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Research Article

On the Convergence of Implicit Iterative Processes for Asymptotically Pseudocontractive Mappings in the Intermediate Sense

Xiaolong Qin,[1,2] Jong Kyu Kim,[3] and Tianze Wang[1]

[1] *School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou 450011, China*
[2] *Department of Mathematics, Hangzhou Normal University, Hangzhou 310036, China*
[3] *Department of Mathematics, Kyungnam University, Masan 631-701, Republic of Korea*

Correspondence should be addressed to Jong Kyu Kim, jongkyuk@kyungnam.ac.kr

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An implicit iterative process is considered. Strong and weak convergence theorems of common fixed points of a finite family of asymptotically pseudocontractive mappings in the intermediate sense are established in a real Hilbert space.

1. Introduction and Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and $T : C \to C$ a mapping. We denote $F(T)$ by the fixed point of the mapping T.

Recall that T is said to be uniformly L-lipschitz if there exists a positive constant L such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall x, y \in C,\ n \ge 1. \tag{1.1}$$

T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\left\|T^n x - T^n y\right\| \leq k_n \left\|x - y\right\|, \quad \forall x, y \in C, \ n \geq 1. \tag{1.3}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. It is known that if C is a nonempty bounded closed convex subset of a Hilbert space space H, then every asymptotically nonexpansive mapping on C has a fixed point. Since 1972, a host of authors have been studing strong and weak convergence problems of the iterative processes for such a class of mappings.

T is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n \to \infty} \sup_{x,y \in C} \left(\left\|T^n x - T^n y\right\| - \left\|x - y\right\|\right) \leq 0. \tag{1.4}$$

Putting

$$\xi_n = \max\left\{0, \sup_{x,y \in C} \left(\left\|T^n x - T^n y\right\| - \left\|x - y\right\|\right)\right\}, \tag{1.5}$$

we see that $\xi_n \to 0$ as $n \to \infty$. Then, (1.4) is reduced to the following:

$$\left\|T^n x - T^n y\right\| \leq \left\|x - y\right\| + \xi_n, \quad \forall x, y \in C, \ n \geq 1. \tag{1.6}$$

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Kirk [2] (see also Bruck et al. [3]) as a generalization of the class of asymptotically nonexpansive mappings. It is known [4] that if C is a nonempty bounded closed convex subset of a Hilbert space space H, then every asymptotically nonexpansive mapping in the intermediate sense on C has a fixed point.

T is said to be strictly pseudocontractive if there exists a constant $\kappa \in [0,1)$ such that

$$\left\|Tx - Ty\right\|^2 \leq \left\|x - y\right\|^2 + \kappa\left\|(I - T)x - (I - T)y\right\|^2, \quad \forall x, y \in C. \tag{1.7}$$

For such a case, T is also said to be a κ-strict pseudocontraction. The class of strict pseudocontractions was introduced by Browder and Petryshyn [5] in 1967. It is clear that every nonexpansive mapping is a 0-strict pseudocontraction.

T is said to be an asymptotically strict pseudocontraction if there exist a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ and a constant $\kappa \in [0,1)$ such that

$$\left\|T^n x - T^n y\right\|^2 \leq k_n \left\|x - y\right\|^2 + \kappa\left\|(I - T^n)x - (I - T^n)y\right\|^2, \quad \forall x, y \in C, \ n \geq 1. \tag{1.8}$$

For such a case, T is also said to be an asymptotically κ-strict pseudocontraction. The class of asymptotically strict pseudocontractions is introduced by Qihou [6] in 1996. It is clear that every asymptotically nonexpansive mapping is an asymptotical 0-strict pseudocontraction.

T is said to be an asymptotically strict pseudocontraction in the intermediate sense if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and a constant $\kappa \in [0, 1)$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^n x - T^n y\|^2 - k_n \|x - y\|^2 - \kappa \|(I - T^n)x - (I - T^n)y\|^2 \right) \leq 0. \tag{1.9}$$

For such a case, T is also said to be an asymptotically κ-strict pseudocontraction in the intermediate sense. Putting

$$\xi_n = \max\left\{ 0, \sup_{x,y\in C} \left(\|T^n x - T^n y\|^2 - k_n \|x - y\|^2 - \kappa \|(I - T^n)x - (I - T^n)y\|^2 \right) \right\}, \tag{1.10}$$

we see that $\xi_n \to 0$ as $n \to \infty$. Then, (1.9) is reduced to the following:

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + \kappa \|(I - T^n)x - (I - T^n)y\|^2 + \xi_n, \quad \forall x, y \in C, \ n \geq 1. \tag{1.11}$$

The class of asymptotically strict pseudocontractions in the intermediate sense was introduced by Sahu et al. [7] as a generalization of the class of asymptotically strict pseudocontractions, see [7] for more details. We also remark that if $k_n = 1$ for each $n \geq 1$ and $\kappa = 0$ in (1.9), then the class of asymptotically κ-strict pseudocontractions in the intermediate sense is reduced to the class of asymptotically nonexpansive mappings in the intermediate sense.

T is said to be pseudocontractive if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C. \tag{1.12}$$

It is easy to see that (1.12) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.13}$$

T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\langle Tx - Ty, x - y \rangle \leq \frac{k_n + 1}{2} \|x - y\|^2, \quad \forall x, y \in C. \tag{1.14}$$

It is easy to see that (1.14) is equivalent to

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + \|(I - T^n)x - (I - T^n)y\|^2, \quad \forall x, y \in C, \ n \geq 1. \tag{1.15}$$

We remark that the class of asymptotically pseudocontractive mappings was introduced by Schu [8] in 1991. For an asymptotically pseudocontractive mapping T, Zhou [9] proved that if

T is also uniformly Lipschitz and uniformly asymptotically regular, then T enjoys a nonempty fixed point set. Moreover, $F(T)$ is closed and convex.

