 0, & x \in B_{2R}(0), \end{cases} \tag{4.67}$$

with $0 \le \eta \le 1$ is sufficiently smooth and $|\nabla \eta| \le C/R$. Then, for sufficiently large n, we have $B_{2R} \subset \Omega_n$ and

$$\begin{aligned} \int_{B_{2R}} \eta^4 |\nabla \omega_n|^2 dy &= -\int_{B_{2R}} 4\eta^3 \omega_n \nabla \omega_n \nabla \eta \, dy + \int_{B_{2R}} \tilde{\alpha}_n(y) \left(\omega_n + l_n \rho_n^{-1} \right)^p \omega \eta^4 dy \\ &\le \frac{1}{2} \int_{B_{2R}} \eta^4 |\nabla \omega_n|^2 dy + \frac{C}{R^2} \int_{B_{2R}} \eta^2 \omega_n^2 dy + L \int_{B_{2R}} \left(\omega_n + l_n \rho_n^{-1} \right)^p \omega \eta^4 dy \\ &\le \frac{1}{2} \int_{B_{2R}} \eta^4 |\nabla \omega_n|^2 dy + C R^{N-2} + C R^N. \end{aligned} \tag{4.68}$$

Then, for sufficiently large $R > 0$, we have

$$\int_{B_R} |\nabla \omega_n|^2 dy \le C R^N, \tag{4.69}$$

where C is independent of n and R. Then, there exists a function $\hat{\omega} \in W^{1,2}_{\text{loc}}(\mathbb{R}^N)$ such that, passing to a subsequence if necessary, as $n \to \infty$

$$\begin{aligned} \tilde{\alpha}_n(y) \longrightarrow \tilde{\alpha}(y) \quad \text{uniformly}, \quad \nabla \omega_n \rightharpoonup \nabla \hat{\omega} \quad \text{in } L^2(B_R), \quad \omega_n \longrightarrow \hat{\omega} \quad \text{in } L^r(B_R) \\ \text{for } 1 \le r \le \frac{2N}{N-2}. \end{aligned} \tag{4.70}$$

Then, we have

$$\begin{aligned} &\int_{B_R} \nabla\widehat{\omega}\nabla\varphi\,\mathrm{d}y = \int_{B_R} \widetilde{\alpha}(y)\widehat{\omega}^p\varphi\,\mathrm{d}y, \quad \text{for any } \varphi\in C_0^1(B_R), \\ &\|\widehat{\omega}\|_{L^\infty(B_R)} = 1, \qquad \widehat{\omega}\ge 0, \quad \text{for } x\in\Omega_n. \end{aligned} \tag{4.71}$$

Moreover, since $\widehat{\omega}\not\equiv 0$, we have $\widehat{\omega}(x) > 0$ for all $x\in B_R$ by the strong maximum principle [43]. Taking balls larger and larger and repeating the argument for the subsequence $\widehat{\omega}_k$ obtained at the previous step, we get a Cantor diagonal subsequence, and for simplicity, we still denote it by $\widehat{\omega}_k$ which converges in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N)$ to a function $\omega\in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N)$; namely,

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla\omega\nabla\varphi\,\mathrm{d}y = \int_{\mathbb{R}^N} \widetilde{\alpha}(y)\omega^p\varphi\,\mathrm{d}y, \quad \text{for any } \varphi\in C_0^1\left(\mathbb{R}^N\right), \\ &\|\omega\|_{L^\infty(\mathbb{R}^N)} = 1, \quad \omega > 0, \quad \text{for } x\in\mathbb{R}^N, \end{aligned} \tag{4.72}$$

which means that (4.72), is a contradiction. Indeed, for the case $1 < p < (N+2)/(N-2)$ with $N > 2$ and the case $p > 1$ with $N \le 2$, thanks to a Liouville-type theorem, Theorem II in [44], and [45, Lemma 3.6], we see that the problem (4.72), has no solution, which is a contradiction and implies that $l_n + \|u_n\|_\infty$ is bounded uniformly. By means of Lemma 4.1, we conclude that the problem (1.1) and (1.2) admits at least one nontrivial periodic solution.

