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Research Article

General Cubic-Quartic Functional Equation

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We obtain the general solution and the generalized Hyers-Ulam stability of the general cubic-quartic functional equation for fixed integers k with $k \neq 0, \pm 1$: $f(x+ky)+f(x-ky) = k^2(f(x+y)+f(x-y))+2(1-k^2)f(x)+((k^4-k^2)/4)(f(2y)-8f(y))+\widetilde{f}(2x)-16\widetilde{f}(x)$, where $\widetilde{f}(x) := f(x)+f(-x)$.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq \delta \tag{1.1}$$

for all $x, y \in E$ and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\left\| f(x) - T(x) \right\| \leq \delta \tag{1.2}$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. In 1978, Rassias [3] proved the following theorem.

Theorem 1.1. *Let $f : E \rightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.3}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.4}$$

for all $x \in E$. If $p < 0$ then inequality (1.3) *holds for all $x, y \neq 0$ and* (1.4) *for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in real t for each fixed $x \in E$, then T is linear.*

In 1990, Rassias during the 27th International Symposium on Functional Equations asked the question whether such a Theorem can also be proved for all real values of p that are greater or equal to one. In 1991, Gajda [4], following the same approach as that of Rassias, provided an affirmative solution to this question for all real values of p that are strictly greater than one. The new concept of stability of the linear mapping that was inspired by Rassias' stability theorem is called Hyers-Ulam-Rassias stability of functional equations.

Jun and Kim [5] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.5}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.5). The function $f(x) = x^3$ satisfies the functional equation (1.5), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function f between real vector spaces X and Y is a solution of (1.5) if and only if there exists a unique function $C : X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$ and C is symmetric for each fixed one variable and is additive for fixed two variables. The stability of the quartic functional equations was studied by Park and Bae [6], when

$$f(x+2y) + f(x-2y) = 4(f(x+y) + f(x-y)) + 24f(y) - 6f(x). \tag{1.6}$$

In fact, they proved that a function f between real vector spaces X and Y is a solution of (1.6) if and only if there exists a unique symmetric multi-additive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f(x) = Q(x, x, x, x)$ for all $x \in X$ (see also [7, 8]). It is straightforward to verify that the function $f(x) = x^4$ satisfies the functional equation (1.6), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem. (see [9–45]).

In 2008, Gordji et al. [17] provided the solution as well as the stability of a mixed type cubic-quartic functional equation. We only mention here the papers [19, 32, 33] concerning the stability of the mixed type functional equations.

In this paper, we deal with the following general cubic-quartic functional equation:

$$f(x+ky)+f(x-ky)=k^2(f(x+y)+f(x-y))+2\left(1-k^2\right)f(x)+\frac{k^4-k^2}{4}\times(f(2y)-8f(y))+\tilde{f}(2x)-16\tilde{f}(x), \quad \text{where } \tilde{f}(x):=f(x)+f(-x). \tag{1.7}$$

Then it follows easily that the function $f(x)=ax^4+bx^3$ satisfies (1.7). We investigate the general solution and the generalized Hyers-Ulam-Rassias stability of the functional equation (1.7).

2. General Solution

In this section, we establish the general solution of functional equation (1.7).

Theorem 2.1. *Let X, Y be vector spaces and let $f : X \to Y$ be a function. Then f satisfies* (1.7) *if and only if there exists a unique symmetric multiadditive function $Q : X\times X\times X\times X \to Y$ and a unique function $C : X\times X\times X \to Y$ such that $f(x)=Q(x,x,x,x)+C(x,x,x)$ for all $x\in X$, where the function C is symmetric for each fixed one variable and is additive for fixed two variables.*

Proof. Let f satisfies (1.7). We decompose f into the even part and odd part by setting

$$f_e(x)=\frac{1}{2}(f(x)+f(-x)), \qquad f_o(x)=\frac{1}{2}(f(x)-f(-x)) \tag{2.1}$$

for all $x\in X$. By (1.7), we have

$$\begin{aligned}
f_e(x+ky)+f_e(x-ky)&=\frac{1}{2}[f(x+ky)+f(-x-ky)+f(x-ky)+f(-x+ky)]\\
&=\frac{1}{2}[f(x+ky)+f(x-ky)]+\frac{1}{2}[f((-x)+(-ky))+f((-x)-(-ky))]\\
&=\frac{1}{2}\Bigg[k^2(f(x+y)+f(x-y))+2\left(1-k^2\right)f(x)\\
&\qquad+\frac{k^4-k^2}{4}(f(2y)-8f(y))+\tilde{f}(2x)-16\tilde{f}(x)\Bigg]\\
&\quad+\frac{1}{2}\Bigg[k^2(f(-x-y)+f(-x+y))+2\left(1-k^2\right)f(-x)+\frac{k^4-k^2}{4}\\
&\qquad\times(f(-2y)-8f(-y))+\tilde{f}(-2x)-16\tilde{f}(-x)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= k^2\left[\frac{1}{2}(f(x+y)+f(-(x+y)))\right]+k^2\left[\frac{1}{2}(f(x-y)+f(-(x-y)))\right] \\
&\quad + 2\left(1-k^2\right)\left[\frac{1}{2}(f(x)+f(-x))\right]+\frac{k^4-k^2}{4}\left[\frac{1}{2}(f(2y)+f(-2y))\right] \\
&\quad - \frac{k^4-k^2}{4}\left[\frac{1}{2}(8f(y)+8f(-y))\right]+\left[\frac{1}{2}\left(\tilde{f}(2x)+\tilde{f}(-2x)\right)\right] \\
&\quad - 16\left[\frac{1}{2}\left(\tilde{f}(x)+\tilde{f}(-x)\right)\right] \\
&= k^2(f_e(x+y)+f_e(x-y))+2\left(1-k^2\right)f_e(x) \\
&\quad + \frac{k^4-k^2}{4}(f_e(2y)-8f_e(y))+\tilde{f}_e(2x)-16\tilde{f}_e(x)
\end{aligned} \tag{2.2}
$$

