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Research Article

The Fixed Point Property in c_0 with an Equivalent Norm

Berta Gamboa de Buen[1] and Fernando Núñez-Medina[2]

[1] *Matemáticas Básicas, Centro de Investigación en Matemáticas (CIMAT), Apartado Postal 402, 36000 Guanajuato, GTO, Mexico*
[2] *Departamento de Matemáticas Aplicadas, Universidad del Papaloapan (UNPA), 68400 Loma Bonita, OAX, Mexico*

Correspondence should be addressed to Berta Gamboa de Buen, gamboa@cimat.mx

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We study the fixed point property (FPP) in the Banach space c_0 with the equivalent norm $\|\cdot\|_D$. The space c_0 with this norm has the weak fixed point property. We prove that every infinite-dimensional subspace of $(c_0, \|\cdot\|_D)$ contains a complemented asymptotically isometric copy of c_0, and thus does not have the FPP, but there exist nonempty closed convex and bounded subsets of $(c_0, \|\cdot\|_D)$ which are not ω-compact and do not contain asymptotically isometric c_0—summing basis sequences. Then we define a family of sequences which are asymptotically isometric to different bases equivalent to the summing basis in the space $(c_0, \|\cdot\|_D)$, and we give some of its properties. We also prove that the dual space of $(c_0, \|\cdot\|_D)$ over the reals is the Bynum space $l_{1\infty}$ and that every infinite-dimensional subspace of $l_{1\infty}$ does not have the fixed point property.

1. Introduction

We start with some notations and terminologies. Let K be a nonempty, convex, closed and bounded subset of a Banach space $(X, \|\cdot\|)$. A mapping $T : K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in K. \tag{1.1}$$

We say that K has the fixed point property for nonexpansive mappings (FPP) if every nonexpansive mapping $T : K \to K$ has a fixed point, that is, a point $x \in K$ such that $Tx = x$. We say that a Banach space $(X, \|\cdot\|)$ has the fixed point property for nonexpansive mappings (FPP) if every nonempty, convex, closed, and bounded subset K of $(X, \|\cdot\|)$ has the FPP, and we say that the Banach space $(X, \|\cdot\|)$ has the weak fixed point property for nonexpansive mappings (ω-FPP) if every nonempty, convex and weakly compact subset K of $(X, \|\cdot\|)$ has the FPP.

In this paper we study the FPP in the Banach space c_0 with the equivalent norm $\|\cdot\|_D$ defined by

$$\|x\|_D = \sup_{i,j\in\mathbb{N}} \left|x_i - x_j\right|, \quad x = \{x_i\} \in c_0. \tag{1.2}$$

The norm $\|\cdot\|_D$ was used by Hagler in [1] to construct a separable Banach space X with non-separable dual such that l_1 does not embed in X and every normalized weakly null sequence in X has a subsequence equivalent to the canonical basis of c_0.

In [2], Dowling et al. gave a characterization of nonempty, convex, closed and bounded subsets of c_0 which are not ω-compact. Specifically, they proved that if K is a convex, closed and bounded subset of c_0, then K is ω-compact if and only if every nonempty, convex, closed and convex subset of K has the FPP. To do that, the authors showed that every closed, convex and bounded subset of c_0 which is not ω-compact contains an asymptotically isometric c_0-summing basic sequence, *aisbc*$_0$ sequence for short, that is, a sequence $\{y_n\}_n \subset c_0$ such that for all $\{t_n\}_n \in l_1$,

$$\sup_{n\in\mathbb{N}} (1+\varepsilon_n)^{-1} \left|\sum_{i=n}^{\infty} t_i\right| \le \left\|\sum_{n=1}^{\infty} t_n y_n\right\| \le \sup_{n\in\mathbb{N}} (1+\varepsilon_n) \left|\sum_{i=n}^{\infty} t_i\right|, \tag{1.3}$$

for some $\{\varepsilon_n\}_n \subset \mathbb{R}$ with $0 \le \varepsilon_{n+1} \le \varepsilon_n$ and $\lim_n \varepsilon_n = 0$. They proved that if a convex, closed and bounded subset K of a Banach space contains an *aisbc*$_0$ sequence, then there exists a nonempty, convex, closed and bounded subset of K without the FPP. The authors used this fact in [3] as a tool to prove that a nonempty, closed, convex and bounded subset of c_0 is ω-compact if and only if it has the FPP.

It is easy to see that $(c_0, \|\cdot\|_D)$ contains c_0 isometrically, and then it contains *aisbc*$_0$ sequences.

First we prove that every infinite-dimensional subspace Y of $(c_0, \|\cdot\|_D)$ has a complemented asymptotically isometric copy of c_0 and hence by a result proved by Dowling et al. in [4], Y does not have the FPP. Also, as an immediate consequence we obtain that Y has an *aisbc*$_0$ sequence. Nevertheless, we exhibit a nonempty closed, convex and bounded subset of $(c_0, \|\cdot\|_D)$, which is not ω-compact and does not contain *aisbc*$_0$ sequences.

Then for every selection of signs $\Theta = \{\theta_i\}$, we define the Θ-basis of c_0 which is equivalent to the summing basis and define the corresponding asymptotically isometric Θ-basic sequence, $ai\Theta bc_{0D}$ sequence for short. We prove that if $\Theta_1 \ne \pm\Theta_2$, then the $ai\Theta_1 bc_{0D}$ and $ai\Theta_2 bc_{0D}$ sequences are different in the sense that there exists a nonempty, closed, convex, and bounded subset of $(c_0, \|\cdot\|_D)$, which is not ω-compact, contains an $ai\Theta_1 bc_{0D}$ sequence, and does not contain $ai\Theta_2 bc_{0D}$ sequences. We also show that the *aisbc*$_0$ and $ai\Theta bc_{0D}$ sequences are different in the last sense for all Θ. Hence, to give a similar result of Theorem 4 of [2] about convex, closed and bounded sets in $(c_0, \|\cdot\|_D)$ without the FPP, it is necessary to consider the $ai\Theta bc_{0D}$ sequences.

Next we prove that if a convex and closed subset K of a Banach space contains an asymptotically isometric c_{0D}-summing basic sequence, that is, an $ai\Theta bc_{0D}$ sequence, where Θ is such that $\theta_i = 1$ for all i, then there exists a nonempty, convex, closed and bounded subset of K without the FPP.

Finally, we show that the dual space of $(c_0, \|\cdot\|_D)$, over the reals, is the Bynum [5] space $l_{1\infty}$. Then, by a result of Dowling et al. in [6], the space $l_{1\infty} = (c_0, \|\cdot\|_D)^*$ has "many" subspaces and contains an asymptotically isometric copy of l_1 and does not have the FPP. In fact, we prove that every infinite dimensional subspace of $l_{1\infty}$ contains an asymptotically isometric copy of l_1 and does not have the FPP.