T is said to be an asymptotically pseudocontractive mapping in the intermediate sense if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\| T^n x - T^n y \right\|^2 - k_n \left\| x - y \right\|^2 - \left\| (I - T^n)x - (I - T^n)y \right\|^2 \right) \leq 0. \tag{1.16}$$

It is easy to see that (1.16) is equivalent to

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\langle T^n x - T^n y, x - y \rangle - \frac{k_n + 1}{2} \left\| x - y \right\|^2 \right) \leq 0. \tag{1.17}$$

Put

$$\xi_n = \max\left\{ 0, \sup_{x,y\in C} \left(\left\| T^n x - T^n y \right\|^2 - k_n \left\| x - y \right\|^2 - \left\| (I - T^n)x - (I - T^n)y \right\|^2 \right) \right\}. \tag{1.18}$$

Then, (1.16) is reduced to the following

$$\left\| T^n x - T^n y \right\|^2 \leq k_n \left\| x - y \right\|^2 + \left\| (I - T^n)x - (I - T^n)y \right\|^2 + \xi_n, \quad \forall n \geq 1,\ x, y \in C. \tag{1.19}$$

It is easy to see that (1.19) is equivalent to

$$\langle T^n x - T^n y, x - y \rangle \leq \frac{k_n + 1}{2} \left\| x - y \right\|^2 + \frac{\xi_n}{2}, \quad \forall n \geq 1,\ x, y \in C. \tag{1.20}$$

The class of asymptotically pseudocontractive mappings in the intermediate sense which includes the class of asymptotically pseudocontractive mappings and the class of asymptotically strict pseudocontractions in the intermediate sense as special cases was introduced by Qin et al. [10].

In 2001, Xu and Ori [11], in the framework of Hilbert spaces, introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_1, T_2, \ldots, T_N\}$ with $\{\alpha_n\}$ a real sequence in $(0, 1)$ and an initial point $x_0 \in C$:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1, \\
x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2, \\
&\ \ \vdots \\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1 x_{N+1}, \\
&\ \ \vdots
\end{aligned} \tag{1.21}$$

which can be written in the following compact form

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n, \quad \forall n \geq 1, \tag{1.22}$$

where $T_n = T_{n(\bmod N)}$ (here the mod N takes values in $\{1, 2, \ldots, N\}$).

They obtained the following weak convergence theorem.

Theorem XO. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, and $T : C \to C$ be a finite family of nonexpansive mappings such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by* (1.22). *If $\{\alpha_n\}$ is chosen so that $\alpha_n \to 0$ as $n \to \infty$, then $\{x_n\}$ converges weakly to a common fixed point of the family of $\{T_i\}_{i=1}^{N}$.*

Subsequently, fixed point problems based on implicit iterative processes have been considered by many authors, see [9, 12–23]. In 2004, Osilike [18] considered the implicit iterative process (1.22) for a finite family of strictly pseudocontractive mappings. To be more precise, he proved the following theorem.

Theorem O. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^{N}$ be N strictly pseudocontractive self-maps of C such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $x_0 \in C$ and let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\alpha_n \to 0$ as $n \to \infty$. Then, the sequence $\{x_n\}$ defined by* (1.22) *converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$.*

In 2008, Qin et al. [20] considered the following implicit iterative process for a finite family of asymptotically strict pseudocontractions:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1) T_1 x_1, \\
x_2 &= \alpha_2 x_1 + (1-\alpha_2) T_2 x_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_1^2 x_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N}) T_N^2 x_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1}) T_1^3 x_{2N+1}, \\
&\vdots
\end{aligned} \tag{1.23}$$

where x_0 is an initial value, $\{\alpha_n\}$ is a sequence in $(0,1)$. Since for each $n \geq 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $h = h(n) \geq 1$ is a positive integer and

$h(n) \to \infty$ as $n \to \infty$. Hence, the above table can be rewritten in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{h(n)} x_n, \quad \forall n \geq 1. \tag{1.24}$$

A weak convergence theorem of the implicit iterative process (1.24) for a finite family of asymptotically strict pseudocontractions was established.

We remark that the implicit iterative process (1.24) has been used to study the class of asymptotically pseudocontractive mappings by Osilike and Akuchu [19]. They obtained strong convergence of the implicit iterative process (1.24), however, there is no weak convergence theorem.

In this paper, motivated by the above results, we reconsider the implicit iterative process (1.24) for asymptotically pseudocontractive mappings in the intermediate sense. Strong and weak convergence theorems of common fixed points of a finite family of asymptotically pseudocontractive mappings in the intermediate sense are established. The results presented in this paper mainly improve and extend the corresponding results announced in Chang et al. [24], Chidume and Shahzad [13], Górnicki [25], Osilike [18], Qin et al. [20], Xu and Ori [11], and Zhou and Chang [23].

In order to prove our main results, we need the following conceptions and lemmas.

Recall that a space X is said to satisfy Opial's condition [26] if, for each sequence $\{x_n\}$ in X, the convergence $x_n \to x$ weakly implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \ (y \neq x). \tag{1.25}$$

Recall that a mapping $T : C \to C$ is semicompact if any sequence $\{x_n\}$ in C satisfying $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ has a convergent subsequence.

Lemma 1.1 (see [27]). *In a real Hilbert space, the following inequality holds*

$$\|ax + (1-a)y\|^2 = a\|x\|^2 + (1-a)\|y\|^2 - a(1-a)\|x-y\|^2, \quad \forall a \in [0,1],\ x, y \in H. \tag{1.26}$$

Lemma 1.2 (see [28]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq n_0, \tag{1.27}$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then, the limit $\lim_{n\to\infty} a_n$ exists.

2. Main Results

Theorem 2.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz continuous and asymptotically pseudocontractive mapping in the intermediate sense with the sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for each $1 \le i \le N$, where $N \ge 1$ is some positive integer. Let $\xi_{(n,i)} = \max\{0, \sup_{x,y\in C}(\|T_i^n x - T_i^n y\|^2 - k_{(n,i)}\|x-y\|^2 - \|(I-T_i^n)x-(I-T_i^n)y\|^2)\}$ for each $1 \le i \le N$. Assume that the common fixed point set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in (1.24). Assume that the control sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the following restrictions:*

(a) $0 < 1-1/L < a \le \alpha_n \le b < 1$, *where* $L = \max\{L_i : 1 \le i \le N\}$, *for all* $n \ge 1$;

(b) $\sum_{n=1}^{\infty}\xi_n < \infty$, *where* $\xi_n = \max\{\xi_{(n,i)} : 1 \le i \le N\}$.

Then, $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.