for all $x, y \in X$. This means that f_e satisfies (1.7), or

$$
\begin{aligned}
f_e(x+ky)+f_e(x-ky) &= k^2(f_e(x+y)+f_e(x-y))+2\left(1-k^2\right)f_e(x) \\
&\quad + \frac{k^4-k^2}{4}(f_e(2y)-8f_e(y))+\tilde{f}_e(2x)-16\tilde{f}_e(x)
\end{aligned} \tag{1.5(e)}
$$

for all $x, y \in X$. Applying the fact that the function f_e is even for all $x, y \in X$, (1.5(e)) can be written in the form

$$
\begin{aligned}
f_e(x+ky)+f_e(x-ky) &= k^2(f_e(x+y)+f_e(x-y))+2\left(1-k^2\right)f_e(x) \\
&\quad + \frac{k^4-k^2}{4}(f_e(2y)-8f_e(y))+2f_e(2x)-32f_e(x)
\end{aligned} \tag{2.3}
$$

for all $x, y \in X$. Now be setting $x = y = 0$ in (2.3), we get $f_e(0) = 0$. Similarly, by setting $y = 0$ in (2.3), we obtain

$$
f_e(2x) = 16f_e(x) \tag{2.4}
$$

for all $x \in X$. Hence (2.3) can be written as

$$
f_e(x+ky)+f_e(x-ky) = k^2(f_e(x+y)+f_e(x-y))+2\left(1-k^2\right)f_e(x)+2\left(k^4-k^2\right)f_e(y) \tag{2.5}
$$

for all $x, y \in X$. By substituting x by $x + y$ in (2.5), we have

$$f_e(x + (1 + k)y) + f_e(x + (1 - k)y) \\ = k^2(f_e(x + 2y) + f_e(x)) + 2\left(1 - k^2\right)f_e(x + y) + 2\left(k^4 - k^2\right)f_e(y) \tag{2.6}$$

for all $x, y \in X$. Substituting $-y$ for y in (2.6), we get by evenness of f

$$f_e(x - (1 + k)y) + f_e(x - (1 - k)y) \\ = k^2(f_e(x - 2y) + f_e(x)) + 2\left(1 - k^2\right)f_e(x - y) + 2\left(k^4 - k^2\right)f_e(y) \tag{2.7}$$

for all $x, y \in X$. Adding (2.6) to (2.7), we obtain

$$f_e(x + (1 + k)y) + f_e(x + (1 - k)y) + f_e(x - (1 + k)y) + f_e(x - (1 - k)y) \\ = k^2(f_e(x + 2y) + f_e(x - 2y)) + 2k^2 f_e(x) + 2\left(1 - k^2\right)(f_e(x + y) + f_e(x - y)) \\ + 4\left(k^4 - k^2\right)f_e(y) \tag{2.8}$$

for all $x, y \in X$. By substituting x by $x - ky$ in (2.5), we have

$$f_e(x) + f_e(x - 2ky) = k^2(f_e(x + (1 - k)y) + f_e(x - (k + 1)y)) + 2\left(1 - k^2\right)f_e(x - ky) \\ + 2\left(k^4 - k^2\right)f_e(y) \tag{2.9}$$

for all $x, y \in X$. Substituting $-x$ for x in (2.9), we get by evenness of f_e

$$f_e(x) + f_e(x + 2ky) = k^2(f_e(x + (k - 1)y) + f_e(x + (k + 1)y)) + 2\left(1 - k^2\right)f_e(x + ky) \\ + 2\left(k^4 - k^2\right)f_e(y) \tag{2.10}$$

for all $x, y \in X$. Adding (2.9) to (2.10), we obtain

$$f_e(x + 2ky) + f_e(x - 2ky) = k^2(f_e(x + (1 - k)y) + f_e(x - (k + 1)y) + f_e(x + (k - 1)y) \\ + f_e(x + (k + 1)y)) + 2\left(1 - k^2\right)(f_e(x - ky) + f_e(x + ky)) \\ + 4\left(k^4 - k^2\right)f_e(y) - 2f_e(x) \tag{2.11}$$

for all $x, y \in X$. Now, by using (2.5), (2.8), and (2.11), we lead to

$$\begin{aligned} f_e(x+2ky) + f_e(x-2ky) = {} & k^4(f_e(x+2y) + f_e(x-2y)) \\ & + 4k^2\left(1-k^2\right)(f_e(x+y) + f_e(x-y)) + 8\left(k^4-k^2\right)f_e(y) \\ & + \left(6k^4 - 8k^2 + 2\right)f_e(x) \end{aligned} \tag{2.12}$$

for all $x, y \in X$. If we replace y by $2y$ in (2.5), we get

$$\begin{aligned} f_e(x+2ky) + f_e(x-2ky) = {} & k^2(f_e(x+2y) + f_e(x-2y)) + 2\left(1-k^2\right)f_e(x) \\ & + 2\left(k^4-k^2\right)f_e(2y) \end{aligned} \tag{2.13}$$