Since we have prove, the boundedness of the solution, then, from Theorem 2.2, u is the nontrivial nonnegative classical periodic solution. □

In what follows, we consider the nonexistence of periodic solutions.

Theorem 4.4. *Assume $\alpha(x)\in C^1(\mathbb{R}^n)$ and $N > 2$. If $p \ge (N+2)/(N-2)$ and Ω is star shaped, then there is no nontrivial and nonnegative periodic solution.*

Proof. If $\alpha(x,t)$ is independent of t, then we deduce that the periodic solution of the problem (1.1) and (1.2) must be a steady state. In fact, multiplying (1.1) by u_t on both sides and integrating over Q_ω yield

$$\frac{1}{2}\iint_{Q_\omega} |u_t|^2\mathrm{d}x + \frac{k}{2}\iint_{Q_\omega} |\nabla u_t|^2\mathrm{d}x = 0, \tag{4.73}$$

which means that u is a steady state and satisfies the steady-state equation

$$-\Delta u = \alpha(x)u^p. \tag{4.74}$$

However, by [46, 47], if $N > 2$, $p \ge (N+2)/(N-2)$, and Ω is star shaped, then the above equation subject to the homogeneous Dirichlet boundary condition has no nontrivial and nonnegative solution. It is a contradiction, whence (1.1) and (1.2) has no nontrivial and nonnegative periodic solution. □

Remark 4.5. Here, it is worth mentioning that, for the case $p \ge (N+2)/(N-2)$, if the domain Ω is an annulus domain, then there may exist nontrivial and nonnegative periodic solution. In fact, if $\alpha(x,t) \equiv \alpha(|x|)$, then the periodic solution is a steady state; namely, it is a solution of the corresponding elliptic equation, while, from the results in [48], there exists radial solutions for this case in an annulus domain.

5. The Singular Case $p = 1$

In this section, we consider the case $p = 1$, in which the problem is written as

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \Delta u + \alpha(x,t)u, \quad x \in \Omega \times \mathbb{R}, \tag{5.1}$$

subject to

$$u(x,t)|_{\partial\Omega} = 0. \tag{5.2}$$

We are going to show the specialty of this case. It is quite different from other cases, in which positive periodic solutions definitely exist or definitely not exist. It will be shown that, for small $\alpha(x,t)$, any solution of the initial boundary value problem decays to zero as time goes to infinity, while, for large $\alpha(x,t)$, all positive solutions blow up at finite time. These imply that there is no positive periodic solution. However, when $\alpha(x,t)$ is independent of t, there may exist positive periodic solution. Here we consider the problem (5.1), (5.2) with the initial value condition

$$u(x,0) = u_0(x), \tag{5.3}$$

where $u_0(x) \geq 0$ for $x \in \Omega$ and satisfies some compatibility conditions.

We have the following theorem.

Theorem 5.1. *Assume that $p = 1$. Let λ_1 be the first eigenvalue of $-\Delta$ with homogeneous boundary condition on Ω.*

(1) *If $L = \sup \alpha(x,t) < \lambda_1$, then all the solutions of the problem* (5.1)–(5.3) *go to 0 as $t \to \infty$, which means that* (1.1) *and* (1.2) *admits no nontrivial nonnegative periodic solutions.*

(2) *If $S = \inf \alpha(x,t) > \lambda_1$, then all the positive solutions of the problem* (5.1)–(5.3) *go to ∞ as $t \to \infty$; that is,* (1.1) *and* (1.2) *admits no positive periodic solutions.*

(3) *If $\alpha(x,t)$ is independent of t and $\alpha(x) \geq \lambda_1$, then there exists a nontrivial and nonnegative periodic solution for* (1.1) *and* (1.2)*. Moreover, if $\alpha \equiv \lambda_1$, then there is a positive periodic solution for* (1.1) *and* (1.2).