Proof. First, we show that the sequence $\{x_n\}$ generated in the implicit iterative process (1.24) is well defined. Define mappings $R_n : C \to C$ by

$$R_n(x) = \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{h(n)}x, \quad \forall x \in C,\ n \ge 1. \tag{2.1}$$

Notice that

$$\begin{aligned}\|R_n(x)-R_n(y)\| &= \left\|\left(\alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{h(n)}x\right) - \left(\alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{h(n)}y\right)\right\| \\ &\le (1-\alpha_n)L\|x-y\| \\ &\le (1-a)L\|x-y\|, \quad \forall x,y \in C.\end{aligned} \tag{2.2}$$

From the restriction (a), we see that R_n is a contraction for each $n \ge 1$. By Banach contraction principle, we see that there exists a unique fixed point $x_n \in C$ such that

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{h(n)}x_n, \quad \forall n \ge 1. \tag{2.3}$$

This shows that the implicit iterative process (1.24) is well defined for uniformly Lipschitz continuous and asymptotically pseudocontractive mappings in the intermediate sense. Let $k_n = \max\{k_{n,i} : 1 \le i \le N\}$. In view of the assumption, we see that $\sum_{n=1}^{\infty}(k_n-1) < \infty$. Fixing $p \in \mathcal{F}$, we see from Lemma 1.1 that

$$\begin{aligned}\|x_n-p\|^2 &= \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\left\|T_{i(n)}^{h(n)}x_n - p\right\|^2 - \alpha_n(1-\alpha_n)\left\|T_{i(n)}^{h(n)}x_n - x_{n-1}\right\|^2 \\ &\le \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\left(k_{h(n)}\|x_n-p\|^2 + \left\|T_{i(n)}^{h(n)}x_n - x_n\right\|^2 + \xi_n\right) \\ &\quad - \alpha_n(1-\alpha_n)\left\|T_{i(n)}^{h(n)}x_n - x_{n-1}\right\|^2\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \|x_{n-1} - p\|^2 + (1-\alpha_n) k_{h(n)} \|x_n - p\|^2 + (1-\alpha_n) \left\| T_{i(n)}^{h(n)} x_n - x_n \right\|^2 \\
&\quad - \alpha_n (1-\alpha_n) \left\| T_{i(n)}^{h(n)} x_n - x_{n-1} \right\|^2 + \xi_n \\
&\leq \alpha_n \|x_{n-1} - p\|^2 + (1-\alpha_n) k_{h(n)} \|x_n - p\|^2 - (1-\alpha_n)^2 \alpha_n \left\| T_{i(n)}^{h(n)} x_n - x_{n-1} \right\|^2 + \xi_n.
\end{aligned} \tag{2.4}$$

From the restriction (a), we see that there exists some n_0 such that

$$(1-\alpha_n) k_{h(n)} < Q < 1, \quad \forall n \geq n_0, \tag{2.5}$$

where $Q = (1-a)(1 + a/(2(1-a)))$. It follows that

$$\begin{aligned}
\|x_n - p\|^2 &\leq \frac{\alpha_n}{1 - (1-\alpha_n) k_{h(n)}} \|x_{n-1} - p\|^2 + \frac{\xi_n}{1 - (1-\alpha_n) k_{h(n)}} \\
&\leq \left(1 + \frac{k_{h(n)} - 1}{1 - Q} \right) \|x_{n-1} - p\|^2 + \frac{\xi_n}{1-Q}, \quad \forall n \geq n_0.
\end{aligned} \tag{2.6}$$

In view of the restriction (b), we obtain from Lemma 1.2 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Hence, the sequence $\{x_n\}$ is bounded. Reconsidering (2.4), we see from the restriction (a) that

$$(1-b)^2 a \left\| T_{i(n)}^{h(n)} x_n - x_{n-1} \right\|^2 \leq \alpha_n \left(\|x_{n-1} - p\|^2 - \|x_n - p\|^2 \right) + \left(k_{h(n)} - 1 \right) \|x_n - p\|^2 + \xi_n. \tag{2.7}$$

This implies that

$$\lim_{n\to\infty} \left\| T_{i(n)}^{h(n)} x_n - x_{n-1} \right\| = 0. \tag{2.8}$$

Notice that

$$\|x_n - x_{n-1}\| \leq \left\| T_{i(n)}^{h(n)} x_n - x_{n-1} \right\|. \tag{2.9}$$

It follows from (2.8) that

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{2.10}$$

Observe that

$$\begin{aligned}
\left\| x_{n-1} - T_{i(n)}^{h(n)} x_{n-1} \right\| &\leq \left\| x_{n-1} - T_{i(n)}^{h(n)} x_n \right\| + \left\| T_{i(n)}^{h(n)} x_n - T_{i(n)}^{h(n)} x_{n-1} \right\| \\
&\leq \left\| x_{n-1} - T_{i(n)}^{h(n)} x_n \right\| + L \|x_n - x_{n-1}\|.
\end{aligned} \tag{2.11}$$

In view of (2.8), and (2.10), we obtain that

$$\lim_{n\to\infty}\left\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\right\|=0. \tag{2.12}$$

Since for any positive integer $n > N$, it can be written as $n = (h(n) - 1)N + i(n)$, where $i(n) \in \{1, 2, \ldots, N\}$. Observe that

$$\begin{aligned}\|x_{n-1}-T_nx_{n-1}\| &\le \left\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\right\|+\left\|T_{i(n)}^{h(n)}x_{n-1}-T_nx_{n-1}\right\|\\ &\le \left\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\right\|+L\left\|T_{i(n)}^{h(n)-1}x_{n-1}-x_{n-1}\right\|\\ &\le \left\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\right\|\\ &\quad+L\Big(\left\|T_{i(n)}^{h(n)-1}x_{n-1}-T_{i(n-N)}^{h(n)-1}x_{n-N}\right\|+\left\|T_{i(n-N)}^{h(n)-1}x_{n-N}-x_{(n-N)-1}\right\|\\ &\qquad+\left\|x_{(n-N)-1}-x_{n-1}\right\|\Big).\end{aligned} \tag{2.13}$$

Since for each $n > N$, $n = (n - N)(\operatorname{mod} N)$, on the other hand, we obtain from $n = (h(n) - 1)N + i(n)$ that $n - N = ((h(n) - 1) - 1)N + i(n) = (h(n - N) - 1)N + i(n - N)$. That is,

$$h(n-N)=h(n)-1,\qquad i(n-N)=i(n). \tag{2.14}$$

Notice that

$$\begin{aligned}\left\|T_{i(n)}^{h(n)-1}x_{n-1}-T_{i(n-N)}^{h(n)-1}x_{n-N}\right\| &= \left\|T_{i(n)}^{h(n)-1}x_{n-1}-T_{i(n)}^{h(n)-1}x_{n-N}\right\|\\ &\le L\|x_{n-1}-x_{n-N}\|,\\ \left\|T_{i(n-N)}^{h(n)-1}x_{n-N}-x_{(n-N)-1}\right\| &= \left\|T_{i(n-N)}^{h(n-N)}x_{n-N}-x_{(n-N)-1}\right\|.\end{aligned} \tag{2.15}$$