for all $x, y \in X$. It follows from (2.12) and (2.13) that

$$\begin{aligned} & k^4(f_e(x+2y) + f_e(x-2y)) + 4k^2\left(1-k^2\right)(f_e(x+y) + f_e(x-y)) + 8\left(k^4-k^2\right)f_e(y) \\ & \quad + \left(6k^4 - 8k^2 + 2\right)f_e(x) \\ & = k^2(f_e(x+2y) + f_e(x-2y)) + 2\left(1-k^2\right)f_e(x) + 2\left(k^4-k^2\right)f_e(2y) \end{aligned} \tag{2.14}$$

for all $x, y \in X$. So we have

$$f_e(x+2y) + f_e(x-2y) = 4(f_e(x+y) + f_e(x-y)) + 24f_e(y) - 6f_e(x) \tag{2.15}$$

for all $x, y \in X$. This means that f_e is a quartic function. Thus there exists a unique symmetric multiadditive function $Q : X \times X \times X \times X \to Y$ such that $f_e(x) = Q(x, x, x, x)$ for all $x \in X$. On the other hand, we can show that f_o satisfies (1.7), or

$$\begin{aligned} f_o(x+ky) + f_o(x-ky) = {} & k^2(f_o(x+y) + f_o(x-y)) + 2\left(1-k^2\right)f_o(x) \\ & + \frac{k^4-k^2}{4}(f_o(2y) - 8f_o(y)) + \tilde{f}_o(2x) - 16\tilde{f}_o(x) \end{aligned} \tag{1.5(o)}$$

for all $x, y \in X$. By oddness of f_o for all $x, y \in X$, (1.5(o)) can be written as

$$\begin{aligned} f_o(x+ky) + f_o(x-ky) = {} & k^2(f_o(x+y) + f_o(x-y)) + 2\left(1-k^2\right)f_o(x) \\ & + \frac{k^4-k^2}{4}(f_o(2y) - 8f_o(y)) \end{aligned} \tag{2.16}$$

for all $x, y \in X$. Now by setting $x = y = 0$ in (3.2), we get $f_o(0) = 0$, and by setting $x = 0$ in (2.16), we obtain

$$f_o(2y) = 8f_o(y) \tag{2.17}$$

for all $y \in X$. Hence (2.16) can be written as

$$f_o(x+ky) + f_o(x-ky) = k^2(f_o(x+y) + f_o(x-y)) + 2\left(1-k^2\right)f_o(x) \tag{2.18}$$

for all $x, y \in X$. Replacing x by $x-y$ in (2.18), we obtain

$$f_o(x+(k-1)y) + f_o(x-(k+1)y) = k^2(f_o(x-2y) + f_o(x)) + 2\left(1-k^2\right)f_o(x-y) \tag{2.19}$$

for all $x, y \in X$. Substituting $-x$ for x in (2.19), we get by oddness of f_o

$$-f_o(x+(1-k)y) - f_o(x+(k+1)y) = k^2(-f_o(x+2y) - f_o(x)) - 2\left(1-k^2\right)f_o(x+y) \tag{2.20}$$

for all $x, y \in X$. If we subtract (2.19) from (2.20), we obtain

$$\begin{aligned} &f_o(x+(k-1)y) + f_o(x-(k+1)y) + f_o(x+(1-k)y) + f_o(x+(k+1)y) \\ &\quad = k^2(f_o(x+2y) + f_o(x-2y)) + 2k^2f_o(x) + 2\left(1-k^2\right)(f_o(x+y) + f_o(x-y)) \end{aligned} \tag{2.21}$$

for all $x, y \in X$. By substituting x by $x+ky$ in (2.18), we have

$$f_o(x) + f_o(x+2ky) = k^2(f_o(x+(k+1)y) + f_o(x+(k-1)y)) + 2\left(1-k^2\right)f_o(x+ky) \tag{2.22}$$

for all $x, y \in X$. Substituting $-y$ for y in (2.22), we get

$$f_o(x) + f_o(x-2ky) = k^2(f_o(x-(k+1)y) + f_o(x-(k-1)y)) + 2\left(1-k^2\right)f_o(x-ky) \tag{2.23}$$

for all $x, y \in X$. Adding (2.22) to (2.23), we obtain

$$\begin{aligned} f_o(x+2ky) + f_o(x-2ky) = k^2(&f_o(x+(k+1)y) + f_o(x+(k-1)y) + f_o(x-(k+1)y) \\ &+f_o(x-(k-1)y)) + 2\left(1-k^2\right)(f_o(x+ky) + f_o(x-ky)) \\ &-2f_o(x) \end{aligned} \tag{2.24}$$

for all $x, y \in X$. Now, by using (2.18), (2.21), and (2.24), we lead to

$$f_o(x+2ky)+f_o(x-2ky)=4k^2\left(1-k^2\right)(f_o(x+y)+f_o(x-y)) \\ +\left(6k^4-8k^2+2\right)f_o(x)+k^4(f_o(x+2y)+f_o(x-2y)) \tag{2.25}$$

for all $x, y \in X$. If we replace y by $2y$ in (2.18), we get

$$f_o(x+2ky)+f_o(x-2ky)=k^2(f_o(x+2y)+f_o(x-2y))+2\left(1-k^2\right)f_o(x) \tag{2.26}$$