Proof. Firstly, we consider the case $L < \lambda_1$. We note that there exist $\tilde{\lambda}$ with $L < \tilde{\lambda} < \lambda_1$ and domain $\tilde{\Omega}$ with $\Omega \subset \tilde{\Omega}$ such that $\tilde{\lambda}$ is the first eigenvalue of $-\Delta$ with homogeneous boundary condition on $\tilde{\Omega}$. Correspondingly, $\tilde{\varphi}$ is the first eigenfunction with $\|\tilde{\varphi}\|_\infty = 1$. A simple calculation yields that $K\widetilde{\varphi}$ is an upper solution of the problem (5.1)–(5.3) for appropriately large $K > 0$. Then, we have $u \leq K\widetilde{\varphi}$. Let $w(x,t)$ be the solution of the following problem:

$$\begin{aligned} \frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} &= \Delta u + \alpha(x,t)u, \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) &= \left(K\widetilde{\varphi}\right)e^{-t}, \quad x \in \partial\Omega, \\ u(x,0) &= K\widetilde{\varphi}(x), \quad x \in \Omega. \end{aligned} \tag{5.4}$$

We conclude that $w(x,t)$ is decreasing on t, and $u(x,t) \leq w(x,t)$. Thus, there exists a function w such that

$$w(x) = \lim_{t\to\infty} w(x,t). \tag{5.5}$$

It follows that $w(x)$ is a steady state of the first equation of (5.4) with homogeneous Dirichlet boundary condition. Clearly, we have $w(x) = 0$ since $\alpha(x,t) < \lambda_1$, which means that $u(x,t)$ goes to 0 uniformly as $t \to \infty$.

Next, let us consider the case $S > \lambda_1$. Take $\widehat{\Omega} \subset \Omega$ such that the first eigenvalue $\widehat{\lambda}$ of $-\Delta$ with homogeneous boundary condition on $\widehat{\Omega}$ satisfies $\lambda_1 < \widehat{\lambda} < S$. Let $\widehat{\phi}$ be the first eigenfunction of $\widehat{\Omega}$. Set

$$\underline{w} = g(t)\widehat{\phi}, \tag{5.6}$$

where $g(t)$ satisfies

$$\begin{aligned} &g'(t) = \frac{S - \widehat{\lambda}}{1 + k\widehat{\lambda}}g(t), \quad t > 0, \\ &g(t) > 0, \quad t > 0, \\ &g(0) = 0. \end{aligned} \tag{5.7}$$

After a direct calculation, we see that $\underline{w}$ is a lower solution of (5.1)–(5.3) for any nontrivial and nonnegative initial value on $\widehat{\Omega}$. Furthermore,

$$\underline{u} = \begin{cases} \underline{w}, & x \in \widehat{\Omega}, \\ 0, & x \in \Omega/\widehat{\Omega} \end{cases} \tag{5.8}$$

is a lower solution of (5.1)–(5.3) for any nontrivial and nonnegative initial value on Ω, while by comparison we see that $u(x,t) \geq \underline{u}$ on Ω. Thus, we have that $\|u\|_\infty$ goes to infinity as $t \to \infty$ since $g(t) \to \infty$ as $t \to \infty$.

However if $\alpha(x,t)$ is independent of t and $\alpha(x) \geq \lambda_1$, then there exists a domain $\Omega' \subset \Omega$, such that one can find a function $\phi > 0$ on Ω' which satisfies

$$-\Delta\phi = \alpha(x)\phi, \quad x \in \Omega,$$
$$\phi\big|_{\partial\Omega'} = 0. \tag{5.9}$$

Then, ϕ is the periodic solution. Specially, when $\alpha(x) = \lambda_1$, then clearly the first eigenfunction is a periodic solution of (1.1) and (1.2). □

Remark 5.2. For $p = 1$ and spatially constant $\alpha(t)$ with $\alpha(t) = \alpha(t+\omega)$, as a fairly obvious observation, which is with no-flux boundary conditions, a spatially constant solution of the following ODE:

$$u'(t) = \alpha(t)u(t),$$
$$u(t) = u(t+\omega), \tag{5.10}$$

is a periodic solution to our problem. However, a simple calculation indicates that the above problem has periodic solutions provided that $\int_0^{\omega} \alpha(s)\mathrm{d}s = 0$, which does not coincide with our assumption $\alpha(x,t) > 0$.

