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*Research Article*

# A Note on the Modified $q$-Bernoulli Numbers and Polynomials with Weight $\alpha$

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A systemic study of some families of the modified $q$-Bernoulli numbers and polynomials with weight $\alpha$ is presented by using the $p$-adic $q$-integration $\mathbb{Z}_p$. The study of these numbers and polynomials yields an interesting $q$-analogue related to Bernoulli numbers and polynomials.

## 1. Introduction

Let $p$ be a fixed prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

The $q$-number $[x]_q$ is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \tag{1.1}$$

see [1–10].

We say that $f$ is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_f(x,y) = \frac{f(x) - f(y)}{x - y} \tag{1.2}$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$. c.f. [11].

For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{f \mid f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable functions$\}$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \tag{1.3}$$

(see [3]).

From (1.3), we can easily derive the following:

$$q^n I_q(f_n) = I_q(f) + (q-1)\sum_{l=0}^{n-1} f(l) q^l + \frac{q-1}{\log q} \sum_{l=0}^{n-1} q^l f'(l), \tag{1.4}$$

where $f_n(x) = f(x+n)$, (see [5, 12]).
In [1, 2], Carlitz defined a set of numbers $B_{k,q}$ inductively by

$$B_{0,q} = 1, \qquad (qB_q + 1)^k - B_{k,\,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.5}$$

with the usual convention about replacing $B_q^k$ by $B_{k,q}$.
These numbers are the $q$-extension of ordinary Bernoulli numbers. But they do not remain finite when $q = 1$. So, Carlitz modified (1.5) as follows:

$$\beta_{0,q} = 1, \qquad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.6}$$

with the usual convention of replacing $\beta^k$ by $\beta_{k,q}$.
In [1], Carlitz also considered the extension of Carlitz's $q$-Bernoulli numbers as follows:

$$\beta_{0,q}^h = \frac{h}{[h]_q}, \qquad q^h\left(q\beta^h + 1\right)^k - \beta_{k,q}^h = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.7}$$

with the usual convention of replacing $(\beta^h)^k$ by $\beta_{k,q}^h$.

In this paper, we construct the modified $q$-Bernoulli numbers with weight $\alpha$, which are different Carlitz's $q$-Bernoulli numbers, by using $p$-adic $q$-integral equation. From (1.4), we derive some interesting identities and relations on the modified $q$-Bernoulli numbers and polynomials.

## 2. The Modified $q$-Bernoulli Numbers and Polynomials with Weight $\alpha$

In this section, we assume $\alpha \in \mathbb{Q}$. Now, we define the modified $q$-Bernoulli numbers with weight $\alpha$ (= $\widetilde{B}_{n,q}^{(\alpha)}$ ) as follows:

$$\begin{aligned}\widetilde{B}_{n,q}^{(\alpha)} &= \int_{\mathbb{Z}_p} q^{-x}[x]_{q^\alpha}^n \, d\mu_q(x) \\ &= \frac{1}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{\alpha l}{[\alpha l]_q}.\end{aligned} \tag{2.1}$$

Thus, by (2.1), we have

$$\widetilde{B}_{n,q}^{(\alpha)} = \frac{1}{(1-q^\alpha)^n}\frac{q-1}{\log q} - n\frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty} q^{\alpha m}[m]_{q^\alpha}^{n-1}. \tag{2.2}$$

Therefore, by (2.1) and (2.2), we obtain the following theorem.

**Theorem 2.1.** *For $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, one has*

$$\begin{aligned}\widetilde{B}_{n,q}^{(\alpha)} &= \frac{\alpha}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{l}{[\alpha l]_q} \\ &= \frac{1}{(1-q^\alpha)^n}\frac{q-1}{\log q} - n\frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty} q^{\alpha m}[m]_{q^\alpha}^{n-1}.\end{aligned} \tag{2.3}$$

Let us define the generating function of the modified $q$-Bernoulli numbers with weight $\alpha$ as follows:

$$F_q^{(\alpha)}(t) = \sum_{n=0}^{\infty}\widetilde{B}_{n,q}^{(\alpha)}\frac{t^n}{n!}. \tag{2.4}$$

Then, by (2.3) and (2.4), we get

$$F_q^{(\alpha)}(t) = \frac{q-1}{\log q}e^{(1/(1-q^\alpha))t} - t\frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty} q^{\alpha m}e^{[m]_{q^\alpha}t}. \tag{2.5}$$