Substituting (2.15) into (2.13), we arrive at

$$\begin{aligned}\|x_{n-1}-T_nx_{n-1}\| &\le \left\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\right\|\\ &\quad+L\Big(L\|x_n-x_{n-N}\|+\left\|T_{i(n-N)}^{h(n-N)}x_{n-N}-x_{(n-N)-1}\right\|+\left\|x_{(n-N)-1}-x_{n-1}\right\|\Big).\end{aligned} \tag{2.16}$$

In view of (2.8), (2.10), and (2.12), we obtain from (2.16) that

$$\lim_{n\to\infty}\|x_{n-1}-T_nx_{n-1}\|=0. \tag{2.17}$$

Notice that

$$\begin{aligned}\|x_n - T_n x_n\| &\le \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_{n-1}\| + \|T_n x_{n-1} - T_n x_n\| \\ &\le (1+L)\|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_{n-1}\|.\end{aligned} \tag{2.18}$$

From (2.10) and (2.17), we arrive at

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{2.19}$$

Notice that

$$\begin{aligned}\left\|x_n - T_{n+j} x_n\right\| &\le \left\|x_n - x_{n+j}\right\| + \left\|x_{n+j} - T_{n+j} x_{n+j}\right\| + \left\|T_{n+j} x_{n+j} - T_{n+j} x_n\right\| \\ &\le (1+L)\left\|x_n - x_{n+j}\right\| + \left\|x_{n+j} - T_{n+j} x_{n+j}\right\|, \quad \forall j \in \{1,2,\ldots,N\}.\end{aligned} \tag{2.20}$$

It follows from (2.10) and (2.19) that

$$\lim_{n\to\infty} \left\|x_n - T_{n+j} x_n\right\| = 0, \quad \forall j \in \{1,2,\ldots,N\}. \tag{2.21}$$

Note that any subsequence of a convergent number sequence converges to the same limit. It follows that

$$\lim_{n\to\infty} \|x_n - T_r x_n\| = 0, \quad \forall r \in \{1,2,\ldots,N\}. \tag{2.22}$$

Since the sequence $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to a point $\overline{x} \in C$. Choose $\alpha \in (0, 1/(1+L))$ and define $y_{\alpha,m,r} = (1-\alpha)\overline{x} + \alpha T_r^m \overline{x}$ for arbitrary but fixed $m \ge 1$. Notice that

$$\begin{aligned}\|x_{n_i} - T_r^m x_{n_i}\| &\le \|x_{n_i} - T_r x_{n_i}\| + \left\|T_r x_{n_i} - T_r^2 x_{n_i}\right\| + \cdots + \left\|T_r^{m-1} x_{n_i} - T_r^m x_{n_i}\right\| \\ &\le [1 + (m-1)L]\|x_{n_i} - T_r x_{n_i}\|, \quad \forall r \in \{1,2,\ldots,N\}.\end{aligned} \tag{2.23}$$

It follows from (2.22) that

$$\lim_{i\to\infty} \|x_{n_i} - T_r^m x_{n_i}\| = 0, \quad \forall r \in \{1,2,\ldots,N\}. \tag{2.24}$$

Note that

$$
\begin{aligned}
&\langle \overline{x} - y_{\alpha,m,r}, y_{\alpha,m,r} - T_r^m y_{\alpha,m,r} \rangle \\
&\quad = \langle \overline{x} - x_{n_i}, y_{\alpha,m,r} - T_r^m y_{\alpha,m} \rangle + \langle x_{n_i} - y_{\alpha,m,r}, y_{\alpha,m,r} - T_r^m y_{\alpha,m,r} \rangle \\
&\quad = \langle \overline{x} - x_{n_i}, y_{\alpha,m,r} - T_r^m y_{\alpha,m,r} \rangle + \langle x_{n_i} - y_{\alpha,m,r}, T_r^m x_{n_i} - T_r^m y_{\alpha,m,r} \rangle \\
&\qquad - \langle x_{n_i} - y_{\alpha,m,r}, x_{n_i} - y_{\alpha,m,r} \rangle + \langle x_{n_i} - y_{\alpha,m,r}, x_{n_i} - T_r^m x_n \rangle \\
&\quad \leq \langle \overline{x} - x_{n_i}, y_{\alpha,m,r} - T_r^m y_{\alpha,m,r} \rangle + \frac{1+k_m}{2} \left\| x_{n_i} - y_{\alpha,m,r} \right\|^2 + \frac{\xi_m}{2} \\
&\qquad - \left\| x_{n_i} - y_{\alpha,m,r} \right\|^2 + \left\| x_{n_i} - y_{\alpha,m,r} \right\| \| x_{n_i} - T_r^m x_n \| \\
&\quad = \langle \overline{x} - x_{n_i}, y_{\alpha,m,r} - T_r^m y_{\alpha,m,r} \rangle + \frac{k_m - 1}{2} \left\| x_{n_i} - y_{\alpha,m,r} \right\|^2 + \frac{\xi_m}{2} \\
&\qquad + \left\| x_{n_i} - y_{\alpha,m,r} \right\| \| x_{n_i} - T_r^m x_{n_i} \|, \quad \forall r \in \{1, 2, \ldots, N\}.
\end{aligned}
\tag{2.25}
$$

Since $x_{n_i} \rightharpoonup \overline{x}$ and (2.24), we arrive at

$$
\langle \overline{x} - y_{\alpha,m}, y_{\alpha,m} - T_r^m y_{\alpha,m} \rangle \leq \frac{k_m - 1}{2} \left\| x_{n_i} - y_{\alpha,m,r} \right\|^2 + \frac{\xi_m}{2}, \quad \forall r \in \{1, 2, \ldots, N\}. \tag{2.26}
$$

On the other hand, we have

$$
\begin{aligned}
&\langle \overline{x} - y_{\alpha,m,r}, (\overline{x} - T_r^m \overline{x}) - (y_{\alpha,m,r} - T_r^m y_{\alpha,m,r}) \rangle \\
&\quad \leq (1+L) \left\| \overline{x} - y_{\alpha,m,r} \right\|^2 = (1+L)\alpha^2 \| \overline{x} - T_r^m \overline{x} \|^2, \quad \forall r \in \{1, 2, \ldots, N\}.
\end{aligned}
\tag{2.27}
$$