for all $x, y \in X$. If we compare (2.25) with (2.26), then we conclude that

$$f_o(x+2y)+f_o(x-2y)=4(f_o(x+y)+f_o(x-y))-6f_o(x) \tag{2.27}$$

for all $x, y \in X$. Replacing x by $2x$ in (2.27), we get

$$f_o(2(x+y))+f_o(2(x-y))=4(f_o(2x+y)+f_o(2x-y))-6f_o(2x) \tag{2.28}$$

for all $x, y \in X$. Finally, it follows from (2.17) and (2.28) that

$$8(f_o(x+y)+f_o(x-y))=4(f_o(2x+y)+f_o(2x-y))-48f_o(x) \tag{2.29}$$

for all $x, y \in X$. By multiplying both sides of (2.29) by $1/4$, we get

$$2(f_o(x+y)+f_o(x-y))=(f_o(2x+y)+f_o(2x-y))-12f_o(x) \tag{2.30}$$

for all $x, y \in X$. This means that f_o is a cubic function and that there exits a unique function $C : X \times X \times X \to Y$ such that $f_o(x) = C(x, x, x)$ for all $x \in X$ and C is symmetric for each fixed one variable and is additive for fixed two variables. Thus for all $x \in X$, we have

$$f(x)=f_e(x)+f_o(x)=C(x,x,x)+Q(x,x,x,x). \tag{2.31}$$

The proof of the converse is trivially. □

The following corollary is an alternative result of above Theorem 2.1.

Corollary 2.2. *Let X, Y be vector spaces, and let $f : X \to Y$ be a function satisfying* (1.7)*. Then the following assertions hold.*

(a) *If f is even function, then f is quartic.*

(b) *If f is odd function, then f is cubic.*

3. Stability

We will investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.7). In the following, let X be a real vector space and let Y be a Banach space. Given $f : X \to Y$, we define the difference operator $D_f : X \times X \to Y$ by

$$D_f(x,y) = f(x+ky) + f(x-ky) - k^2(f(x+y) + f(x-y)) - 2\left(1-k^2\right)f(x) - \frac{k^4-k^2}{4}(f(2y) - 8f(y)) - \tilde{f}(2x) + 16\tilde{f}(x) \tag{3.1}$$

for all $x, y \in X$.

Theorem 3.1. *Let $j \in \{-1, 1\}$ be fixed and let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\sum_{i=(1+j)/2}^{\infty} k^{4ij}\varphi\left(\frac{x}{k^{ij}}, \frac{y}{k^{ij}}\right) < \infty \tag{3.2}$$

for all $x, y \in X$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x,y)\| \le \varphi(x,y) \tag{3.3}$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{n\to\infty} k^{4nj} f\left(\frac{x}{k^{nj}}\right) \tag{3.4}$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quartic function satisfying

$$\|f(x) - Q(x)\| \le \frac{1}{k^4}\tilde{\varphi}_e(x) \tag{3.5}$$

for all $x \in X$, where

$$\tilde{\varphi}_e(x) = \sum_{i=(1+j)/2}^{\infty} k^{4ij}\left[\frac{1}{2}\varphi\left(0, \frac{x}{k^{ij}}\right) + \frac{k^4-k^2}{16}\varphi\left(\frac{x}{k^{ij}}, 0\right)\right]. \tag{3.6}$$

Proof. Let $j = 1$. It follows from (3.3) and using evenness of f that

$$\left\| f(x+ky) + f(x-ky) - k^2(f(x+y) + f(x-y)) - 2\left(1-k^2\right)f(x) - \frac{k^4-k^2}{4}(f(2y) - 8f(y)) - 2f(2x) + 32f(x) \right\| \le \varphi(x,y) \tag{3.7}$$

for all $x, y \in X$. Replacing x and y by 0 and x in (3.7), respectively, we see that

$$\left\| 2f(kx) + \left(2k^4 - 4k^2\right) f(x) + \frac{k^2 - k^4}{4} f(2x) \right\| \le \varphi(0, x) \tag{3.8}$$

for all $x \in X$. If we divide both sides of (3.8) by 2, we get

$$\left\| f(kx) + \left(k^4 - 2k^2\right) f(x) + \frac{k^2 - k^4}{8} f(2x) \right\| \le \frac{1}{2} \varphi(0, x) \tag{3.9}$$

for all $x \in X$. Putting $y = 0$ in (3.7), we obtain

$$\| 2f(2x) - 32f(x) \| \le \varphi(x, 0) \tag{3.10}$$

for all $x \in X$. If we multiply both sides of (3.10) by $(k^4 - k^2)/16$, then we have

$$\left\| \frac{k^4 - k^2}{8} f(2x) - 2\left(k^4 - k^2\right) f(x) \right\| \le \frac{k^4 - k^2}{16} \varphi(x, 0) \tag{3.11}$$

for all $x \in X$. It follows from (3.9) and (3.11) that

$$\left\| f(kx) - k^4 f(x) \right\| \le \frac{1}{2} \varphi(0, x) + \frac{k^4 - k^2}{16} \varphi(x, 0) \tag{3.12}$$

for all $x \in X$. Let

$$\psi_e(x) = \frac{1}{2} \varphi(0, x) + \frac{k^4 - k^2}{16} \varphi(x, 0) \tag{3.13}$$

for all $x \in X$. Thus by (3.12), we get

$$\left\| f(kx) - k^4 f(x) \right\| \le \psi_e(x) \tag{3.14}$$

for all $x \in X$. If we replace x in (3.14) by x/k^{n+1} and multiply both sides of (3.14) by k^{4n}, we see that

