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Research Article

Regularity Criteria for a Turbulent Magnetohydrodynamic Model

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We establish some regularity criteria for a turbulent magnetohydrodynamic model. As a corollary, we prove that the smooth solution exists globally when the spatial dimension n satisfies $3 \le n \le 8$.

1. Introduction

In this paper, we study the following simplified turbulent MHD model [1]:

$$\partial_t v - \Delta v + (u \cdot \nabla) u + \nabla \pi = (B \cdot \nabla) B, \tag{1.1}$$

$$\partial_t H - \Delta H + (u \cdot \nabla) B - (B \cdot \nabla) u = 0, \tag{1.2}$$

$$u = \left(1 - \alpha^2 \Delta\right) u, \quad H = \left(1 - \alpha^2 \Delta\right) B, \quad \alpha > 0, \tag{1.3}$$

$$\operatorname{div} v = \operatorname{div} u = \operatorname{div} H = \operatorname{div} B = 0, \tag{1.4}$$

$$(v, H)(0) = (v_0, H_0) \quad \text{in } \mathbb{R}^n \ (n \ge 3). \tag{1.5}$$

Here v is the fluid velocity field, u is the "filtered" fluid velocity, π is the pressure, H is the magnetic field, and B is the "filtered" magnetic field. $\alpha > 0$ is the length scale parameter that represents the width of the filter. For simplicity we will take $\alpha = 1$.

When $n = 3$, the global well-posedness of the problem has been proved in [1]. When $H = B = 0$, (1.1) and (1.4) is the well-known Bardina model. Very recently, the authors [2] have proved that the Bardina model has a unique global-in-time weak solution when

$3 \leq n \leq 8$. Here we wold like to point out that by the same arguments, we can prove the following.

Theorem 1.1. *Let* $3 \leq n \leq 8$. *Let* $(u_0, B_0) \in H^1(\mathbb{R}^n)$ *with* $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ *in* $\mathbb{R}^n$. *Then for any* $T > 0$, *the problem* (1.1)–(1.5) *has a unique weak solution satisfying*

$$\begin{aligned}
&\frac{1}{2}\int u^2 + |\nabla u|^2 + B^2 + |\nabla B|^2 dx + \int_0^T \int |\nabla u|^2 + |\Delta u|^2 + |\nabla B|^2 + |\Delta B|^2 dx\, dt \\
&\quad \leq \frac{1}{2}\int u_0^2 + |\nabla u_0|^2 + B_0^2 + |\nabla B_0|^2 dx.
\end{aligned} \tag{1.6}$$

The proof for Theorem 1.1 is similar to that for the Bardina model in [2], so we omit it here.

The aim of this paper is to study the regularity of the weak solutions. We will prove

Theorem 1.2. *Let* $n \geq 3$. *Let* $(v_0, H_0) \in H^s(\mathbb{R}^n)$ *with* $s > 1$ *and* $\operatorname{div} v_0 = \operatorname{div} H_0 = 0$ *in* $\mathbb{R}^n$. *Let* (v, H) *be a local smooth solution to the problem* (1.1)–(1.5) *satisfying*

$$(v, H) \in L^\infty(0, T; H^s) \cap L^2\left(0, T; H^{s+1}\right), \tag{1.7}$$

for any fixed $T > 0$. *Then* (v, H) *can be extended beyond* $T > 0$ *provided that one of the following condition is satisfied:*

$$(1)\ (u, B) \in L^p(0, T; L^q) \quad \textit{with}\ \frac{2}{p} + \frac{n}{q} = 3,\ \frac{n}{3} < q \leq n, \tag{1.8}$$

$$(2)\ (\nabla u, \nabla B) \in L^p(0, T; L^q(\mathbb{R}^n)) \quad \textit{with}\ \frac{2}{p} + \frac{n}{q} = 4,\ \frac{n}{4} < q \leq \frac{n}{2}. \tag{1.9}$$

By (1.6) and (1.8), as a corollary, we have the following

Corollary 1.3. *Let* $3 \leq n \leq 8$. *Let* $(v_0, H_0) \in H^s(\mathbb{R}^n)$ *with* $s > 1$ *and* $\operatorname{div} v_0 = \operatorname{div} H_0 = 0$ *in* $\mathbb{R}^n$. *Then for any* $T > 0$, *the problem* (1.1)–(1.5) *has a unique smooth solution* (v, H) *satisfying* (1.7).