Notice that

$$
\begin{aligned}
\| \overline{x} - T_r^m \overline{x} \|^2 &= \langle \overline{x} - T_r^m \overline{x}, \overline{x} - T_r^m \overline{x} \rangle \\
&= \frac{1}{\alpha} \langle \overline{x} - y_{\alpha,m,r}, \overline{x} - T_r^m \overline{x} \rangle \\
&= \frac{1}{\alpha} \langle \overline{x} - y_{\alpha,m,r}, (\overline{x} - T_r^m \overline{x}) - (y_{\alpha,m,r} - T_r^m y_{\alpha,m,r}) \rangle \\
&\quad + \frac{1}{\alpha} \langle \overline{x} - y_{\alpha,m,r}, y_{\alpha,m,r} - T_r^m y_{\alpha,m,r} \rangle, \quad \forall r \in \{1, 2, \ldots, N\}.
\end{aligned}
\tag{2.28}
$$

Substituting (2.26) and (2.27) into (2.28), we arrive at

$$
\alpha[1 - (1+L)\alpha] \| \overline{x} - T_r^m \overline{x} \|^2 \leq \frac{k_m - 1}{2} \left\| x_{n_i} - y_{\alpha,m,r} \right\|^2 + \frac{\xi_m}{2}, \quad \forall r \in \{1, 2, \ldots, N\}, \; m \geq 1. \tag{2.29}
$$

Letting $m \to \infty$ in (2.29), we see that $T_r^m \overline{x} \to \overline{x}$ for each $1 \leq r \leq N$. Since T_r is uniformly L_r-Lipschitz, we can obtain that $\overline{x} = T_r \overline{x}$ for each $1 \leq r \leq N$. This means that $\overline{x} \in F$.

Next we show that $\{x_n\}$ converges weakly to $\overline{x}$. Supposing the contrary, we see that there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $x^* \in C$, where $x^* \neq \overline{x}$. Similarly, we can show $x^* \in \mathcal{F}$. Notice that we have proved that $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in \mathcal{F}$. Assume that $\lim_{n\to\infty}\|x_n - \overline{x}\| = d$ where d is a nonnegative number. By virtue of the Opial property of H, we see that

$$\begin{aligned} d &= \liminf_{n_i \to \infty}\|x_{n_i} - \overline{x}\| < \liminf_{n_i \to \infty}\|x_{n_i} - x^*\| \\ &= \liminf_{n_j \to \infty}\left\|x_{n_j} - x^*\right\| < \liminf_{n_j \to \infty}\left\|x_{n_j} - \overline{x}\right\| = d. \end{aligned} \tag{2.30}$$

This is a contradiction. Hence $\overline{x} = x^*$. This completes the proof. □

For the class of asymptotically pseudocontractive mappings, we have, from Theorem 2.1, the following results immediately.

Corollary 2.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz continuous and asymptotically pseudocontractive mapping with the sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for each $1 \leq i \leq N$, where $N \geq 1$ is some positive integer. Assume that the common fixed point set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in* (1.24)*. Assume that the control sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the following restrictions $0 < 1 - 1/L < a \leq \alpha_n \leq b < 1$, where $L = \max\{L_i : 1 \leq i \leq N\}$, for all $n \geq 1$. Then, $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.*

For the class of asymptotically nonexpansive mappings in the intermediate sense, we can obtain from Theorem 2.1 the following results immediately.

Corollary 2.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz continuous and asymptotically nonexpansive mapping in the intermediate sense for each $1 \leq i \leq N$, where $N \geq 1$ is some positive integer. Let $\xi_{(n,i)} = \max\{0, \sup_{x,y\in C}(\|T_i^n x - T_i^n y\|^2 - \|x-y\|^2)\}$ for each $1 \leq i \leq N$. Assume that the common fixed point set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in* (1.24)*. Assume that the control sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the following restrictions:*

(a) *$0 < 1 - 1/L < a \leq \alpha_n \leq b < 1$, where $L = \max\{L_i : 1 \leq i \leq N\}$, for all $n \geq 1$;*

(b) *$\sum_{n=1}^{\infty} \xi_n < \infty$, where $\xi_n = \max\{\xi_{(n,i)} : 1 \leq i \leq N\}$.*

Then $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.

For the class of asymptotically nonexpansive mappings, we can conclude from Theorem 2.1 the following results immediately.

Corollary 2.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be an asymptotically nonexpansive mapping with the sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for each $1 \leq i \leq N$, where $N \geq 1$ is some positive integer. Assume that the common fixed point set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in* (1.24)*. Assume that the control* *sequence* $\{\alpha_n\}$ *in* $[0,1]$ *satisfies the following restriction* $0 < 1-1/L < a \leq \alpha_n \leq b < 1$*, where* $L = \max\{\sup_{n\geq 1}\{k_{n,i}\} : 1 \leq i \leq N\}$*, for all* $n \geq 1$*. Then,* $\{x_n\}$ *converges weakly to some point in* $\mathcal{F}$*.*

Next, we give strong convergence theorems with the help of semicompactness.

Theorem 2.5. *Let* C *be a nonempty closed convex subset of a Hilbert space* H*. Let* $T_i : C \to C$ *be a uniformly* L_i*-Lipschitz continuous and asymptotically pseudocontractive mapping in the intermediate sense with the sequence* $\{k_{n,i}\} \subset [1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ *for each* $1 \leq i \leq N$*, where* $N \geq 1$ *is some positive integer. Let* $\xi_{(n,i)} = \max\{0, \sup_{x,y\in C}(\|T_i^n x - T_i^n y\|^2 - k_{(n,i)}\|x-y\|^2 - \|(I-T_i^n)x-(I-T_i^n)y\|^2)\}$ *for each* $1 \leq i \leq N$*. Assume that the common fixed point set* $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ *is nonempty. Let* $\{x_n\}_{n=0}^{\infty}$ *be a sequence generated in* (1.24)*. Assume that the control sequence* $\{\alpha_n\}$ *in* $[0,1]$ *satisfies the following restrictions:*

(a) $0 < 1-1/L < a \leq \alpha_n \leq b < 1$*, where* $L = \max\{L_i : 1 \leq i \leq N\}$*, for all* $n \geq 1$*;*

(b) $\sum_{n=1}^{\infty}\xi_n < \infty$*, where* $\xi_n = \max\{\xi_{(n,i)} : 1 \leq i \leq N\}$*.*