$$\left\| k^{4(n+1)} f\left(\frac{x}{k^{n+1}}\right) - k^{4n} f\left(\frac{x}{k^n}\right) \right\| \le k^{4n} \psi_e\left(\frac{x}{k^{n+1}}\right) \tag{3.15}$$

for all $x \in X$ and all nonnegative integers n. So

$$\begin{aligned}\left\| k^{4(n+1)} f\left(\frac{x}{k^{n+1}}\right) - k^{4m} f\left(\frac{x}{k^{m}}\right) \right\| &\le \sum_{i=m}^{n} \left\| k^{4(i+1)} f\left(\frac{x}{k^{i+1}}\right) - k^{4i} f\left(\frac{x}{k^{i}}\right) \right\| \\ &\le \sum_{i=m}^{n} k^{4i} \psi_e\left(\frac{x}{k^{i+1}}\right)\end{aligned} \tag{3.16}$$

for all nonnegative integers n and m with $n \ge m$ and all $x \in X$. By (3.2), we infer that

$$\sum_{i=m}^{n} k^{4i} \psi_e\left(\frac{x}{k^{i+1}}\right) < \infty, \qquad \lim_{n\to\infty} k^{4n} \psi_e\left(\frac{x}{k^{n+1}}\right) = 0 \tag{3.17}$$

for all $x \in X$. It follows from (3.16) and (3.17) that the sequence $\{k^{4n} f(x/k^n)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{k^{4n} f(x/k^n)\}$ converges for all $x \in X$. So one can define a mapping $Q : X \to Y$ by (3.4) for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.16), we obtain (3.5). It follows from (3.4), (3.15), and (3.17) that

$$\left\| Q(x) - k^4 Q\left(\frac{x}{k}\right) \right\| = \lim_{n\to\infty} \left\| k^{4n} f\left(\frac{x}{k^n}\right) - k^{4(n+1)} f\left(\frac{x}{k^{(n+1)}}\right) \right\| \le \lim_{n\to\infty} k^{4n} \psi_e\left(\frac{x}{k^{n+1}}\right) = 0 \tag{3.18}$$

for all $x \in X$. So

$$Q(kx) = k^4 Q(x) \tag{3.19}$$

for all $x \in X$. On the other hand, it follows from (3.2), (3.3), and (3.4) that

$$\| D_Q(x,y) \| = \lim_{n\to\infty} k^{4n} \left\| D_f\left(\frac{x}{k^n}, \frac{y}{k^n}\right) \right\| \le \lim_{n\to\infty} k^{4n} \varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) = 0 \tag{3.20}$$

for all $x, y \in X$. Therefore, by Corollary 2.2, the function $Q : X \to Y$ is quartic.

To prove the uniqueness of Q, let $Q' : X \to Y$ be a another quartic function satisfying (3.5). Since

$$\lim_{n\to\infty} k^{4n} \sum_{i=1}^{\infty} k^{4i} \varphi\left(\frac{x}{k^{n+i}}, \frac{y}{k^{n+i}}\right) = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} k^{4i} \varphi\left(\frac{x}{k^{i}}, \frac{y}{k^{i}}\right) = 0 \tag{3.21}$$

for all $x, y \in X$, hence

$$\lim_{n\to\infty} k^{4n} \widetilde{\varphi}_e\left(\frac{x}{k^n}\right) = 0 \tag{3.22}$$

for all $x \in X$. So it follows from (3.5) and (3.22) that

$$\|Q(x) - Q'(x)\| = \lim_{n\to\infty} k^{4n}\left\|f\left(\frac{x}{k^n}\right) - Q'\left(\frac{x}{k^n}\right)\right\| \le \lim_{n\to\infty}\frac{k^{4n}}{k^4}\widetilde{\varphi}_e\left(\frac{x}{k^n}\right) = 0 \tag{3.23}$$

for all $x \in X$. Hence $Q = Q'$.

For $j = -1$, the proof of the theorem is similar. □

Theorem 3.2. *Let $j \in \{-1, 1\}$ be fixed, and let $\varphi : X \times X \to [0,\infty)$ be a function such that*

$$\widetilde{\varphi}_c(x,y) = \sum_{i=(1+j)/2}^{\infty} 2^{3ij}\varphi\left(\frac{x}{2^{ij}}, \frac{y}{2^{ij}}\right) < \infty \tag{3.24}$$

for all $x, y \in X$. Suppose that an odd function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.3). *Then the limit*

$$C(x) := \lim_{n\to\infty} 2^{3nj} f\left(\frac{x}{2^{nj}}\right) \tag{3.25}$$

exists for all $x \in X$ and $C : X \to Y$ is a unique cubic function satisfying

$$\|f(x) - C(x)\| \le \frac{1}{2(k^4 - k^2)}\widetilde{\varphi}_c(0,x) \tag{3.26}$$

for all $x \in X$.