2. The Space $(c_0, \|\cdot\|_D)$

In the sequel, we will denote by $\{e_n\}$ the canonical basis of c_0 and by $\{\xi_n\}$ the summing basis of c_0, that is, $\xi_n = \sum_{i=1}^{n} e_i,\ n \in \mathbb{N}$.

García Falset proved in [7] that a Banach space with strongly bimonotone basis and with the weak Banach-Saks property has the ω-FPP. It is easy to see that the canonical basis of c_0 is strongly bimonotone in $(c_0, \|\cdot\|_D)$. On the other hand, since c_0 has the weak Banach-Saks property and $\|\cdot\|_D$ and $\|\cdot\|_\infty$ are equivalent, we get that $(c_0, \|\cdot\|_D)$ has the weak Banach-Saks property. Hence we have that $(c_0, \|\cdot\|_D)$ has the ω-FPP.

To study the FPP in the space $(c_0, \|\cdot\|_D)$ using $aisbc_0$ sequences, we would expect that nonempty, convex, closed and bounded subsets K of $(c_0, \|\cdot\|_D)$, which are not ω-compact, contain an $aisbc_0$ sequence. This fact is true for some ω-compact sets in $(c_0, \|\cdot\|_D)$, since the space c_0 embeds isometrically in $(c_0, \|\cdot\|_D)$. In fact we have the following proposition.

Proposition 1. *Let $\{u_k\}_k \subset (c_0, \|\cdot\|_D)$ be a block basis of $\{e_n\}$ with $u_k = \sum_{i=p_k}^{q_k} a_i e_i,\ 1 \le p_1 \le q_1 < p_2 \le q_2 < \cdots$. If $\|u_k\|_\infty = 1 = a_{i^k}$, for some $p_k \le i^k \le q_k$, and $y_k = (1/2)(u_{2k} - u_{2k-1})$, then the space $\overline{\operatorname{span}}\{y_k\}$ is isometric to $(c_0, \|\cdot\|_\infty)$.*

Proof. Since $\|u_k\|_\infty = 1 = a_{i^k}$ for every $k \in \mathbb{N}$, then $|a_j| \le 1$ for all $j \in \mathbb{N}$ and

$$\max_{p_{2k-1} \le i \le q_{2k-1},\, p_{2k} \le j \le q_{2k}} \left| a_i + a_j \right| = a_{i^{2k-1}} + a_{i^{2k}} = 2. \tag{2.1}$$

Hence, it is straightforward to see that $\|\sum_{k=1}^{n} t_k y_k\|_D = \|\sum_{k=1}^{n} t_k e_k\|_\infty$. □

In the following theorem, we will show, using some results proved by Dowling et al. [4, 8], that every infinite-dimensional subspace Y of c_{0D} fails to have the FPP.

Theorem 2. *Let Y be an infinite-dimensional subspace of c_{0D}. Then Y has a complemented asymptotically isometric copy of c_0 and thus Y does not have the FPP.*

Proof. Let $\{\varepsilon_k\}_k \subset (0,1)$ be a sequence such that $\varepsilon_{k+1} < \varepsilon_k,\ k \in \mathbb{N}$ and $\varepsilon_k \to 0$. As in [9] we construct sequences $\{n_k\} \subset \mathbb{N}$ and $\{y_k\}_k \subset Y$ such that $n_k < n_{k+1},\ y_k = \sum_{i=n_k}^{\infty} \alpha_i^k e_i,\ \|y_k\|_\infty = 1$, and

$$\sup_{i \ge n_{k+1}} \left| \alpha_i^j \right| < \frac{\varepsilon_{k+2}}{4k} \quad \forall j = 1, \ldots, k, \text{ and every } k \in \mathbb{N}. \tag{2.2}$$

Since $\|y_k\|_\infty = 1$, taking $-y_k$ instead of y_k, if necessary, we can suppose that there exists $n_k \le r^k < n_{k+1}$ such that

$$\alpha_{r^k}^k = 1. \tag{2.3}$$

Define $x_k = (y_{2k-1} - y_{2k})/2$. Then, by (2.3) and (2.2), we get that $1 - (\varepsilon_k/2) < \|x_k\|_D < 1 + (\varepsilon_k/2)$ and

$$\begin{aligned}\sum_{k=1}^{\infty} t_k x_k &= \frac{1}{2}\sum_{k=1}^{\infty} t_k \left(\sum_{i=n_{2k-1}}^{\infty} \alpha_i^{2k-1} e_i - \sum_{i=n_{2k}}^{\infty} \alpha_i^{2k} e_i \right) \\ &= \frac{1}{2}\sum_{k=1}^{\infty} t_k \left(\sum_{i=n_{2k-1}}^{\infty} \left(\alpha_i^{2k-1} - \alpha_i^{2k}\right) e_i \right) \\ &= \frac{1}{2}\sum_{k=1}^{\infty} \left(\sum_{i=n_{2k-1}}^{n_{2k+1}-1} \left(\sum_{j=1}^{k} t_j \left(\alpha_i^{2j-1} - \alpha_i^{2j}\right) e_i \right)\right),\end{aligned} \tag{2.4}$$

where $\alpha_i^{2k} = 0$ for $i = n_{2k-1}, \dots n_{2k} - 1,\ k \in \mathbb{N}$. Then by (2.3) and (2.2), if $k > 1$, we get

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} t_n x_n\right\|_D &\geq \frac{1}{2} \max_{n_{2k-1}\leq r<n_{2k}, n_{2k}\leq s<n_{2k+1}} \left| \sum_{j=1}^{k} t_j \left(\alpha_r^{2j-1} - \alpha_r^{2j} - \alpha_s^{2j-1} + \alpha_s^{2j}\right)\right| \\ &\geq \frac{1}{2}\left| \sum_{j=1}^{k} t_j \left(\alpha_{r^{2k-1}}^{2j-1} - \alpha_{r^{2k-1}}^{2j} - \alpha_{r^{2k}}^{2j-1} + \alpha_{r^{2k}}^{2j}\right)\right| \\ &\geq \frac{1}{2}|t_k|\left|\alpha_{r^{2k-1}}^{2k-1} - \alpha_{r^{2k}}^{2k-1} + \alpha_{r^{2k}}^{2k}\right| - \frac{1}{2}\sum_{j=1}^{k-1}|t_j|\left|\alpha_{r^{2k-1}}^{2j-1} - \alpha_{r^{2k-1}}^{2j} - \alpha_{r^{2k}}^{2j-1} + \alpha_{r^{2k}}^{2j}\right| \\ &\geq \frac{1}{2}|t_k|\left(\left|\alpha_{r^{2k-1}}^{2k-1} + \alpha_{r^{2k}}^{2k}\right| - \left|\alpha_{r^{2k}}^{2k-1}\right|\right) \\ &\quad - \frac{1}{2}\sum_{j=1}^{k-1}|t_j|\left(\left|\alpha_{r^{2k-1}}^{2j-1}\right| + \left|\alpha_{r^{2k-1}}^{2j}\right| + \left|\alpha_{r^{2k}}^{2j-1}\right| + \left|\alpha_{r^{2k}}^{2j}\right|\right) \\ &\geq \frac{1}{2}|t_k|(2-\varepsilon_k) - \frac{1}{2}\sum_{j=1}^{k-1}|t_j|\frac{\varepsilon_k}{k} \\ &\geq |t_k|\left(1 - \frac{\varepsilon_k}{2}\right) - \max_{1\leq j\leq k}|t_j|\frac{\varepsilon_k}{2}.\end{aligned} \tag{2.5}$$