If one of $\{T_1, T_2, \ldots, T_N\}$ *is semicompact, then the sequence* $\{x_n\}$ *converges strongly to some point in* $\mathcal{F}$*.*

Proof. Without loss of generality, we may assume that T_1 is semicompact. From (2.22), we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges strongly to $x \in C$. For each $r \in \{1,2,\ldots,N\}$, we get that

$$\|x - T_r x\| \leq \|x - x_{n_i}\| + \|x_{n_i} - T_r x_{n_i}\| + \|T_r x_{n_i} - T_r x\|. \tag{2.31}$$

Since T_r is Lipschitz continuous, we obtain from (2.22) that $x \in \bigcap_{r=1}^{N} F(T_r) = \mathcal{F}$. In view of Theorem 2.1, we obtain that $\lim_{n\to\infty}\|x_n - x\|$ exists. Therefore, we can obtain the desired conclusion immediately. □

For the class of asymptotically pseudocontractive mappings, we have from Theorem 2.5 the following results immediately.

Corollary 2.6. *Let* C *be a nonempty closed convex subset of a Hilbert space* H*. Let* $T_i : C \to C$ *be a uniformly* L_i*-Lipschitz continuous and asymptotically pseudocontractive mapping with the sequence* $\{k_{n,i}\} \subset [1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ *for each* $1 \leq i \leq N$*, where* $N \geq 1$ *is some positive integer. Assume that the common fixed point set* $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ *is nonempty. Let* $\{x_n\}_{n=0}^{\infty}$ *be a sequence generated in* (1.24)*. Assume that the control sequence* $\{\alpha_n\}$ *in* $[0,1]$ *satisfies the following restrictions:* $0 < 1-1/L < a \leq \alpha_n \leq b < 1$*, where* $L = \max\{L_i : 1 \leq i \leq N\}$*, for all* $n \geq 1$*. If one of* $\{T_1, T_2, \ldots, T_N\}$ *is semicompact, then the sequence* $\{x_n\}$ *converges strongly to some point in* $F\mathcal{F}$*.*

For the class of asymptotically nonexpansive mappings in the intermediate sense, we can obtain from Theorem 2.5 the following results immediately.

Corollary 2.7. *Let* C *be a nonempty closed convex subset of a Hilbert space* H*. Let* $T_i : C \to C$ *be a uniformly* L_i*-Lipschitz continuous and asymptotically nonexpansive mapping in the intermediate sense for each* $1 \leq i \leq N$*, where* $N \geq 1$ *is some positive integer. Let* $\xi_{(n,i)} = \max\{0, \sup_{x,y\in C}(\|T_i^n x - T_i^n y\|^2 - \|x-y\|^2)\}$ *for each* $1 \leq i \leq N$*. Assume that the common fixed* *point set* $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ *is nonempty. Let* $\{x_n\}_{n=0}^{\infty}$ *be a sequence generated in* (1.24). *Assume that the control sequence* $\{\alpha_n\}$ *in* $[0,1]$ *satisfies the following restrictions:*

(a) $0 < 1 - 1/L < a \leq \alpha_n \leq b < 1$, *where* $L = \max\{L_i : 1 \leq i \leq N\}$, *for all* $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \xi_n < \infty$, *where* $\xi_n = \max\{\xi_{(n,i)} : 1 \leq i \leq N\}$.

If one of $\{T_1, T_2, \ldots, T_N\}$ *is semicompact, then the sequence* $\{x_n\}$ *converges strongly to some point in* $\mathcal{F}$.

For the class of asymptotically nonexpansive mappings, we can conclude from Theorem 2.5 the following results immediately.

Corollary 2.8. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* $T_i : C \to C$ *be an asymptotically nonexpansive mapping with the sequence* $\{k_{n,i}\} \subset [1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ *for each* $1 \leq i \leq N$, *where* $N \geq 1$ *is some positive integer. Assume that the common fixed point set* $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ *is nonempty. Let* $\{x_n\}_{n=0}^{\infty}$ *be a sequence generated in* (1.24). *Assume that the control sequence* $\{\alpha_n\}$ *in* $[0,1]$ *satisfies the following restriction:* $0 < 1 - 1/L < a \leq \alpha_n \leq b < 1$, *where* $L = \max\{\sup_{n\geq 1}\{k_{n,i}\} : 1 \leq i \leq N\}$, *for all* $n \geq 1$. *If one of* $\{T_1, T_2, \ldots, T_N\}$ *is semicompact, then the sequence* $\{x_n\}$ *converges strongly to some point in* $\mathcal{F}$.

In 2005, Chidume and Shahzad [13] introduced the following conception. Recall that a family $\{T_i\}_{i=1}^{N} : C \to C$ with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ is said to satisfy Condition (B) on C if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(m) > 0$ for all $m \in (0,\infty)$ such that for all $x \in C$

$$\max_{1\leq i\leq N}\{\|x - T_i x\|\} \geq f(d(x,\mathcal{F})). \tag{2.32}$$

Next, we give strong convergence theorems with the help of Condition (B).

Theorem 2.9. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* $T_i : C \to C$ *be a uniformly* L_i*-Lipschitz continuous and asymptotically pseudocontractive mapping in the intermediate sense with the sequence* $\{k_{n,i}\} \subset [1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_{n,i} - 1) < \infty$ *for each* $1 \leq i \leq N$, *where* $N \geq 1$ *is some positive integer. Let* $\xi_{(n,i)} = \max\{0, \sup_{x,y\in C}(\|T_i^n x - T_i^n y\|^2 - k_{(n,i)}\|x - y\|^2 - \|(I - T_i^n)x - (I - T_i^n)y\|^2)\}$ *for each* $1 \leq i \leq N$. *Assume that the common fixed point set* $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ *is nonempty. Let* $\{x_n\}_{n=0}^{\infty}$ *be a sequence generated in* (1.24). *Assume that the control sequence* $\{\alpha_n\}$ *in* $[0,1]$ *satisfies the following restrictions:*

(a) $0 < 1 - 1/L < a \leq \alpha_n \leq b < 1$, *where* $L = \max\{L_i : 1 \leq i \leq N\}$, *for all* $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \xi_n < \infty$, *where* $\xi_n = \max\{\xi_{(n,i)} : 1 \leq i \leq N\}$.

If $\{T_1, T_2, \ldots, T_N\}$ *satisfies Condition* (B), *then the sequence* $\{x_n\}$ *converges strongly to some point in* $\mathcal{F}$.