Proof. Let $j = 1$. It follows from (3.3) and using oddness of f that

$$\begin{aligned}&\Big\|f(x+ky) + f(x-ky) - k^2(f(x+y) + f(x-y)) - 2\left(1-k^2\right)f(x)\\ &-\frac{k^4-k^2}{4}(f(2y) - 8f(y))\Big\| \le \varphi(x,y)\end{aligned} \tag{3.27}$$

for all $x, y \in X$. Replacing x and y by 0 and x in (3.27), respectively, we see that

$$\left\|\frac{k^4-k^2}{4}(f(2x) - 8f(x))\right\| \le \varphi(0,x) \tag{3.28}$$

for all $x \in X$. If we multiply both sides of (3.28) by $4/(k^4 - k^2)$, we get

$$\|f(2x) - 8f(x)\| \le \frac{4}{k^4 - k^2}\varphi(0,x) \tag{3.29}$$

for all $x \in X$. If we replace x in (3.29) by $x/2^{n+1}$ and multiply both sides of (3.29) by 2^{3n}, we see that

$$\left\| 2^{3(n+1)} f\left(\frac{x}{2^{n+1}}\right) - 2^{3n} f\left(\frac{x}{2^n}\right) \right\| \le 2^{3n} \frac{4}{k^4 - k^2} \varphi\left(0, \frac{x}{2^{n+1}}\right) \tag{3.30}$$

for all $x \in X$ and all nonnegative integers n. So

$$\begin{aligned} \left\| 2^{3(n+1)} f\left(\frac{x}{2^{n+1}}\right) - 2^{3m} f\left(\frac{x}{2^m}\right) \right\| &\le \sum_{i=m}^{n} \left\| 2^{3(i+1)} f\left(\frac{x}{2^{i+1}}\right) - 2^{3i} f\left(\frac{x}{2^i}\right) \right\| \\ &\le \frac{4}{k^4 - k^2} \sum_{i=m}^{n} 2^{3i} \varphi\left(0, \frac{x}{2^{i+1}}\right) \end{aligned} \tag{3.31}$$

for all nonnegative integers n and m with $n \ge m$ and all $x \in X$. By (3.24), we infer that

$$\sum_{i=m}^{n} 2^{3i} \varphi\left(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}\right) < \infty, \qquad \lim_{n \to \infty} 2^{3n} \varphi\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) = 0 \tag{3.32}$$

for all $x, y \in X$. It follows from (3.31) and (3.32) that the sequence $\{2^{3n} f(x/2^n)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^{3n} f(x/2^n)\}$ converges for all $x \in X$. So one can define a mapping $C : X \to Y$ by (3.25) for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.31), we obtain (3.26). It follows from (3.25), (3.30), and (3.32) that

$$\left\| C(x) - 2^3 C\left(\frac{x}{2}\right) \right\| = \lim_{n \to \infty} \left\| 2^{3n} f\left(\frac{x}{2^n}\right) - 2^{3(n+1)} f\left(\frac{x}{2^{(n+1)}}\right) \right\| \le \lim_{n \to \infty} \frac{4}{k^4 - k^2} 2^{3n} \varphi\left(0, \frac{x}{2^{n+1}}\right) = 0 \tag{3.33}$$

for all $x \in X$. So

$$C(2x) = 2^3 C(x) \tag{3.34}$$

for all $x \in X$. On the other hand, it follows from (3.3), (3.24), and (3.25) that

$$\| D_c(x, y) \| = \lim_{n \to \infty} 2^{3n} \left\| D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \le \lim_{n \to \infty} 2^{3n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{3.35}$$

for all $x, y \in X$. Therefore by Corollary 2.2, the function $C : X \to Y$ is cubic.

To prove the uniqueness of C, let $C' : X \to Y$ be a another cubic function satisfying (3.26). Since

$$\lim_{n \to \infty} 2^{3n} \sum_{i=1}^{\infty} 2^{3i} \varphi\left(\frac{x}{2^{n+i}}, \frac{y}{2^{n+i}}\right) = \lim_{n \to \infty} \sum_{i=n+1}^{\infty} 2^{3i} \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) = 0 \tag{3.36}$$

for all $x, y \in X$, hence

$$\lim_{n\to\infty} 2^{3n}\widetilde{\varphi}_c\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{3.37}$$

for all $x, y \in X$. So it follows from (3.26) and (3.37) that

$$\left\|C(x) - C'(x)\right\| = \lim_{n\to\infty} 2^{3n}\left\|f\left(\frac{x}{2^n}\right) - C'\left(\frac{x}{2^n}\right)\right\| \le \lim_{n\to\infty}\frac{1}{2(k^4-k^2)}2^{3n}\widetilde{\varphi}_c\left(0, \frac{x}{2^n}\right) = 0 \tag{3.38}$$

for all $x \in X$. Hence $C = C'$.

For $j = -1$, the proof of the theorem is similar. □

Theorem 3.3. *Let $j \in \{1, -1\}$ be fixed. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.3)*. If the upper bound $\phi : X \times X \to [0, \infty)$ is a mapping such that*

$$\sum_{i=(1+j)/2}^{\infty}\left[\left(\frac{1+j}{2}\right)k^{4ij}\varphi\left(\frac{x}{k^{ij}}, \frac{y}{k^{ij}}\right) + \left(\frac{1-j}{2}\right)2^{3ij}\varphi\left(\frac{x}{2^{ij}}, \frac{y}{2^{ij}}\right)\right] < \infty, \tag{3.39}$$

for all $x, y \in X$, then there exists a unique quartic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ satisfying

$$\left\|f(x) - Q(x) - C(x)\right\| \le \frac{1}{2k^4}\left[\widetilde{\varphi}_e(x) + \widetilde{\varphi}_e(-x)\right] + \frac{1}{4(k^4-k^2)}\left[\widetilde{\varphi}_c(0,x) + \widetilde{\varphi}_c(0,-x)\right] \tag{3.40}$$

for all $x \in X$, where

$$\begin{gathered}\widetilde{\varphi}_e(x) = \sum_{i=(1+j)/2}^{\infty} k^{4ij}\left[\frac{1}{2}\varphi\left(0, \frac{x}{k^{ij}}\right) + \frac{k^4-k^2}{16}\varphi\left(\frac{x}{k^{ij}}, 0\right)\right],\\ \widetilde{\varphi}_c(x,y) = \sum_{i=(1+j)/2}^{\infty} 2^{3nj}\varphi\left(\frac{x}{2^{ij}}, \frac{y}{2^{ij}}\right).\end{gathered} \tag{3.41}$$