In the viewpoint of (2.1), we define the modified $q$-Bernoulli numbers with weight $\alpha$ as follows:

$$\begin{aligned}\widetilde{B}_{n,q}^{(\alpha)}(x) &= \int_{\mathbb{Z}_p} q^{-y}[x+y]_{q^\alpha}^n \, d\mu_q(y) \\ &= \sum_{l=0}^{n}\binom{n}{l}[x]_{q^\alpha}^{n-l} q^{\alpha l x}\widetilde{B}_{l,q}^{(\alpha)} \\ &= \left([x]_{q^\alpha} + q^{\alpha x}\widetilde{B}_{q}^{(\alpha)}\right)^n, \quad \text{for } n \in \mathbb{Z}_+,\end{aligned} \tag{2.6}$$

with the usual convention of replacing $(\widetilde{B}_q^{(\alpha)})^n$ by $\widetilde{B}_{n,q}^{(\alpha)}$.
From (2.6), we note that

$$\begin{aligned}\widetilde{B}_{n,q}^{(\alpha)}(x) &= \frac{\alpha}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{l}{[\alpha l]_q} \\ &= \frac{1}{(1-q^\alpha)^n}\frac{q-1}{\log q} - n\frac{\alpha}{[\alpha]_q}q^{\alpha x}\sum_{m=0}^{\infty} q^{\alpha m}[m+x]_{q^\alpha}^{n-1}.\end{aligned} \tag{2.7}$$

Therefore, by (2.7), we obtain the following theorem.

**Theorem 2.2.** *For $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned}\widetilde{B}_{n,q}^{(\alpha)}(x) &= \frac{\alpha}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{l}{[\alpha l]_q} \\ &= \frac{1}{(1-q^\alpha)^n}\frac{q-1}{\log q} - n\frac{\alpha}{[\alpha]_q}q^{\alpha x}\sum_{m=0}^{\infty} q^{\alpha m}[m+x]_{q^\alpha}^{n-1}.\end{aligned} \tag{2.8}$$

Let $F_q^{(\alpha)}(t,x) = \sum_{n=0}^{\infty}\widetilde{B}_{n,q}^{(\alpha)}(x)(t^n/n!)$ be the generating function of the modified $q$-Bernoulli polynomials with weight $\alpha$.
Then, by (2.7), we get

$$F_q^{(\alpha)}(t,x) = \frac{q-1}{\log q}e^{(1/(1-q^\alpha))t} - t\frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty} q^{\alpha(m+x)}e^{[m+x]_{q^\alpha}t}. \tag{2.9}$$

Therefore, by (2.9), we obtain the following corollary

**Corollary 2.3.** *Let $F_q^{(\alpha)}(t,x) = \sum_{n=0}^{\infty}\widetilde{B}_{n,q}^{(\alpha)}(x)(t^n/n!)$. Then one has*

$$F_q^{(\alpha)}(t,x) = \frac{q-1}{\log q}e^{(1/(1-q^\alpha))t} - t\frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty} q^{\alpha(m+x)}e^{[m+x]_{q^\alpha}t}. \tag{2.10}$$

In particular, $F_q^{(\alpha)}(t,0) = F_q^{(\alpha)}(t)$.

From Corollary 2.3, we can derive the following equation:

$$F_q^{\alpha}(t,1) - F_q^{\alpha}(t) = t\frac{\alpha}{[\alpha]_q}. \tag{2.11}$$

By (2.5) and (2.11), we get

$$\widetilde{B}_{0,q}^{(\alpha)} = \frac{q-1}{\log q}, \qquad \widetilde{B}_{n,q}^{(\alpha)}(1) - \widetilde{B}_{n,q}^{(\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

**Theorem 2.4.** *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{0,q}^{(\alpha)} = \frac{q-1}{\log q}, \qquad \widetilde{B}_{n,q}^{(\alpha)}(1) - \widetilde{B}_{n,q}^{(\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q}, & \textit{if } n = 1, \\ 0, & \textit{if } n > 1. \end{cases} \tag{2.13}$$

By using (2.6), we obtain the following corollary.

**Corollary 2.5.** *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{0,q}^{(\alpha)} = \frac{q-1}{\log q}, \qquad \left(q^{\alpha}\widetilde{B}_q^{(\alpha)} + 1\right)^n - \widetilde{B}_{n,q}^{(\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q}, & \textit{if } n = 1, \\ 0, & \textit{if } n > 1. \end{cases} \tag{2.14}$$

*with the usual convention of replacing $(\widetilde{B}_q^{(\alpha)})^n$ by $\widetilde{B}_{n,q}^{(\alpha)}$.*

From (1.4), we can derive the following equation:

$$\int_{\mathbb{Z}_p} f(x+n)q^{-x}\,d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)q^{-x}\,d\mu_q(x) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} f'(l). \tag{2.15}$$

Thus, by (1.6), (2.6), and (2.15), we get

$$\widetilde{B}_{m,q}^{(\alpha)}(n) - \widetilde{B}_{m,q}^{(\alpha)} = \frac{\alpha}{[\alpha]_q} m \sum_{l=0}^{n-1} [l]_{q^{\alpha}}^{m-1} q^{\alpha l}, \quad n \in \mathbb{N},\ m \in \mathbb{Z}_+. \tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