Proof. In view of Condition (B), we obtain from (2.22) that $f(d(x_n,\mathcal{F})) \to 0$, which implies $d(x_n,\mathcal{F}) \to 0$. Next, we show that the sequence $\{x_n\}$ is Cauchy. In view of (2.6), for any positive integers m, n, where $m > n > n_0$, we obtain that

$$\|x_m - p\| \leq B\|x_n - p\| + B\sum_{i=n+1}^{\infty}\frac{\xi_i}{1-Q} + \frac{\xi_m}{1-Q}, \tag{2.33}$$

where $B = \exp\{\sum_{n=1}^{\infty}((k_{h(n)} - 1)/(1-Q))\}$. It follows that

$$\|x_n - x_m\| \le \|x_n - p\| + \|x_m - p\| \le (1+B)\|x_n - p\| + B\sum_{i=n+1}^{\infty}\frac{\xi_i}{1-Q} + \frac{\xi_m}{1-Q}. \tag{2.34}$$

It follows that $\{x_n\}$ is a Cauchy sequence in C,so $\{x_n\}$ converges strongly to some $\overline{q} \in C$. Since T_r is Lipschitz for each $r \in \{1,2,\ldots,N\}$, we see that $\mathcal{F}$ is closed. This in turn implies that $\overline{q} \in \mathcal{F}$. This completes the proof. □

For the class of asymptotically pseudocontractive mappings, we have from Theorem 2.9 the following results immediately.

Corollary 2.10. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz continuous and asymptotically pseudocontractive mapping with the sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for each $1 \le i \le N$, where $N \ge 1$ is some positive integer. Assume that the common fixed point set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in* (1.24)*. Assume that the control sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the following restrictions: $0 < 1 - 1/L < a \le \alpha_n \le b < 1$, where $L = \max\{L_i : 1 \le i \le N\}$, for all $n \ge 1$. If $\{T_1, T_2, \ldots, T_N\}$ satisfies Condition (B), then the sequence $\{x_n\}$ converges strongly to some point in $\mathcal{F}$.*

For the class of asymptotically nonexpansive mappings in the intermediate sense, we can obtain from Theorem 2.9 the following results immediately.

Corollary 2.11. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz continuous and asymptotically nonexpansive mapping in the intermediate sense for each $1 \le i \le N$, where $N \ge 1$ is some positive integer. Let $\xi_{(n,i)} = \max\{0, \sup_{x,y\in C}(\|T_i^n x - T_i^n y\|^2 - \|x-y\|^2)\}$ for each $1 \le i \le N$. Assume that the common fixed point set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in* (1.24)*. Assume that the control sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the following restrictions:*

(a) *$0 < 1 - 1/L < a \le \alpha_n \le b < 1$, where $L = \max\{L_i : 1 \le i \le N\}$, for all $n \ge 1$;*

(b) *$\sum_{n=1}^{\infty}\xi_n < \infty$, where $\xi_n = \max\{\xi_{(n,i)} : 1 \le i \le N\}$.*

If $\{T_1, T_2, \ldots, T_N\}$ satisfies Condition (B), then the sequence $\{x_n\}$ converges strongly to some point in $\mathcal{F}$.

For the class of asymptotically nonexpansive mappings, we can conclude from Theorem 2.9 the following results immediately.

Corollary 2.12. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be an asymptotically nonexpansive mapping with the sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for each $1 \le i \le N$, where $N \ge 1$ is some positive integer. Assume that the common fixed point set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in* (1.24)*. Assume that the control sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the following restriction: $0 < 1 - 1/L < a \le \alpha_n \le b < 1$, where $L = \max\{\sup_{n\ge 1}\{k_{n,i}\} : 1 \le i \le N\}$, for all $n \ge 1$. If $\{T_1, T_2, \ldots, T_N\}$ satisfies Condition (B), then the sequence $\{x_n\}$ converges strongly to some point in $\mathcal{F}$.*

Finally, we give the following strong convergence criteria.

Theorem 2.13. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz continuous and asymptotically pseudocontractive mapping in the intermediate sense with the sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for each $1 \le i \le N$, where $N \ge 1$ is some positive integer. Let $\xi_{(n,i)} = \max\{0, \sup_{x,y\in C}(\|T_i^n x - T_i^n y\|^2 - k_{(n,i)}\|x-y\|^2 - \|(I-T_i^n)x-(I-T_i^n)y\|^2)\}$ for each $1 \le i \le N$. Assume that the common fixed point set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in* (1.24). *Assume that the control sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the following restrictions:*

(a) $0 < 1-1/L < a \le \alpha_n \le b < 1$, *where* $L = \max\{L_i : 1 \le i \le N\}$, *for all* $n \ge 1$;

(b) $\sum_{n=1}^{\infty}\xi_n < \infty$, *where* $\xi_n = \max\{\xi_{(n,i)} : 1 \le i \le N\}$.

Then, the sequence $\{x_n\}$ converges strongly to some point in $\mathcal{F}$ if and only if $\liminf_{n\to\infty} d(x_n,\mathcal{F}) = 0$.

Proof. The necessity is obvious. We only show the sufficiency. Assume that

$$\liminf_{n\to\infty} d(x_n,\mathcal{F}) = 0. \tag{2.35}$$

In view of Lemma 1.2, we can obtain from (2.6) that $\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0$. The desired results can be obtain from Theorem 2.9 immediately. □

For the class of asymptotically pseudocontractive mappings, we have from Theorem 2.13 the following results immediately.

Corollary 2.14. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz continuous and asymptotically pseudocontractive mapping with the sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for each $1 \le i \le N$, where $N \ge 1$ is some positive integer. Assume that the common fixed point set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in* (1.24). *Assume that the control sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the following restrictions $0 < 1-1/L < a \le \alpha_n \le b < 1$, where $L = \max\{L_i : 1 \le i \le N\}$, for all $n \ge 1$. Then, the sequence $\{x_n\}$ converges strongly to some point in $\mathcal{F}$ if and only if* $\liminf_{n\to\infty} d(x_n,\mathcal{F}) = 0$.

For the class of asymptotically nonexpansive mappings in the intermediate sense, we can obtain from Theorem 2.13 the following results immediately.