On the other hand, if $n_{2k-1} \leq r < n_{2k+1},\ n_{2m-1} \leq s < n_{2m+1},\ k \leq m$, using (2.2), we get

$$\begin{aligned}&\frac{1}{2}\left| \sum_{j=1}^{k} t_j \left(\alpha_r^{2j-1} - \alpha_r^{2j}\right) - \sum_{j=1}^{m} t_j \left(\alpha_s^{2j-1} - \alpha_s^{2j}\right)\right| \\ &\quad\leq \frac{1}{2}\left[|t_k|\left(\left|\alpha_r^{2k-1}\right| + \left|\alpha_r^{2k}\right|\right) + |t_m|\left(\left|\alpha_s^{2m-1}\right| + \left|\alpha_s^{2m}\right|\right)\right] \\ &\qquad + \frac{1}{2}\left[\sum_{j=1}^{k-1}|t_j|\left(\left|\alpha_r^{2j-1}\right| + \left|\alpha_r^{2j}\right|\right) + \sum_{j=1}^{m-1}|t_j|\left(\left|\alpha_s^{2j-1}\right| + \left|\alpha_s^{2j}\right|\right)\right]\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\left[|t_k|(1+\varepsilon_k)+|t_m|(1+\varepsilon_m)+\sum_{j=1}^{k-1}\left|t_j\right|\frac{\varepsilon_k}{k-1}+\sum_{j=1}^{m-1}\left|t_j\right|\frac{\varepsilon_m}{m-1}\right]\\
&\le \frac{1}{2}\left[|t_k|(1+\varepsilon_k)+|t_m|(1+\varepsilon_m)+\max_{1\le j<k}\left|t_j\right|\varepsilon_k+\max_{1\le j<m}\left|t_j\right|\varepsilon_m\right]\\
&\le \frac{1}{2}\left[\max_{1\le j\le k}\left|t_j\right|(1+\varepsilon_k)+\max_{1\le j\le m}\left|t_j\right|(1+\varepsilon_m)\right]\\
&\le \sup_{n\in\mathbb{N}}\left((1+\varepsilon_n)\max_{1\le j\le n}\left|t_j\right|\right)\le \sup_{n\in\mathbb{N}}(1+\varepsilon_n)|t_n|.
\end{aligned} \tag{2.6}$$

Then we obtain

$$\sup_{n\in\mathbb{N}}\left(|t_n|\left(1-\frac{\varepsilon_n}{2}\right)-\max_{1\le j\le n}\left|t_j\right|\frac{\varepsilon_n}{2}\right)\le\left\|\sum_{n=1}^{\infty}t_nx_n\right\|_D\le\sup_{n\in\mathbb{N}}(1+\varepsilon_n)|t_n|. \tag{2.7}$$

Now, define $z_n = x_n/(1+\varepsilon_n)$ and $m = (1-\varepsilon_1)/(1+\varepsilon_1)$; then $(1-\varepsilon_n)/(1+\varepsilon_n) \le \|z_n\|_D$ and $\lim_n\|z_n\|_D = 1$. On the other hand,

$$\begin{aligned}
(1+\varepsilon_1)m\sup_{n\in\mathbb{N}}|t_n| = (1-\varepsilon_1)\sup_{n\in\mathbb{N}}|t_n| &= \left(1-\frac{\varepsilon_1}{2}\right)\sup_{n\in\mathbb{N}}|t_n|-\frac{\varepsilon_1}{2}\sup_{n\in\mathbb{N}}|t_n|\\
&\le \sup_{n\in\mathbb{N}}\left(|t_n|\left(1-\frac{\varepsilon_n}{2}\right)-\max_{1\le j\le n}\left|t_j\right|\frac{\varepsilon_n}{2}\right).
\end{aligned} \tag{2.8}$$

Thus

$$m\sup_{n\in\mathbb{N}}|t_n|\le\sup_{n\in\mathbb{N}}\left(|t_n|\left(1-\frac{\varepsilon_n}{2}\right)-\max_{1\le j\le n}\left|t_j\right|\frac{\varepsilon_n}{2}\right)\le\left\|\sum_{n=1}^{\infty}t_nz_n\right\|_D\le\sup_{n\in\mathbb{N}}|t_n|. \tag{2.9}$$

Then by Theorem 2 of [8] Y contains an asymptotically isometric copy of c_0 and since Y does not contain a copy of l_1, by Corollary 11 of [8] it contains a complemented asymptotically isometric copy of c_0. Finally by Proposition 11 of [4], Y does not have the FPP. □

As a consequence of the last theorem, we get that every infinite-dimensional subspace of $(c_0, \|\cdot\|_D)$ contains an *aisbc*$_0$ sequence. Nevertheless, the following result gives an example of a nonempty, convex, closed and bounded subset of $(c_0, \|\cdot\|_D)$ which is not weakly compact and without *aisb*c$_0$ sequences.

Proposition 3. *Let $\{\xi_n\}$ be the c_0 summing basis. Then*

$$C=\left\{\sum_{n=1}^{\infty}\lambda_n\xi_n:\lambda_n\ge 0,\ \sum_{n=1}^{\infty}\lambda_n=1\right\} \tag{2.10}$$

does not have aisbc$_0$ sequences with the norm $\|\cdot\|_D$.

Proof. Suppose that $\{y_n\}$ is an $aisbc_0$ sequence in C with $\|\cdot\|_D$ for some sequence $\{\varepsilon_n\}$. Then $y_n = \sum_{i=1}^{\infty}\lambda_i^n\xi_i$ for some sequence $\{\lambda_i^n\}$ such that $\lambda_i^n \geq 0$ and $\sum_{i=1}^{\infty}\lambda_i^n = 1$. Fix $0 < \varepsilon < 1/4$. Passing to a subsequence we can suppose that $\varepsilon_{n+1} \leq \varepsilon_n < (1/2)-2\varepsilon$ and $1/(1+\varepsilon_n) > 1-\varepsilon$, $n \in \mathbb{N}$.