When $n = 9$ or 10, we can get a better result as follows.

Theorem 1.4. *Let* $n = 9$ *or 10 and let* $(v_0, H_0) \in L^2(\mathbb{R}^n)$ *and* $\operatorname{div} v_0 = \operatorname{div} H_0 = 0$ *in* $\mathbb{R}^n$. *Let* (v, H) *be a local smooth solution to the problem* (1.1)–(1.5) *satisfying*

$$(v, H) \in L^\infty\left(0, T; L^2\right) \cap L^2\left(0, T; H^1\right), \tag{1.10}$$

for any fixed $T > 0$. Then (v, H) can be extended beyond $T > 0$ if one of the following conditions is satisfied:

$$\text{(1) } u \in C\Big([0,T]; L^{n/3}\Big), \tag{1.11}$$

$$\text{(2) } u \in L^p(0,T;L^q) \quad \textit{with } \frac{2}{p} + \frac{n}{q} = 3,\ \frac{n}{3} < q \le n, \tag{1.12}$$

$$\text{(3) } \nabla u \in C\Big([0,T]; L^{n/4}\Big), \tag{1.13}$$

$$\text{(4) } \nabla u \in L^p(0,T;L^q) \quad \textit{with } \frac{2}{p} + \frac{n}{q} = 4 \textit{ with } \frac{n}{4} < q \le \frac{n}{2}. \tag{1.14}$$

Remark 1.5. If we delete the harmless lower order terms $\partial_t u - \Delta u$ and $\partial_t B - \Delta B$ in (1.1) and (1.2), then we have

$$\begin{aligned} -\partial_t \Delta u + \Delta^2 u + (u \cdot \nabla)u + \nabla \pi &= (B \cdot \nabla)B, \\ -\partial_t \Delta B + \Delta^2 B + (u \cdot \nabla)B - (B \cdot \nabla)u &= 0, \end{aligned} \tag{1.15}$$

then the system (1.15) has the following property: if (u, B, π) is a solution of (1.15), then for all $\lambda > 0$,

$$(u_\lambda, B_\lambda, \pi_\lambda)(x,t) = \Big(\lambda^3 u, \lambda^3 B, \lambda^6 \pi\Big)\Big(\lambda x, \lambda^2 t\Big) \tag{1.16}$$

is also a solution. In this sense, our conditions (1.8) and (1.11)–(1.14) are scaling invariant (optimal). Equations (1.8) and (1.12) do not hold true for $q > n$. But we also can establish regularity criteria for $q > n$ in nonscaling invariant forms.

In Section 2, we will prove Theorem 1.2. In Section 3, we will prove Theorem 1.4.

2. Proof of Theorem 1.2

Since it is easy to prove that the problem (1.1)–(1.5) has a unique local smooth solution, we only need to establish the a priori estimates. The proof of the case $n \le 4$ is easier and similar and thus we omit the details here, we only deal with the case $n \ge 5$.

Testing (1.1) by u, using (1.3) and (1.4), we find that

$$\frac{1}{2}\frac{d}{dt}\int u^2 + |\nabla u|^2 dx + \int |\nabla u|^2 + |\Delta u|^2 dx = \int (B \cdot \nabla)B \cdot u\, dx. \tag{2.1}$$

Testing (1.2) by B, using (1.3) and (1.4), we see that

$$\frac{1}{2}\frac{d}{dt}\int B^2+|\nabla B|^2dx+\int|\nabla B|^2+|\Delta B|^2dx=\int(B\cdot\nabla)u\cdot B\,dx=-\int(B\cdot\nabla)B\cdot u\,dx. \tag{2.2}$$

Summing up (2.1) and (2.2), we easily get (1.6).

(I) Let (1.8) hold true.

In the following calculations, we will use the product estimates due to Kato and Ponce [3]:

$$\|\Lambda^s(fg)\|_{L^p}\le C\left(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{q_1}}+\|f\|_{L^{p_2}}\|\Lambda^s g\|_{L^{q_2}}\right), \tag{2.3}$$

with $s>0$, $\Lambda:=(-\Delta)^{1/2}$ and $1/p=1/p_1+1/q_1=1/p_2+1/q_2$.

The proof of the case $q=n$ is easier and similar, we omit the details here. Now we assume $n/3<q<n$.