Corollary 2.15. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz continuous and asymptotically nonexpansive mapping in the intermediate sense for each $1 \le i \le N$, where $N \ge 1$ is some positive integer. Let $\xi_{(n,i)} = \max\{0, \sup_{x,y\in C}(\|T_i^n x - T_i^n y\|^2 - \|x-y\|^2)\}$ for each $1 \le i \le N$. Assume that the common fixed point set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in* (1.24). *Assume that the control sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the following restrictions:*

(a) $0 < 1-1/L < a \le \alpha_n \le b < 1$, *where* $L = \max\{L_i : 1 \le i \le N\}$, *for all* $n \ge 1$;

(b) $\sum_{n=1}^{\infty}\xi_n < \infty$, *where* $\xi_n = \max\{\xi_{(n,i)} : 1 \le i \le N\}$.

Then, the sequence $\{x_n\}$ converges strongly to some point in $\mathcal{F}$ if and only if $\liminf_{n\to\infty} d(x_n,\mathcal{F}) = 0$.

For the class of asymptotically nonexpansive mappings, we can conclude from Theorem 2.13 the following results immediately.

Corollary 2.16. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to C$ be an asymptotically nonexpansive mapping with the sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for each $1 \leq i \leq N$, where $N \geq 1$ is some positive integer. Assume that the common fixed point set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in* (1.24). *Assume that the control sequence $\{\alpha_n\}$ in $[0,1]$ satisfies the following restriction: $0 < 1 - 1/L < a \leq \alpha_n \leq b < 1$, where $L = \max\{\sup_{n\geq 1}\{k_{n,i}\} : 1 \leq i \leq N\}$, for all $n \geq 1$. Then, the sequence $\{x_n\}$ converges strongly to some point in $\mathcal{F}$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

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References

[1] K. Goebel and W. A. Kirk, "A fixed point theorem for asymptotically nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 35, pp. 171–174, 1972.

[2] W. A. Kirk, "Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type," *Israel Journal of Mathematics*, vol. 17, pp. 339–346, 1974.

[3] R. Bruck, T. Kuczumow, and S. Reich, "Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property," *Colloquium Mathematicum*, vol. 65, no. 2, pp. 169–179, 1993.

[4] H. K. Xu, "Existence and convergence for fixed points of mappings of asymptotically nonexpansive type," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 16, no. 12, pp. 1139–1146, 1991.

[5] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[6] L. Qihou, "Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 26, no. 11, pp. 1835–1842, 1996.

[7] D. R. Sahu, H.-K. Xu, and J.-C. Yao, "Asymptotically strict pseudocontractive mappings in the intermediate sense," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 10, pp. 3502–3511, 2009.

[8] J. Schu, "Iterative construction of fixed points of asymptotically nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 158, no. 2, pp. 407–413, 1991.

[9] H. Zhou, "Demiclosedness principle with applications for asymptotically pseudo-contractions in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 9, pp. 3140–3145, 2009.

[10] X. Qin, S. Y. Cho, and J. K. Kim, "Convergence theorems on asymptotically pseudocontractive mappings in the intermediate sense," *Fixed Point Theory and Applications*, vol. 2010, Article ID 186874, 14 pages, 2010.

[11] H.-K. Xu and R. G. Ori, "An implicit iteration process for nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 22, no. 5-6, pp. 767–773, 2001.

[12] L. C. Ceng, D. R. Sahu, and J. C. Yao, "Implicit iterative algorithms for asymptotically nonexpansive mappings in the intermediate sense and Lipschitz-continuous monotone mappings," *Journal of Computational and Applied Mathematics*, vol. 233, no. 11, pp. 2902–2915, 2010.

[13] C. E. Chidume and N. Shahzad, "Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 62, no. 6, pp. 1149–1156, 2005.

[14] F. Cianciaruso, G. Marino, and X. Wang, "Weak and strong convergence of the Ishikawa iterative process for a finite family of asymptotically nonexpansive mappings," *Applied Mathematics and Computation*, vol. 216, no. 12, pp. 3558–3567, 2010.

[15] W. Guo and Y. J. Cho, "On the strong convergence of the implicit iterative processes with errors for a finite family of asymptotically nonexpansive mappings," *Applied Mathematics Letters*, vol. 21, no. 10, pp. 1046–1052, 2008.

[16] S. H. Khan, I. Yildirim, and M. Ozdemir, "Convergence of an implicit algorithm for two families of nonexpansive mappings," *Computers & Mathematics with Applications*, vol. 59, no. 9, pp. 3084–3091, 2010.

[17] J. K. Kim, Y. M. Nam, and J. Y. Sim, "Convergence theorems of implicit iterative sequences for a finite family of asymptotically quasi-nonxpansive type mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. e2839–e2848, 2009.

[18] M. O. Osilike, "Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps," *Journal of Mathematical Analysis and Applications*, vol. 294, no. 1, pp. 73–81, 2004.

[19] M. O. Osilike and B. G. Akuchu, "Common fixed points of a finite family of asymptotically pseudocontractive maps," *Fixed Point Theory and Applications*, vol. 2004, no. 2, pp. 81–88, 2004.

[20] X. Qin, Y. Su, and M. Shang, "On the convergence of implicit iteration process for a finite family of k-strictly asymptotically pseudocontractive mappings," *Kochi Journal of Mathematics*, vol. 3, pp. 67–76, 2008.

[21] X. Qin, S. M. Kang, and R. P. Agarwal, "On the convergence of an implicit iterative process for generalized asymptotically quasi-nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2010, Article ID 714860, 19 pages, 2010.

[22] X. Qin, Y. J. Cho, and M. Shang, "Convergence analysis of implicit iterative algorithms for asymptotically nonexpansive mappings," *Applied Mathematics and Computation*, vol. 210, no. 2, pp. 542–550, 2009.

[23] Y. Zhou and S.-S. Chang, "Convergence of implicit iteration process for a finite family of asymptotically nonexpansive mappings in Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 23, no. 7-8, pp. 911–921, 2002.

[24] S. S. Chang, Y. J. Cho, and H. Zhou, "Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings," *Journal of the Korean Mathematical Society*, vol. 38, no. 6, pp. 1245–1260, 2001.

[25] J. Górnicki, "Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces," *Commentationes Mathematicae Universitatis Carolinae*, vol. 30, no. 2, pp. 249–252, 1989.

[26] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.

[27] J. Reinermann, "Über Fixpunkte kontrahierender Abbildungen und schwach konvergente Toeplitz-Verfahren," *Archiv der Mathematik*, vol. 20, pp. 59–64, 1969.

[28] K.-K. Tan and H. K. Xu, "Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process," *Journal of Mathematical Analysis and Applications*, vol. 178, no. 2, pp. 301–308, 1993.