Assume first that there exists $M \in \mathbb{N}$ such that for every $n \geq M$, $\sum_{i=M+1}^{\infty}\lambda_i^n \leq (1/2)-\varepsilon$. Let $u_n = \sum_{i=1}^{M}\lambda_i^n\xi_i$ and $v_n = \sum_{i=M+1}^{\infty}\lambda_i^n\xi_i$; then $y_n = u_n + v_n$. Since $\{u_n\} \subset [\xi_i]_{i=1}^{M}$ is bounded and $\dim [\xi_i]_{i=1}^{M} = M$, passing to another subsequence we can suppose that $u_n \to u$ for some $u \in C$. Then, there exist $n_1, n_2 \in \mathbb{N}$ with $M \leq n_1 < n_2$ such that

$$\|u_{n_1} - u_{n_2}\|_D < \varepsilon. \tag{2.11}$$

Since $\sum_{i=M+1}^{\infty}\lambda_i^n \leq (1/2)-\varepsilon$, $n \geq M$, we also get

$$\begin{aligned}\|v_{n_1} - v_{n_2}\|_D &= \max_{M+1\leq r\leq k<\infty}\left|\sum_{i=r}^{k}\lambda_i^{n_1} - \sum_{i=r}^{k}\lambda_i^{n_2}\right| \\ &\leq \sum_{i=M+1}^{\infty}\lambda_i^{n_1} + \sum_{i=M+1}^{\infty}\lambda_i^{n_2} \leq 1-2\varepsilon.\end{aligned} \tag{2.12}$$

Hence $\|y_{n_1} - y_{n_2}\|_D \leq 1-\varepsilon$. On the other hand, since $\{y_n\}$ is an $aisbc_0$ sequence, we have that $\|y_{n_1} - y_{n_2}\|_D \geq 1/(1+\varepsilon_{n_2})$, which contradicts the fact that $1/(1+\varepsilon_{n_2}) > 1-\varepsilon$.

Assume now that for all $M \in \mathbb{N}$, there exist $n \geq M$ such that $\sum_{i=M+1}^{\infty}\lambda_i^n > (1/2)-\varepsilon$. Denote each y_n by $\{\alpha_i^n\} = \sum_{i=1}^{\infty}\alpha_i^n e_n$, where $\{e_n\}$ is the canonical basis of c_0. Then $\alpha_i^n = \sum_{j=i}^{\infty}\lambda_j^n$. Since $y_1, y_2 \in c_0$, there exists $M \in \mathbb{N}$ such that

$$\alpha_i^1, \alpha_i^2 < \frac{\varepsilon}{2}, \quad i \geq M. \tag{2.13}$$

By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=M+1}^{\infty}\lambda_i^{n_0} > (1/2)-\varepsilon$. Then

$$\frac{3}{2} - 2\varepsilon \leq \left\|y_1 + y_2 - y_{n_0}\right\|_D. \tag{2.14}$$

On the other hand, since $\{y_n\}$ is an $aisbc_0$ sequence, we have that $\|y_1 + y_2 - y_{n_0}\|_D \leq 1+\varepsilon_1$, which contradicts the fact that $\varepsilon_1 < (1/2)-2\varepsilon$. □

In view of the last proposition and motivated by the behavior of the c_0 summing basic sequence with the norm $\|\cdot\|_D$, we will define the asymptotically isometric c_{0D}-summing basic sequence. First we consider the following definition.

Definition 4. Let $\{x_n\}$ be a bounded basic sequence in a Banach space X. We say that $\{x_n\}$ is a convexly closed sequence if the set

$$C = \left\{\sum_{n=1}^{\infty}t_n x_n : t_n \geq 0,\ \sum_{n=1}^{\infty}t_n = 1\right\} \tag{2.15}$$

is closed, that is, if $\overline{\text{conv}}\{x_n\} = C$.

Note that subsequences of convexly closed sequences are again convexly closed and that every basic sequence equivalent to a convexly closed sequence is convexly closed.

It is easy to see that the c_0 summing basis, the canonical basis of l_1, and $aisbc_0$ sequences are convexly closed. Moreover, a weakly null basic sequence in a Banach space is not a convexly closed sequence. Hence the canonical basis of c_0 and the canonical basis of l_p, $1 < p < \infty$, are not convexly closed.

Definition 5. Let $\{x_n\}$ be a sequence in a Banach space X. We say that $\{x_n\}$ is an asymptotically isometric c_{0D}-summing basic sequence, $aisbc_{0D}$ sequence for short, if $\{x_n\}$ is convexly closed and there exists $\{\varepsilon_n\} \subset (0,\infty)$ such that $\varepsilon_n \searrow 0$ and

$$\sup_{1\le n\le m<\infty}(1+\varepsilon_m)^{-1}\left|\sum_{k=n}^{m}t_k\right| \le \left\|\sum_{n=1}^{\infty}t_nx_n\right\| \le \sup_{1\le n\le m<\infty}(1+\varepsilon_m)\left|\sum_{k=n}^{m}t_k\right|, \quad \forall\{t_n\}\in l_1. \tag{2.16}$$

Now, we prove that the analogous of the operator defined in [2] is still contractive and then Banach spaces containing $aisbc_{0D}$ sequences does not have the FPP.

Proposition 6. *Let K be a nonempty, convex, closed and bounded subset of a Banach space X. Let $\{\varepsilon_n\} \subset (0,\infty)$ be a sequence such that $\varepsilon_n \to 0$ and $\varepsilon_n < 2^{-1}4^{-n}$, $n \ge 2$. If K contains an $aisbc_{0D}$ sequence with this $\{\varepsilon_n\}$, then there exists a nonempty, convex and closed subset C of K and $T : C \to C$ affine, nonexpansive, and fixed-point-free. Moreover, T is contractive.*

Proof. Let $\{x_n\}$ be an $aisbc_{0D}$ sequence in K with $\{\varepsilon_n\} \subset (0,\infty)$ such that $\varepsilon_n < 2^{-1}4^{-n}$, $n \ge 2$. Set

$$C = \overline{\operatorname{conv}}\{x_n\} = \left\{\sum_{n=1}^{\infty}t_nx_n : t_n \ge 0,\ n\in\mathbb{N}\ y\sum_{n=1}^{\infty}t_n = 1\right\} \subset K. \tag{2.17}$$