Proof. Let $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in X$. Then $f_e(0) = 0$ and f_e is even function satisfying $\|D_{f_e}(x,y)\| \le (1/2)[\phi(x,y) + \phi(-x,-y)]$ for all $x, y \in X$. By Theorem 3.1, there exists a unique quartic function $Q : X \to Y$ satisfying

$$\left\|f_e(x) - Q(x)\right\| \le \frac{1}{2k^4}\left[\widetilde{\varphi}_e(x) + \widetilde{\varphi}_e(-x)\right] \tag{3.42}$$

for all $x \in X$, where

$$\widetilde{\varphi}_e(x) = \sum_{i=(1+j)/2}^{\infty} k^{4ij}\left[\frac{1}{2}\varphi\left(0, \frac{x}{k^{ij}}\right) + \frac{k^4-k^2}{16}\varphi\left(\frac{x}{k^{ij}}, 0\right)\right] \tag{3.43}$$

for all $x \in X$. Let now $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$. Then $f_e(0) = 0$ and f_o is an odd function satisfying $\|D_{f_o}(x,y)\| \le (1/2)[\phi(x,y) + \phi(-x,-y)]$ for all $x, y \in X$. Hence, in view of Theorem 3.2, there exists a unique cubic function $C : X \to Y$ such that

$$\|f_o(x) - Q(x)\| \le \frac{1}{4(k^4 - k^2)} [\tilde{\varphi}_c(0,x) + \tilde{\varphi}_c(0,-x)] \tag{3.44}$$

for all $x \in X$, where

$$\tilde{\varphi}_c(x,y) = \sum_{i=(1+j)/2}^{\infty} 2^{3nj} \varphi\left(\frac{x}{2^{ij}}, \frac{y}{2^{ij}}\right) \tag{3.45}$$

for all $x, y \in X$. On the other hand, we have $f(x) = f_e(x) + f_o(x)$ for all $x \in X$. Then by combining (3.42) and (3.44), it follows that

$$\begin{aligned} \|f(x) - C(x) - Q(x)\| &\le \|f_e(x) - Q(x)\| + \|f_o(x) - C(x)\| \\ &\le \frac{1}{2k^4}[\tilde{\varphi}_e(x) + \tilde{\varphi}_e(-x)] + \frac{1}{4(k^4 - k^2)}[\tilde{\varphi}_c(0,x) + \tilde{\varphi}_c(0,-x)] \end{aligned} \tag{3.46}$$

for all $x \in X$. □

We are going to investigate the Hyers-Ulam-Rassias stability problem for functional equation (1.7).

Corollary 3.4. *Let $p \in (-\infty, 3) \cup (4, +\infty)$, $\theta > 0$. Suppose $f : X \to Y$ satisfies $f(0) = 0$ and inequality*

$$\|D_f(x,y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{3.47}$$

for all $x, y \in X$. Then there exist a unique quartic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ satisfying

$$\|f(x) - Q(x) - C(x)\| \le \begin{cases} \theta\|x\|^p\left(\dfrac{1}{k^4}\left(\left(\dfrac{1}{2} + \dfrac{k^4 - k^2}{16}\right)\left(\dfrac{1}{k^{p-4} - 1}\right)\right) + \dfrac{1}{2(k^4 - k^2)}\left(\dfrac{1}{2^{p-3} - 1}\right)\right), & p > 4, \\ \theta\|x\|^p\left(\dfrac{1}{k^4}\left(\left(\dfrac{1}{2} + \dfrac{k^4 - k^2}{16}\right)\left(\dfrac{1}{1 - k^{p-4}}\right)\right) + \dfrac{1}{2(k^4 - k^2)}\left(\dfrac{1}{1 - 2^{p-3}}\right)\right), & p < 3, \end{cases} \tag{3.48}$$

for all $x \in X$.

Proof. In Theorem 3.3, put $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. □

Similarly, one can solve Ulam stability problem for functional equation (1.7) when the norm of the Cauchy difference is controlled by the mixed type product-sum function

$$(x, y) \longmapsto \theta\left(\|x\|_X^u \|y\|_X^v + \|x\|^p + \|y\|^p\right). \tag{3.49}$$

Corollary 3.5. *Let u, v, p be real numbers such that $u+v, p \in (-\infty, 3) \cup (4, +\infty)$ and $\theta > 0$. Suppose $f : X \to Y$ satisfies $f(0) = 0$ and inequality*

$$\|D_f(x, y)\| \leq \theta\left(\|x\|_X^u \|y\|_X^v + \|x\|^p + \|y\|^p\right) \tag{3.50}$$

for all $x, y \in X$. Then there exist a unique quartic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ satisfying

$$\|f(x) - Q(x) - C(x)\| \\ \leq \begin{cases} \theta\|x\|^p\left(\dfrac{1}{k^4}\left(\left(\dfrac{1}{2} + \dfrac{k^4 - k^2}{16}\right)\left(\dfrac{1}{k^{p-4} - 1}\right)\right) + \dfrac{1}{2(k^4 - k^2)}\left(\dfrac{1}{2^{p-3} - 1}\right)\right), & p > 4, \\ \theta\|x\|^p\left(\dfrac{1}{k^4}\left(\left(\dfrac{1}{2} + \dfrac{k^4 - k^2}{16}\right)\left(\dfrac{1}{1 - k^{p-4}}\right)\right) + \dfrac{1}{2(k^4 - k^2)}\left(\dfrac{1}{1 - 2^{p-3}}\right)\right), & p < 3, \end{cases} \tag{3.51}$$

for all $x \in X$.