Applying Λ^s to (1.1), testing by $\Lambda^s v$, using (1.4), we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\Lambda^s v|^2dx+\int\left|\Lambda^{s+1}v\right|^2dx&=\int(\Lambda^s((B\cdot\nabla)B)-\Lambda^s((u\cdot\nabla)u))\Lambda^s v\,dx\\&=\int\Lambda^s\operatorname{div}(B\otimes B-u\otimes u)\Lambda^s v\,dx.\end{aligned} \tag{2.4}$$

Similarly, applying Λ^s to (1.2), testing by $\Lambda^s H$, using (1.4), we infer that

$$\frac{1}{2}\frac{d}{dt}\int|\Lambda^s H|^2dx+\int\left|\Lambda^{s+1}H\right|^2dx=\int\Lambda^s\operatorname{curl}(u\times B)\cdot\Lambda^s H\,dx. \tag{2.5}$$

Summing up (2.4) and (2.5), using (2.3), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\Lambda^s v|^2+|\Lambda^s H|^2dx+\int\left|\Lambda^{s+1}v\right|^2+\left|\Lambda^{s+1}H\right|^2dx\\&\quad=\int\Lambda^s\operatorname{div}(B\otimes B-u\otimes u)\Lambda^s v\,dx+\int\Lambda^s\operatorname{curl}(u\times B)\cdot\Lambda^s H\,dx\\&\quad\le C\left(\|B\|_{L^q}\left\|\Lambda^{s+1}B\right\|_{L^{t_1}}+\|u\|_{L^q}\left\|\Lambda^{s+1}u\right\|_{L^{t_1}}\right)\|\Lambda^s v\|_{L^{t_2}}\\&\qquad+C\left(\|u\|_{L^q}\left\|\Lambda^{s+1}B\right\|_{L^{t_1}}+\|B\|_{L^q}\left\|\Lambda^{s+1}u\right\|_{L^{t_1}}\right)\|\Lambda^s H\|_{L^{t_2}}\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\|u\|_{L^q}+\|B\|_{L^q}\right)\left(\left\|\Lambda^{s+1}u\right\|_{L^{t_1}}+\left\|\Lambda^{s+1}B\right\|_{L^{t_1}}\right)\left(\|\Lambda^s v\|_{L^{t_2}}+\|\Lambda^s H\|_{L^{t_2}}\right)\\
&= C\|(u,B)\|_{L^q}\left\|\Lambda^{s+1}(u,B)\right\|_{L^{t_1}}\|\Lambda^s(v,H)\|_{L^{t_2}}\\
&\qquad\qquad\qquad\left(\frac{1}{q}+\frac{1}{t_1}+\frac{1}{t_2}=1,\ 2\le t_2\le\frac{2n}{n-2}<t_1\le\frac{2n}{n-4}\right)\\
&\le C\|(u,B)\|_{L^q}\left\|\Lambda^{s-1}(v,H)\right\|_{L^{t_1}}\|\Lambda^s(v,H)\|_{L^{t_2}}\\
&\le C\|(u,B)\|_{L^q}\|\Lambda^s(v,H)\|_{L^2}^{1-\theta_1}\left\|\Lambda^{s+1}(v,H)\right\|_{L^2}^{\theta_1}\|\Lambda^s(v,H)\|_{L^2}^{1-\theta_2}\left\|\Lambda^{s+1}(v,H)\right\|_{L^2}^{\theta_2}\\
&= C\|(u,B)\|_{L^q}\|\Lambda^s(v,H)\|_{L^2}^{2-\theta_1-\theta_2}\left\|\Lambda^{s+1}(v,H)\right\|_{L^2}^{\theta_1+\theta_2}\\
&\le\frac{1}{2}\left\|\Lambda^{s+1}(v,H)\right\|_{L^2}^2+C\|(u,B)\|_{L^q}^{2/(2-\theta_1-\theta_2)}\|\Lambda^s(v,H)\|_{L^2}^2,
\end{aligned}
\tag{2.6}
$$

which implies

$$
\|(v,H)\|_{L^\infty(0,T;H^s)}+\|(v,H)\|_{L^2(0,T;H^{s+1})}\le C. \tag{2.7}
$$

Here we have used the following Gagliardo-Nirenberg inequalities:

$$
\begin{gathered}
\left\|\Lambda^{s-1}(v,H)\right\|_{L^{t_1}}\le C\|\Lambda^s(v,H)\|_{L^2}^{1-\theta_1}\left\|\Lambda^{s+1}(v,H)\right\|_{L^2}^{\theta_1},\\
\left(-\frac{n}{t_1}=(1-\theta_1)\left(1-\frac{n}{2}\right)+\theta_1\left(2-\frac{n}{2}\right)\right)\\
\|\Lambda^s(v,H)\|_{L^{t_2}}\le C\|\Lambda^s(v,H)\|_{L^2}^{1-\theta_2}\left\|\Lambda^{s+1}(v,H)\right\|_{L^2}^{\theta_2},\\
\left(-\frac{n}{t_2}=(1-\theta_2)\left(-\frac{n}{2}\right)+\theta_2\left(1-\frac{n}{2}\right)\right).
\end{gathered}
\tag{2.8}
$$

(II) Let (1.9) hold true.

In the following calculations, we will use the following commutator estimates due to Kato and Ponce [3]:

$$
\|\Lambda^s(fg)-f\Lambda^s g\|_{L^p}\le C\left(\|\nabla f\|_{L^{p_1}}\left\|\Lambda^{s-1}g\right\|_{L^{q_1}}+\|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right), \tag{2.9}
$$

with $s>0$ and $1/p=1/p_1+1/q_1=1/p_2+1/q_2$.

The proof of the case $q=n/2$ is easier and similar, we omit the details here. Now we assume $n/4<q<n/2$.

Applying Λ^s to (1.1), testing by $\Lambda^s u$, and using (1.3) and (1.4), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Lambda^s u|^2+\left|\Lambda^{s+1}u\right|^2 dx+\int \left|\Lambda^{s+1}u\right|^2+\left|\Lambda^{s+2}u\right|^2 dx \\
&\quad=-\int[\Lambda^s(u\cdot\nabla u)-u\cdot\nabla\Lambda^s u]\Lambda^s u\,dx+\int[\Lambda^s(B\cdot\nabla B)-B\cdot\nabla\Lambda^s B]\Lambda^s u\,dx \\
&\qquad+\int(B\cdot\nabla)\Lambda^s B\cdot\Lambda^s u\,dx.
\end{aligned}\tag{2.10}$$

Applying Λ^s to (1.2), testing by $\Lambda^s B$, using (1.3) and (1.4), we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Lambda^s B|^2+\left|\Lambda^{s+1}B\right|^2 dx+\int \left|\Lambda^{s+1}B\right|^2+\left|\Lambda^{s+2}B\right|^2 dx \\
&\quad=-\int[\Lambda^s(u\cdot\nabla B)-u\nabla\Lambda^s B]\Lambda^s B\,dx+\int[\Lambda^s(B\cdot\nabla u)-B\cdot\nabla\Lambda^s u]\Lambda^s B\,dx \\
&\qquad+\int(B\cdot\nabla)\Lambda^s u\cdot\Lambda^s B\,dx.
\end{aligned}\tag{2.11}$$

Summing up (2.10) and (2.11), noting that the last terms of (2.10) and (2.11) disappeared, and using (2.9), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Lambda^s u|^2+\left|\Lambda^{s+1}u\right|^2+|\Lambda^s B|^2+\left|\Lambda^{s+1}B\right|^2 dx \\
&\quad+\int \left|\Lambda^{s+1}u\right|^2+\left|\Lambda^{s+2}u\right|^2+\left|\Lambda^{s+1}B\right|^2+\left|\Lambda^{s+2}B\right|^2 dx \\
&=-\int[\Lambda^s(u\cdot\nabla u)-u\cdot\nabla\Lambda^s u]\Lambda^s u\,dx+\int[\Lambda^s(B\cdot\nabla B)-B\cdot\nabla\Lambda^s B]\Lambda^s u\,dx \\
&\quad-\int[\Lambda^s(u\cdot\nabla B)-u\cdot\nabla\Lambda^s B]\Lambda^s B\,dx+\int[\Lambda^s(B\cdot\nabla u)-B\cdot\nabla\Lambda^s u]\Lambda^s B\,dx \\
&\le C\|\nabla u\|_{L^q}\|\Lambda^s u\|^2_{L^{2q/(q-1)}}+C\|\nabla B\|_{L^q}\|\Lambda^s B\|_{L^{2q/(q-1)}}\|\Lambda^s u\|_{L^{2q/(q-1)}} \\
&\quad+C\|\nabla B\|_{L^q}\|\Lambda^s B\|^2_{L^{2q/(q-1)}}+C\|\nabla u\|_{L^q}\|\Lambda^s B\|^2_{L^{2q/(q-1)}} \\
&\le C\|(\nabla u,\nabla B)\|_{L^q}\|\Lambda^s(u,B)\|^2_{L^{2q/(q-1)}} \\
&\le C\|(\nabla u,\nabla B)\|_{L^q}\left\|\Lambda^{s+1}(u,B)\right\|^{2(1-\theta)}_{L^2}\left\|\Lambda^{s+2}(u,B)\right\|^{2\theta}_{L^2} \\
&\le \frac{1}{2}\left\|\Lambda^{s+2}(u,B)\right\|^2_{L^2}+C\|(\nabla u,\nabla B)\|^{1/(1-\theta)}_{L^q}\left\|\Lambda^{s+1}(u,B)\right\|^2_{L^2},
\end{aligned}\tag{2.12}$$