Thus C is nonempty, convex, closed and bounded. Define $Tx_n = \sum_{j=1}^{\infty}((x_{n+j})/2^j)$, $n\in\mathbb{N}$, and extend T linearly to C, that is, if $x = \sum_{n=1}^{\infty}t_nx_n \in C$ then define $T(\sum_{n=1}^{\infty}t_nx_n) = \sum_{n=1}^{\infty}t_nTx_n$. It is easy to see that $T(C) \subset C$ and that T is affine and fixed-point-free, see [2]. We only need to show that T is a contractive mapping. Let $x, y \in C$, with $x \ne y$. Then $x = \sum_{n=1}^{\infty}t_nx_n$ and $y = \sum_{n=1}^{\infty}s_nx_n$, with $t_n, s_n \ge 0$, and $\sum_{n=1}^{\infty}t_n = \sum_{n=1}^{\infty}s_n = 1$. Let $\beta_n = t_n - s_n$, $n\in\mathbb{N}$, such that $\sum_{n=1}^{\infty}\beta_n = 0$. As in [2] we have

$$T(x) - T(y) = \sum_{n=1}^{\infty}B_nx_n, \tag{2.18}$$

where $B_1 = 0$ and $B_n = (\beta_1/2^{n-1}) + (\beta_2/2^{n-2}) + \cdots + (\beta_{n-1}/2)$, $n \ge 2$. Consequently,

$$\|T(x) - T(y)\| = \left\|\sum_{n=1}^{\infty}B_nx_n\right\| \le \sup_{1\le n\le m<\infty}(1+\varepsilon_m)\left|\sum_{k=n}^{m}B_k\right|. \tag{2.19}$$

Take $n, m \in \mathbb{N}$ with $n \le m$. Since

$$\begin{aligned}
\sum_{k=n}^{m} B_k = {} & \frac{\beta_1}{2^{n-1}} + \frac{\beta_2}{2^{n-2}} + \cdots + \frac{\beta_{n-1}}{2} \\
& + \frac{\beta_1}{2^{n}} + \frac{\beta_2}{2^{n-1}} + \cdots + \frac{\beta_{n-1}}{2^2} + \frac{\beta_n}{2} + \cdots \\
& + \frac{\beta_1}{2^{m-1}} + \frac{\beta_2}{2^{m-2}} + \cdots + \frac{\beta_{n-1}}{2^{m-(n-1)}} + \frac{\beta_n}{2^{m-n}} + \frac{\beta_{n+1}}{2^{m-(n+1)}} + \cdots + \frac{\beta_{m-1}}{2} \\
= {} & \frac{1}{2}\left(\beta_{n-1} + \beta_n + \cdots + \beta_{m-1}\right) \\
& + \frac{1}{2^2}\left(\beta_{n-2} + \beta_{n-1} + \cdots + \beta_{m-2}\right) + \cdots \\
& + \frac{1}{2^{n-1}}\left(\beta_1 + \beta_2 + \cdots + \beta_{m-(n-1)}\right) + \cdots \\
& + \frac{1}{2^{m-2}}\left(\beta_1 + \beta_2\right) + \frac{1}{2^{m-1}}\left(\beta_1\right),
\end{aligned} \tag{2.20}$$

we have

$$\begin{aligned}
(1+\varepsilon_m)\left|\sum_{k=n}^{m} B_k\right| \le {} & (1+\varepsilon_m)\Bigg(\frac{1+2\varepsilon_{m-1}}{2}\frac{1}{1+2\varepsilon_{m-1}}\left|\beta_{n-1} + \beta_n + \cdots + \beta_{m-1}\right| \\
& + \frac{1+2\varepsilon_{m-2}}{2^2}\frac{1}{1+2\varepsilon_{m-2}}\left|\beta_{n-2} + \beta_{n-1} + \cdots + \beta_{m-2}\right| + \cdots \\
& + \frac{1+2\varepsilon_{m-(n-1)}}{2^{n-1}}\frac{1}{1+2\varepsilon_{m-(n-1)}}\left|\beta_1 + \beta_2 + \cdots + \beta_{m-(n-1)}\right| + \cdots \\
& + \frac{1+2\varepsilon_2}{2^{m-2}}\frac{1}{1+2\varepsilon_2}\left|\beta_1 + \beta_2\right| + \frac{1+2\varepsilon_1}{2^{m-1}}\frac{1}{1+2\varepsilon_1}\left|\beta_1\right|\Bigg) \\
\le {} & \left(\sup_{1\le i\le j\le m}\left(1+2\varepsilon_j\right)^{-1}\left|\sum_{k=i}^{j}\beta_k\right|\right) Q_{nm},
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
Q_{nm} = {} & (1+\varepsilon_m)\Bigg(\frac{1+2\varepsilon_{m-1}}{2} + \frac{1+2\varepsilon_{m-2}}{2^2} + \cdots + \\
& + \frac{1+2\varepsilon_{m-(n-1)}}{2^{n-1}} + \frac{1+2\varepsilon_{m-n}}{2^{n}} + \cdots + \frac{1+2\varepsilon_2}{2^{m-2}} + \frac{1+2\varepsilon_1}{2^{m-1}}\Bigg) \\
\le {} & \left(1 + \frac{1}{2\cdot 4^m}\right)\left[\left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-1}}\right) + \left(\frac{1}{2\cdot 4^{m-1}} + \cdots + \frac{1}{2^{m-1}\cdot 4^1}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left(1+\frac{1}{2\cdot 4^m}\right)\left[\left(1-\frac{1}{2^{m-1}}\right)+\left(\frac{1}{2^{2m-1}}+\frac{1}{2^{2m-2}}+\cdots+\frac{1}{2^{m+1}}\right)\right] \\
&< \left(1+\frac{1}{4^m}\right)\left[\left(1-\frac{1}{2^{m-1}}\right)+\frac{1}{2^m}\right] < 1.
\end{aligned}
\tag{2.22}
$$

Then we get

$$
\begin{aligned}
\sup_{1\le n\le m<\infty}(1+\varepsilon_m)\left|\sum_{k=n}^{m}B_k\right| &\le \sup_{1\le n\le m<\infty}(1+2\varepsilon_m)^{-1}\left|\sum_{k=n}^{m}\beta_k\right| < \sup_{1\le n\le m<\infty}(1+\varepsilon_m)^{-1}\left|\sum_{k=n}^{m}\beta_k\right| \\
&\le \left\|\sum_{n=1}^{\infty}\beta_n x_n\right\| = \|x-y\|.
\end{aligned}
\tag{2.23}
$$

Thus T is contractive. □

Next for any sequence of signs we will define a basis in c_0 equivalent to $\{\xi_n\}$, the summing basis of c_0, and a sequence asymptotically isometric to it.