Applying Corollary 3.4, one can obtain the stability of the functional equation (1.7) in the following form.

Corollary 3.6. *Let ϵ be a positive real number. Suppose $f : X \to Y$ satisfies $f(0) = 0$ and $\|D_f(x, y)\| \leq \epsilon$ for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ satisfying*

$$\|f(x) - Q(x) - C(x)\| \leq \epsilon\left(\frac{1}{k^4}\left(\left(\frac{1}{2} + \frac{k^4 - k^2}{16}\right)\left(\frac{k^4}{k^4 - 1}\right)\right) + \frac{1}{2(k^4 - k^2)}\left(\frac{8}{8 - 1}\right)\right) \tag{3.52}$$

for all $x \in X$.

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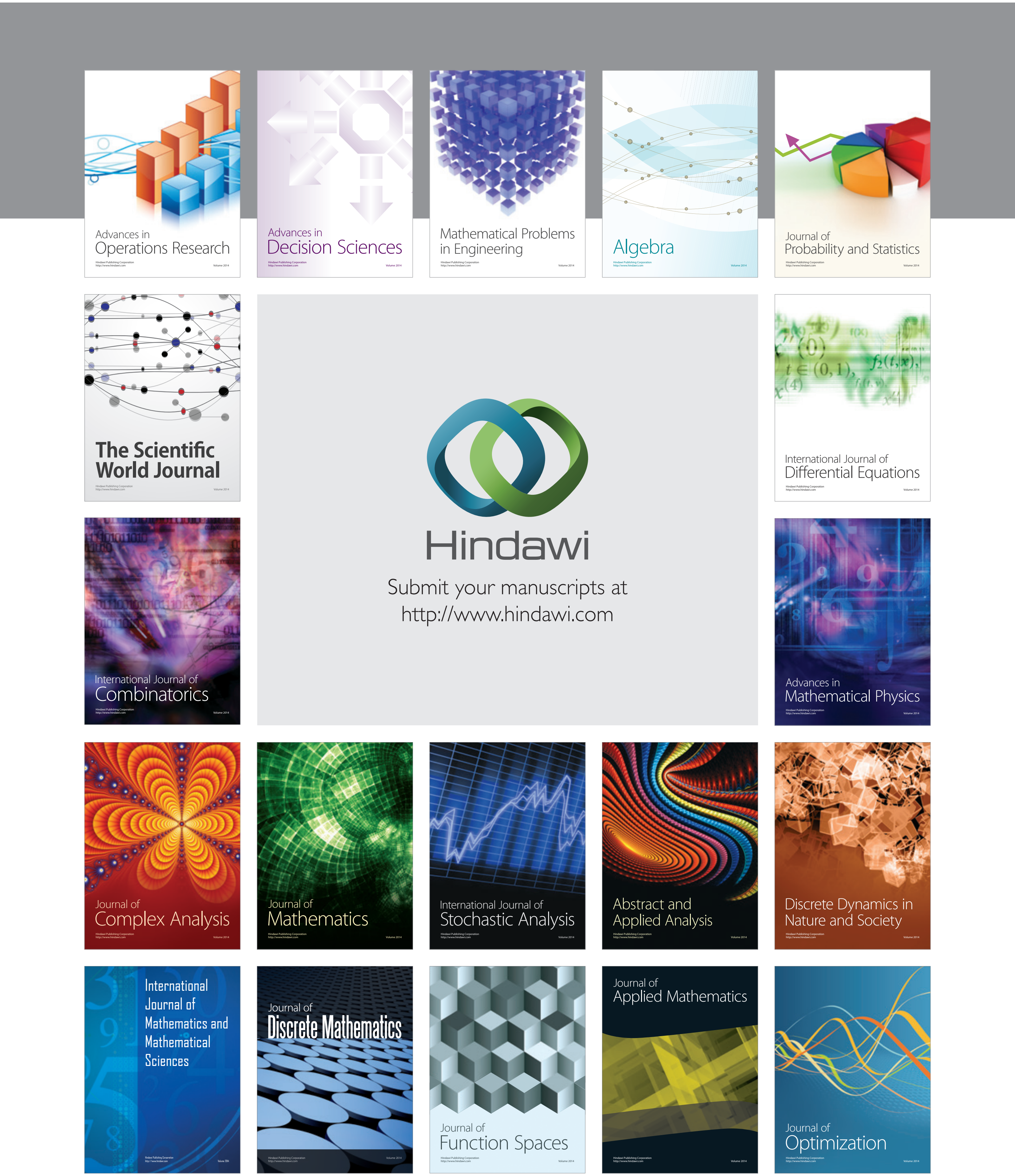
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