which yields

$$\|(u,B)\|_{L^\infty(0,T;H^{s+1})} + \|(u,B)\|_{L^2(0,T;H^{s+2})} \le C. \tag{2.13}$$

Here we have used the following Gagliardo-Nirenberg inequality:

$$\|\Lambda^s(u,B)\|_{L^{2q/(q-1)}} \le C\left\|\Lambda^{s+1}(u,B)\right\|_{L^2}^{1-\theta}\left\|\Lambda^{s+2}(u,B)\right\|_{L^2}^{\theta}, \tag{2.14}$$

with

$$\begin{gathered} -\frac{q-1}{2q}n = (1-\theta)\left(1-\frac{n}{2}\right) + \theta\left(2-\frac{n}{2}\right), \\ \frac{2n}{n-2} \le \frac{2q}{q-1} \le \frac{2n}{n-4}. \end{gathered} \tag{2.15}$$

This completes the proof.

3. Proof of Theorem 1.4

We only need to prove the a priori estimates.

Testing (1.1) and (1.2) by (v,H), using (1.3) and (1.4), and summing up the results, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int v^2 + H^2 dx + \int |\nabla v|^2 + |\nabla H|^2 dx \\ &\quad = \int (u\cdot\nabla)u\cdot\Delta u\,dx - \int (B\cdot\nabla)B\cdot\Delta u\,dx + \int (u\cdot\nabla)B\cdot\Delta B\,dx - \int (B\cdot\nabla)u\cdot\Delta B\,dx \\ &\quad = I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.1}$$

Using (1.4), we see that

$$\begin{aligned} I_1 &= \sum_{i,j}\int u_i\partial_i u\partial_j^2 u\,dx = -\sum_{i,j}\int \partial_j u_i\partial_i u\partial_j u\,dx, \\ I_2 &= -\sum_{i,j}\int B_i\partial_i B\partial_j^2 u\,dx = \sum_{i,j}\int \partial_j B_i\partial_i B\partial_j u\,dx - \sum_{i,j}\int B_i\partial_j B\partial_i\partial_j u\,dx, \\ I_3 &= \sum_{i,j}\int u_i\partial_i B\partial_j^2 B\,dx = -\sum_{i,j}\int \partial_j u_i\partial_i B\partial_j B\,dx, \\ I_4 &= -\sum_{i,j}\int B_i\partial_i u\partial_j^2 B\,dx = \sum_{i,j}\int \partial_j B_i\partial_i u\partial_j B\,dx + \sum_{i,j}\int B_i\partial_j B\partial_i\partial_j u\,dx. \end{aligned} \tag{3.2}$$

Inserting the above estimates into (3.1), noting that the last term of I_2 and I_4 disappeared, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int v^2 + H^2 dx + \int |\nabla v|^2 + |\nabla H|^2 dx &= -\sum_{i,j}\int \partial_i u \partial_j u \partial_j u_i \, dx + \sum_{i,j}\int \partial_j u_i \partial_i B \partial_j B_i \, dx \\ &\quad - \sum_{i,j}\int \partial_j u_i \partial_i B \partial_j B \, dx + \sum_{i,j}\int \partial_i u \partial_j B \partial_j B_i \, dx \\ &= J.\end{aligned} \tag{3.3}$$

The proofs of the cases (1.11) and (1.13) are similar, we omit the details here.