Let $\{e_n\}$ be the canonical basis of c_0 and for any selection of signs $\Theta = \{\theta_i\}_i$, that is, $\theta_i \in \{-1,1\}$, $i\in\mathbb{N}$, let $\{\zeta_n^{\Theta}\}_n$ be the sequence defined by

$$
\zeta_n^{\Theta} = \sum_{k=1}^{n}\theta_k e_k, \quad n\in\mathbb{N}. \tag{2.24}
$$

Since $\|\sum_{n=1}^{m}t_n\xi_n\|_\infty = \|\sum_{n=1}^{m}t_n\zeta_n^{\Theta}\|_\infty$ for all $\{t_n\}_{n=1}^{m}\subset\mathbb{K}$, we get that $\{\zeta_n^{\Theta}\}$ is a basis of c_0 equivalent to the c_0 summing basis. The sequence $\{\zeta_n^{\Theta}\}$ is called the Θ-basis of c_0. Let $\Theta_0 = \{\theta_i\}$, where $\theta_i = 1$, $i\in\mathbb{N}$. Then the Θ_0-basis of c_0 is the c_0 summing basis. If we define $C = \{\sum_{n=1}^{\infty}t_n\zeta_n^{\Theta} : t_n\ge 0 \text{ and } \sum_{n=1}^{\infty}t_n = 1\}$, then C is nonempty, convex and bounded. Since $\|\cdot\|_\infty$ and $\|\cdot\|_D$ are equivalent, we have that $\{\zeta_n^{\Theta}\}$ is convexly closed in $(c_0,\|\cdot\|_D)$.

The set $C = \{\sum_{n=1}^{\infty}t_n\zeta_n^{\Theta} : t_n\ge 0 \text{ and } \sum_{n=1}^{\infty}t_n = 1\}$ is not ω-compact. The following result shows that the set C contains neither $aisbc_{0D}$ sequences nor $aisbc_0$ sequences with the norm $\|\cdot\|_D$ if $\Theta\ne\pm\Theta_0$.

Proposition 7. *For $\Theta\ne\pm\Theta_0$, let $\{\zeta_n^{\Theta}\}$ be the Θ-basis of c_0 considered in $(c_0,\|\cdot\|_D)$. If*

$$
C = \left\{\sum_{n=1}^{\infty}t_n\zeta_n^{\Theta} : t_n\ge 0,\ \sum_{n=1}^{\infty}t_n = 1\right\}, \tag{2.25}
$$

*then the set C contains neither **aisbc$_{0D}$** sequences nor **aisbc$_0$** sequences with the norm $\|\cdot\|_D$.*

Proof. Let $\{y_k\}\subset C$. Then $y_k = \sum_{n=1}^{\infty}\lambda_n^k\zeta_n^{\Theta}$ for some $\lambda_n^k\ge 0$ and $\sum_{n=1}^{\infty}\lambda_n^k = 1$. Suppose that $\{y_k\}$ is an $aisbc_{0D}$ sequence (resp. an $aisbc_0$ sequence) with the norm $\|\cdot\|_D$. Let $n_0 = \min\{n : \theta_n\ne\theta_1\}$. If there exists $0<\rho<1$ such that $\sum_{i=1}^{n_0-1}\lambda_i^k \le 1-\rho$ for all $k\ge 1$, then for all $k\ge 1$,

$$
\|y_k\|_D \ge \sum_{n=1}^{\infty}\lambda_n^k + \sum_{n=n_0}^{\infty}\lambda_n^k \ge 1+\rho. \tag{2.26}
$$

Since $\{y_k\}$ is an $aisbc_{0D}$ sequence (resp. an $aisbc_0$ sequence) with the norm $\|\cdot\|_D$, then $\|y_k\|_D \leq 1 + \varepsilon_k \to 1$ and this is impossible. Now, if $\limsup_k \sum_{i=1}^{n_0-1} \lambda_i^k = 1$, as in the proof of Proposition 3, we obtain a subsequence $\{y_{k_i}\}$ of $\{y_k\}$ with $\|y_{k_i} - y_{k_{i+1}}\|_D \to 0$. Since $\{y_n\}$ is an $aisbc_{0D}$ with the norm $\|\cdot\|_D$, then $(1+\varepsilon_{k_i})^{-1} \leq \|y_{k_i} - y_{k_{i+1}}\|_D$ (resp. $(1+\varepsilon_{k_{i+1}})^{-1} \leq \|y_{k_i} - y_{k_{i+1}}\|_D$) and making $i \to \infty$ we get that $1 \leq 0$. This contradiction proves the result. □

Although the set C of the last proposition has neither $aisbc_{0D}$ sequences nor $aisbc_0$ sequences, for some Θ it does not have the FPP.

For $\Theta = \{\theta_i\}$, let F_n be the set such that if $i, j \in F_n$, then $\theta_i = \theta_j$, and if $i \in F_{n+1}$ and $j \in F_n$, then $\theta_i \neq \theta_j$. Denote by r_n the cardinality of F_n. If $r_n < \infty$, define $p_0 = 0$, $p_{n-1} = \min F_n - 1$, and $p_n = \max F_n$.

Proposition 8. *Let $\Theta \neq \pm\Theta_0$. Then*

$$C = \left\{ \sum_{n=1}^{\infty} t_n \zeta_n^{\Theta} : t_n \geq 0,\ \sum_{n=1}^{\infty} t_n = 1 \right\} \tag{2.27}$$

does not have the FPP in the following cases.

(1) *There exists $k \geq 1$ such that $r_n \leq r_{n+k} < \infty$, $n \in \mathbb{N}$.*

(2) *$r_1 = 1$ and $r_2 = \infty$.*

(3) *There exists $\{i_n\}$, with $i_1 > 1$, such that for any $k, l \in \mathbb{N}$ with $i_{k-1} < l < i_k$ we have $\theta_l = \theta_{i_k}$ and also $\theta_k \neq \theta_{i_k}$ for all $k \geq 2$ or $\theta_k = \theta_{i_k}$ for all $k \geq 2$.*

Proof. Let $\Theta \neq \pm\Theta_0$.

(1) If there exists $k \geq 1$ such that $r_n \leq r_{n+k} < \infty$, $n \in \mathbb{N}$, define $q_n = \sum_{j=n-(k-2)}^{n+1} r_j$, $n \geq k$ and $T : C \to C$ by

$$T\sum_{n=1}^{\infty} t_n \zeta_n^{\Theta} = T\sum_{n=1}^{\infty} \sum_{i=p_{n-1}+1}^{p_n} t_i \zeta_i^{\Theta} = \sum_{n=k}^{\infty} \sum_{i=w_n}^{p_{n+1}} t_{i-q_n} \zeta_i^{\Theta}, \tag{2.28}$$

where $w_n = p_n + r_{n+1} - r_{n+1-k} + 1$. The idea is to translate the coefficients of $\sum_{n=1}^{\infty} t_n \zeta_n^{\Theta}$ in the block F_n into the last r_n terms of the block F_{n+k}. Then it is easy to see that T does not have fixed points. To prove that T is nonexpansive first observe that if k is even the signs of the θ_i and θ_j with $i \in F_n$ and $j \in F_{n+k}$ are the same and are different if k is odd. Now let $x = \sum_{n=1}^{\infty} t_n \zeta_n^{\Theta}$, $y = \sum_{n=1}^{\infty} s_n \zeta_n^{\Theta}$, and $x - y = \sum_{n=1}^{\infty} \alpha_n \zeta_n^{\Theta}$. Then $\alpha_n = t_n - s_n$ and $\sum_{n=1}^{\infty} \alpha_n = 0$. Hence