**Theorem 2.6.** *For $n \in \mathbb{N}, m \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{m,q}^{(\alpha)}(n) - \widetilde{B}_{m,q}^{(\alpha)} = \frac{\alpha}{[\alpha]_q} m \sum_{l=0}^{n-1} [l]_{q^{\alpha}}^{m-1} q^{\alpha l}. \tag{2.17}$$

From (2.6), we note that

$$\begin{aligned}\widetilde{B}_{n,q}^{(\alpha)}(x) &= \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n q^{-y}\, d\mu_q(y) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{y=0}^{p^N-1} [x+y]_{q^\alpha}^n \\ &= \frac{1-q}{1-q^d} \sum_{a=0}^{d-1} \lim_{N\to\infty} \frac{1}{[p^N]_{q^d}} \sum_{y=0}^{p^N-1} [a+x+dy]_{q^\alpha}^n \\ &= \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} \int_{\mathbb{Z}_p} \left[\frac{a+x}{d}+y\right]_{q^{\alpha d}}^n q^{-dy}\, d\mu_{q^d}(y) \\ &= \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} \widetilde{B}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right).\end{aligned} \tag{2.18}$$

Therefore, by (2.18), we obtain the following distribution relation for the modified $q$-Bernoulli polynomials with weight $\alpha$.

**Theorem 2.7.** *For $d \in \mathbb{N}, n \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{n,q}^{(\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} \widetilde{B}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right). \tag{2.19}$$

To derive the relation of reflection symmetry of the modified $q$-Bernoulli polynomials with weight $\alpha$, we evaluate the following $p$-adic $q$-integral on $\mathbb{Z}_p$:

$$\begin{aligned}\widetilde{B}_{n,q^{-1}}^{(\alpha)}(1-x) &= \int_{\mathbb{Z}_p} [1-x+x_1]_{q^{-\alpha}}^n q^{x_1}\, d\mu_{q^{-1}}(x_1) \\ &= \frac{1}{(1-q^{-\alpha})^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x-1} \frac{\alpha l}{[\alpha l]_q} \\ &= \frac{(-1)^n}{q} \frac{q^{\alpha n}}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\alpha l}{[\alpha l]_q} \\ &= q^{\alpha n-1}(-1)^n \widetilde{B}_{n,q}^{(\alpha)}(x).\end{aligned} \tag{2.20}$$

Therefore, by (2.20), we obtain the following reflection symmetry relation of the modified $q$-Bernoulli polynomials with weight $\alpha$.

**Theorem 2.8.** *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{n,q^{-1}}^{(\alpha)}(1-x) = q^{\alpha n-1}(-1)^n \widetilde{B}_{n,q}^{(\alpha)}(x). \tag{2.21}$$

From (1.3), we note that

$$\frac{1}{q}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n}q^{-x}\,d\mu_q(x) = (-1)^n q^{\alpha n-1}\int_{\mathbb{Z}_p}[x-1]_{q^{\alpha}}^{n}q^{-x}\,d\mu_q(x)$$
$$= (-1)^n q^{\alpha n-1}\widetilde{B}_{n,q}^{(\alpha)}(-1)$$
$$= \widetilde{B}_{n,q^{-1}}^{(\alpha)}(2), \tag{2.22}$$

and, by (2.6), we get

$$\begin{aligned}
\widetilde{B}_{n,q}^{(\alpha)}(2) &= \left(q^{2\alpha}\widetilde{B}_{q}^{(\alpha)}+[2]_{q^{\alpha}}\right)^n = \left(q^{\alpha}\left(q^{\alpha}\widetilde{B}_{q}^{(\alpha)}+1\right)+1\right)^n \\
&= \sum_{l=0}^{n}\binom{n}{l}q^{\alpha l}\left(q^{\alpha}\widetilde{B}_{q}^{(\alpha)}+1\right)^l \\
&= \widetilde{B}_{0,q}^{(\alpha)} + nq^{\alpha}\left(q^{\alpha}\widetilde{B}_{q}^{(\alpha)}+1\right)^1 + \sum_{l=2}^{n}\binom{n}{l}q^{\alpha l}\left(q^{\alpha}\widetilde{B}_{q}^{(\alpha)}+1\right)^l \\
&= \frac{(q-1)}{\log q} + nq^{\alpha}\left(\frac{\alpha}{[\alpha]_q}+\widetilde{B}_{1,q}^{(\alpha)}\right) + \sum_{l=2}^{n}\binom{n}{l}q^{\alpha l}\widetilde{B}_{l,q}^{(\alpha)} \\
&= nq^{\alpha}\frac{\alpha}{[\alpha]_q} + \sum_{l=0}^{n}\binom{n}{l}q^{\alpha l}\widetilde{B}_{l,q}^{(\alpha)}.
\end{aligned} \tag{2.23}$$

Let $n \in \mathbb{N}$ with $n \geq 2$. Then, by (2.12) and (2.23), we obtain the following theorem.