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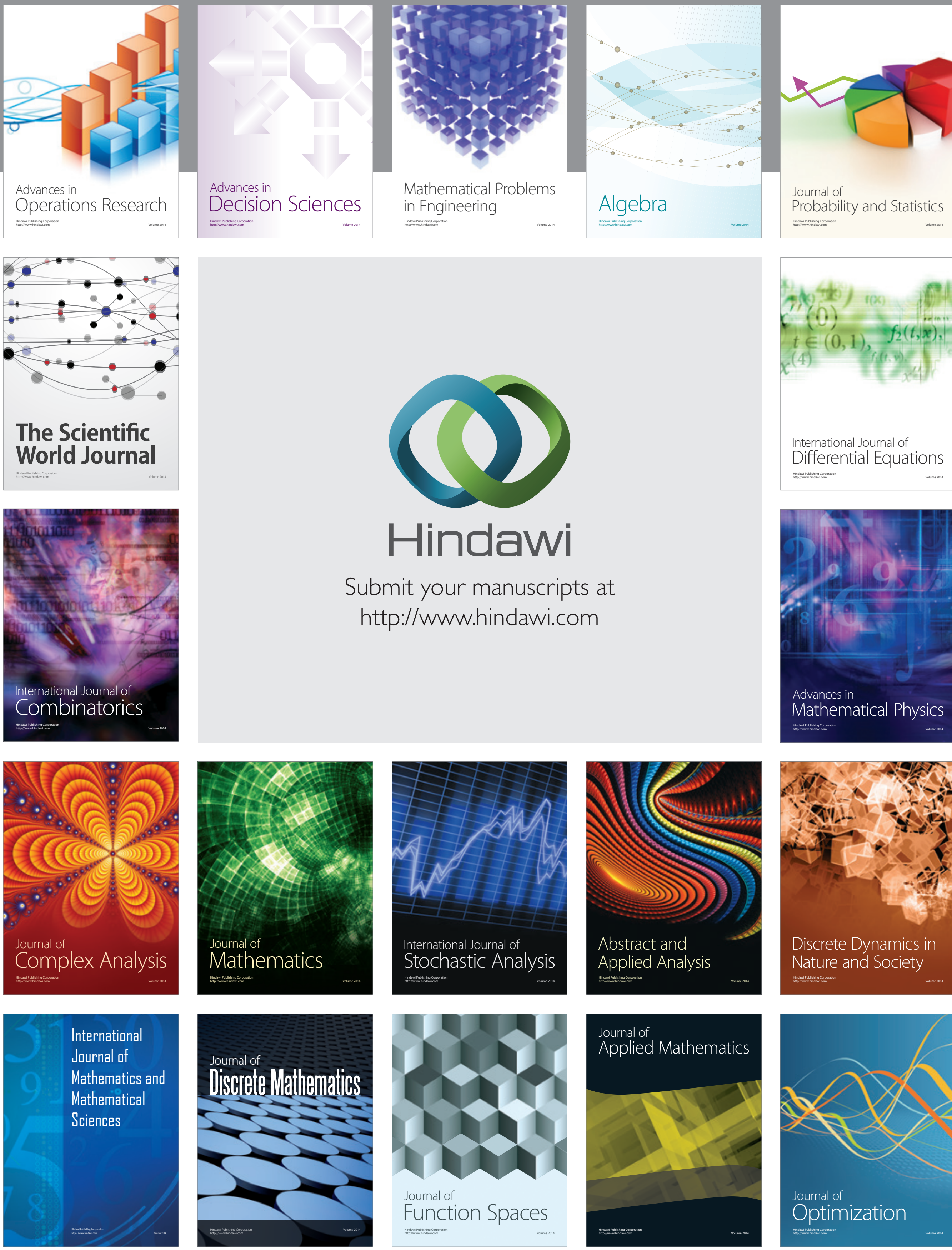
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