$$x - y = \sum_{n=1}^{\infty} \sum_{i=p_{n-1}+2}^{p_n} \theta_{p_n} \left(\sum_{n=i}^{\infty} \alpha_n \right) e_i, \tag{2.29}$$

$$T(x-y) = \sum_{n=k}^{\infty} \left(\theta_{p_{n+1}} \sum_{n=i-q_n}^{\infty} \alpha_n \right) \left(\sum_{i=p_n+1}^{w_n} e_i \right) \sum_{i=w_n+1}^{p_{n+1}} \theta_{p_{n+1}} \left(\sum_{n=i-q_n}^{\infty} \alpha_n \right) e_i \tag{2.30}$$

are the expressions of $x - y$ and $T(x - y)$ with respect to the canonical basis. Since the coefficients in (2.29) and (2.30) are the same, or the same with opposite signs, with perhaps some repetitions in (2.30), T is an isometry.

(2) Suppose now that $r_1 = 1$ and $r_2 = \infty$. In this case, define $T\sum_{n=1}^{\infty} t_n\zeta_n^{\Theta} = \sum_{n=1}^{\infty} t_n\zeta_{n+1}^{\Theta}$. Clearly T is nonexpansive and fixed-point-free.

(3) In this case it is straightforward to see that the operator $T : C \to C$ defined by $T\sum_{n=1}^{\infty} t_n\zeta_n^{\Theta} = \sum_{n=1}^{\infty} t_n\zeta_{i_n}^{\Theta}$ is nonexpansive and does not have fixed points. □

Proposition 9. *Let $\Theta \neq \pm\Theta_0$. Suppose Θ does not satisfy the hypotheses of the above proposition, and let $\{i_n\}$ be a sequence with $i_1 > 1$. Then the operator $T : C \to C$ defined by $T\sum_{n=1}^{\infty} t_n\zeta_n^{\Theta} = \sum_{n=1}^{\infty} t_n\zeta_{i_n}^{\Theta}$ is expansive.*

Proof. Since Θ does not satisfy the hypotheses (1) and (2) of the above proposition, there are three possibilities.

(I) $r_n < \infty$ for every $n \geq 2$; then for every k there exists n such that $r_{n+k} < r_n$.

(II) $r_2 = \infty$; then $r_1 > 1$.

(III) There exists $k > 2$ such that $r_k = \infty$.

Let $\{i_n\}$ be fixed with $i_1 > 1$ and denote $i_0 = 0$. Since Θ does not satisfy the hypotheses (3) of the above proposition, there exist k and l with $i_{k-1} < l < i_k$ such that $\theta_l \neq \theta_{i_k}$ or there exists $k_1 \geq 2$ with $\theta_{k_1} = \theta_{i_{k_1}}$ and there exists $k_2 \geq 2$ with $\theta_{k_2} \neq \theta_{i_{k_2}}$.

Case 1. For every k there exists n such that $r_{n+k} < r_n$.

There are two subcases.

Subcase 1.1. There are k and l with $i_{k-1} < l < i_k$ such that $\theta_l \neq \theta_{i_k}$.

Let $x = (1/8)\zeta_1^{\Theta} + (3/8)\zeta_{k-1}^{\Theta} + (1/2)\zeta_k^{\Theta}$ and $y = (1/16)\zeta_1^{\Theta} + (3/16)\zeta_{k-1}^{\Theta} + (3/4)\zeta_k^{\Theta}$. Then $x - y = (1/16)\zeta_1^{\Theta} + (3/16)\zeta_{k-1}^{\Theta} - (1/4)\zeta_k^{\Theta} = -(1/16)\sum_{i=2}^{k-1}\theta_i e_i - (1/4)\theta_k e_k$ and $\|x - y\|_D \leq 5/16$. On the other hand, $Tx - Ty = (1/16)\zeta_{i_1}^{\Theta} + (3/16)\zeta_{i_{k-1}}^{\Theta} - (1/4)\zeta_{i_k}^{\Theta} = -(1/16)\sum_{j=i_1+1}^{i_{k-1}}\theta_j e_j - (1/4)\sum_{j=i_{k-1}+1}^{i_k}\theta_j e_j$ and $\|Tx - Ty\|_D = 1/2$.

Subcase 1.2. For any $k \in \mathbb{N}$ and l with $i_{k-1} < l < i_k$, we have $\theta_l = \theta_{i_k}$.

There are two subsubcases. (1) $\theta_1 = \theta_{i_1}$ and (2) $\theta_1 \neq \theta_{i_1}$.

(1) $\theta_1 = \theta_{i_1}$

If $\theta_k = \theta_{i_k}$ for every k; then we would have $F_1 = \mathbb{N}$, which implies $\Theta = \pm\Theta_0$. Then there is k such that $\theta_k \neq \theta_{i_k}$. Let $s = \min\{l : \theta_l \neq \theta_{i_l}\}$. Then $s > 1$.

There are two possibilities: (A) there exists $r > s$ such that $\theta_r = \theta_{i_r}$ and (B) $\theta_k \neq \theta_{i_k}$ for all $k \geq s$.

(A) Let $k + 1 = \min\{r > s : \theta_r = \theta_{i_r}\}$. We need to consider the following cases.

(a) $\theta_k = \theta_{k+1}$.

Let $x = (1/2)\zeta_{k-1}^{\Theta} + (1/2)\zeta_{k+1}^{\Theta}$ and $y = (3/4)\zeta_{k-1}^{\Theta} + (1/4)\zeta_{k+1}^{\Theta}$. Then $x - y = -(1/4)\ \zeta_{k-1}^{\Theta} + (1/4)\zeta_{k+1}^{\Theta} = \theta_{k+1}((1/4)e_k + (1/4)e_{k+1})$ and $\|x - y\|_D = 1/4$. On the other hand, $Tx - Ty = -(1/4)\ \zeta_{i_{k-1}}^{\Theta} + (1/4)\ \zeta_{i_{k+1}}^{\Theta} = (1/4)\sum_{j=i_{k-1}+1}^{i_k}\theta_j e_j + (1/4)\sum_{j=i_k+1}^{i_{k+1}}\theta_j e_j = -(1/4)\theta_{k+1}\sum_{j=i_{k-1}+1}^{i_k} e_j + (1/4)\theta_{k+1}\sum_{j=i_k+1}^{i_{k+1}} e_j$ and $\|Tx - Ty\|_D = 1/2$.