**Theorem 2.9.** *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$\widetilde{B}_{n,q}^{(\alpha)}(2) - nq^{\alpha}\frac{\alpha}{[\alpha]_q} = \left(q^{\alpha}\widetilde{B}_{q}^{(\alpha)}+1\right)^n = \widetilde{B}_{n,q}^{(\alpha)}. \tag{2.24}$$

*In particular,*

$$\frac{1}{q}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n}q^{-x}\,d\mu_q(x) = \widetilde{B}_{n,q^{-1}}^{(\alpha)}(2) = \frac{n}{q}\frac{\alpha}{[\alpha]_q} + \widetilde{B}_{n,\,q^{-1}}^{(\alpha)}. \tag{2.25}$$


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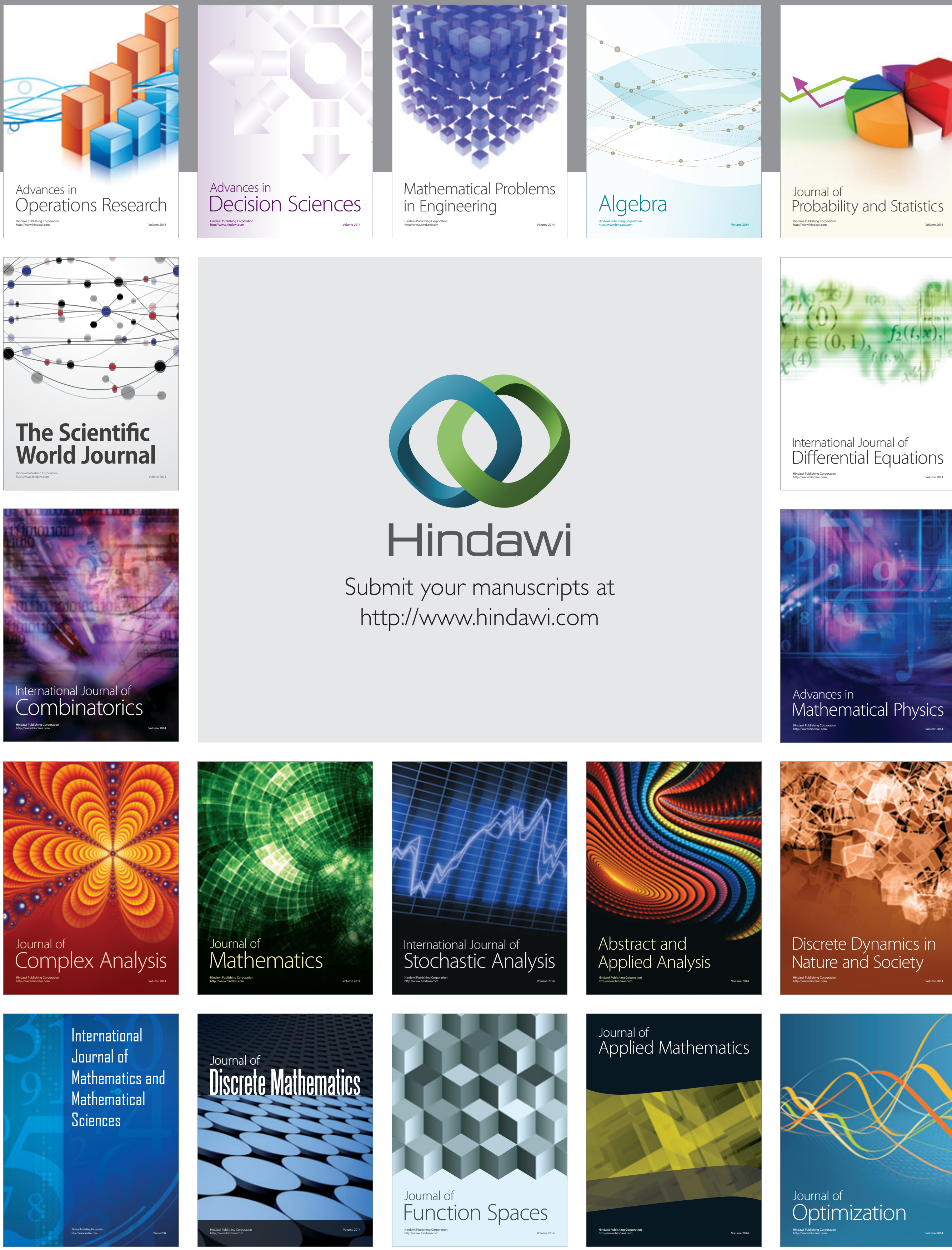
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