(I) Let (1.14) hold true,

$$\begin{aligned} J &\leq C\|\nabla u\|_{L^q}\|\nabla(u,B)\|^2_{L^{2q/(q-1)}} \\ &\leq C\|\nabla u\|_{L^q}\|\Delta(u,B)\|^{2(1-\theta)}_{L^2}\|\Delta\nabla(u,B)\|^{2\theta}_{L^2} \\ &\leq C\|\nabla u\|_{L^q}\|(v,H)\|^{2(1-\theta)}_{L^2}\|\nabla(v,H)\|^{2\theta}_{L^2} \\ &\leq \frac{1}{2}\|\nabla(v,H)\|^2_{L^2} + C\|\nabla u\|^{1/(1-\theta)}_{L^q}\|(v,H)\|^2_{L^2}. \end{aligned} \tag{3.4}$$

Inserting the above estimates into (3.3) and using the Gronwall inequality yields

$$\|(v,H)\|_{L^\infty(0,T;L^2)} + \|(v,H)\|_{L^2(0,T;H^1)} \leq C. \tag{3.5}$$

Here we have used the Gagliardo-Nirenberg inequality:

$$\|w\|_{L^{2q/(q-1)}} \leq C\|\nabla w\|^{1-\theta}_{L^2}\|\Delta w\|^{\theta}_{L^2} \tag{3.6}$$

with

$$\begin{aligned} -\frac{q-1}{2q}n &= (1-\theta)\left(1-\frac{n}{2}\right) + \theta\left(2-\frac{n}{2}\right), \\ \frac{2n}{n-2} &\leq \frac{2q}{q-1} \leq \frac{2n}{n-4}. \end{aligned} \tag{3.7}$$

(II) Let (1.12) hold true.

Integrating by parts, using (1.4) we have

$$
\begin{aligned}
J &= \sum_{i,j}\int u\partial_i\partial_j u\partial_j u_i\,dx - \sum_{i,j}\int u_i\partial_j\left(\partial_i B\partial_j B_i\right)dx + \sum_{i,j}\int u_i\partial_j\left(\partial_i B\partial_j B\right)dx - \sum_{i,j}\int u\partial_i\partial_j B\partial_j B_i\,dx \\
&\le C\|u\|_{L^q}\|\nabla(u,B)\|_{L^{t_1}}\|\Delta(u,B)\|_{L^{t_2}} \quad \left(\frac{1}{q}+\frac{1}{t_1}+\frac{1}{t_2}=1\right) \\
&\le C\|u\|_{L^q}\|\Delta(u,B)\|_{L^2}^{1-\theta_1}\|\nabla\Delta(u,B)\|_{L^2}^{\theta_1}\|\Delta(u,B)\|_{L^2}^{1-\theta_2}\|\nabla\Delta(u,B)\|_{L^2}^{\theta_2} \\
&= C\|u\|_{L^q}\|\Delta(u,B)\|_{L^2}^{2-\theta_1-\theta_2}\|\nabla\Delta(u,B)\|_{L^2}^{\theta_1+\theta_2} \\
&\le C\|u\|_{L^q}\|(v,H)\|_{L^2}^{2-\theta_1-\theta_2}\|\nabla(v,H)\|_{L^2}^{\theta_1+\theta_2} \\
&\le \frac{1}{2}\|\nabla(v,H)\|_{L^2}^2 + C\|u\|_{L^q}^{2/(2-\theta_1-\theta_2)}\|(v,H)\|_{L^2}^2,
\end{aligned}
\tag{3.8}
$$

which yields (3.5).

Here we have used the Gagliardo-Nirenberg inequalities:

$$
\begin{gathered}
\|\nabla(u,B)\|_{L^{t_1}} \le C\|\Delta(u,B)\|_{L^2}^{1-\theta_1}\|\nabla\Delta(u,B)\|_{L^2}^{\theta_1}, \\
\left(-\frac{n}{t_1}=(1-\theta_1)\left(1-\frac{n}{2}\right)+\theta_1\left(2-\frac{n}{2}\right)\right) \\
\|\Delta(u,B)\|_{L^{t_2}} \le C\|\Delta(u,B)\|_{L^2}^{1-\theta_2}\|\nabla\Delta(u,B)\|_{L^2}^{\theta_2}, \\
\left(-\frac{n}{t_2}=(1-\theta_2)\left(-\frac{n}{2}\right)+\theta_2\left(1-\frac{n}{2}\right)\right).
\end{gathered}
\tag{3.9}
$$

This completes the proof.

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