(b) $\theta_k \neq \theta_{k+1}$.

Let $x = (1/2)\zeta_{k-1}^{\Theta} + (1/2)\zeta_{k+1}^{\Theta}$ and $y = (3/4)\zeta_k^{\Theta} + (1/4)\zeta_{k+1}^{\Theta}$. Then $x - y = (1/2)\ \zeta_{k-1}^{\Theta} - (3/4)\ \zeta_k^{\Theta} + (1/4)\zeta_{k+1}^{\Theta} = -(1/2)\theta_k e_k + (1/4)\theta_{k+1}e_{k+1} = \theta_{k+1}((1/2)e_k + (1/4)e_{k+1})$ and $\|x-y\|_D = 1/2$. On the other hand, $Tx - Ty = (1/2)\ \zeta_{i_{k-1}}^{\Theta} - (3/4)\ \zeta_{i_k}^{\Theta} + (1/4)\ \zeta_{i_{k+1}}^{\Theta} = -(1/2)\sum_{j=i_{k-1}+1}^{i_k}\theta_j e_j + (1/4)\sum_{j=i_k+1}^{i_{k+1}}\theta_j e_j = -(1/2)\ \theta_k\sum_{j=i_{k-1}+1}^{i_k} e_j + (1/4)\theta_k\sum_{j=i_k+1}^{i_{k+1}} e_j$ and $\|Tx-Ty\|_D = 3/4$.

(B) $\theta_k \neq \theta_{i_k}$ for all $k \geq s$. By hypothesis we have that $s > 2$. There are two cases.

(a) $\theta_{s-1} = \theta_s$. Then $\theta_{i_{s-1}} \neq \theta_{i_s}$. Let $x = (1/4)\zeta_{s-2}^{\Theta} + (1/2)\zeta_{s-1}^{\Theta} + (1/4)\zeta_s^{\Theta}$ and $y = (1/4)\zeta_{s-1}^{\Theta} + (3/4)\zeta_s^{\Theta}$. Then $x - y = (1/4)\zeta_{s-2}^{\Theta} + (1/4)\zeta_{s-1}^{\Theta} - (1/2)\zeta_s^{\Theta} = -\theta_s((1/4)e_{s-1} + (1/2)e_s)$ and $\|x-y\|_D = 1/2$. On the other hand, $Tx - Ty = (1/4)\ \zeta_{i_{s-2}}^{\Theta} + (1/4)\ \zeta_{i_{s-1}}^{\Theta} - (1/2)\ \zeta_{i_s}^{\Theta} = -(1/4)\theta_s\sum_{j=i_{s-2}+1}^{i_{s-1}} e_j + (1/2)\theta_s\sum_{j=i_{s-1}+1}^{i_s} e_j$ and $\|Tx-Ty\|_D = 3/4$.

(b) $\theta_{s-1} \neq \theta_s$. Then $\theta_{i_{s-1}} = \theta_{i_s}$. Let $x = (1/4)\zeta_{s-2}^{\Theta} + (1/4)\zeta_{s-1}^{\Theta} + (1/2)\zeta_s^{\Theta}$ and $y = (3/4)\zeta_{s-1}^{\Theta} + (1/4)\zeta_s^{\Theta}$. Then $x - y = (1/4)\zeta_{s-2}^{\Theta} - (1/2)\zeta_{s-1}^{\Theta} + (1/4)\zeta_s^{\Theta} = -(1/4)\theta_{s-1}e_{s-1} + (1/4)\theta_s e_s = \theta_s((1/4)e_{s-1} + (1/4)e_s)$ and $\|x-y\|_D = 1/4$. On the other hand, $Tx - Ty = (1/4)\ \zeta_{i_{s-2}}^{\Theta} - (1/2)\ \zeta_{i_{s-1}}^{\Theta} + (1/4)\ \zeta_{i_s}^{\Theta} = -(1/4)\theta_{i_{s-1}}\sum_{j=i_{s-2}+1}^{i_{s-1}} e_j + (1/4)\ \theta_{i_s}\sum_{j=i_{s-1}+1}^{i_s} e_j = \theta_{s-1}(-(1/4)\sum_{j=i_{s-2}+1}^{i_{s-1}} e_j + (1/4)\sum_{j=i_{s-1}+1}^{i_s} e_j)$ and $\|Tx-Ty\|_D = 1/2$.

(2) $\theta_1 \neq \theta_{i_1}$

In this case there exists k such that $\theta_k = \theta_{i_k}$. If $s = \min\{l : \theta_l = \theta_{i_l}\}$, then $s > 1$.

Hence consider the cases: (A) there exists $r > s$ such that $\theta_r \neq \theta_{i_r}$ and (B) $\theta_k = \theta_{i_k}$ for all $k \geq s$ and proceed as in the Case (1).

Case 2. $r_2 = \infty$ and $r_1 > 1$.

Then $\theta_{p_1} \neq \theta_{i_{p_1}}$ with $1 < p_1$. Hence we can proceed as in Subcase 1.2(1)(A) above taking $k = p_1$.

Case 3. There is $s > 1$ such that $r_{s+1} = \infty$.

Then $\theta_{p_s} \neq \theta_{i_{p_s}}$ with $1 < p_s$. Hence we can proceed as in Subcase 1.2(1)(A) above taking $k = p_s$. □

Next, for every selection of signs $\Theta \neq \pm\Theta_0$, we will define the asymptotically isometric c_{0D}-Θ-basic sequences. To this end, let us consider the following notation.

Let

$$\mathfrak{S}_{\Theta} = \{(n,m) : \theta_n = \theta_m\},$$
$$\mathfrak{D}_{\Theta} = \{(n,m) : \theta_n \neq \theta_m\}. \tag{2.31}$$

Definition 10. Let $\{x_n\}$ be a sequence in a Banach space X. We say that $\{x_n\}$ is an asymptotically isometric c_{0D}-Θ-basic sequence ($ai\Theta bc_{0D}$ sequence for short) if $\{x_n\}$ is convexly closed and there exists $\{\varepsilon_n^{\Theta}\} \subset (0,(1/2))$ such that $\varepsilon_n^{\Theta} \searrow 0$, and

$$\begin{aligned} L\left(\left\{\varepsilon_n^{\Theta}\right\},\{t_n\},\mathfrak{S}_{\Theta}\right) \vee L\left(\left\{\varepsilon_n^{\Theta}\right\},\{t_n\},\mathfrak{D}_{\Theta}\right) &\leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \\ &\leq R\left(\left\{\varepsilon_n^{\Theta}\right\},\{t_n\},\mathfrak{S}_{\Theta}\right) \vee R\left(\left\{\varepsilon_n^{\Theta}\right\},\{t_n\},\mathfrak{D}_{\Theta}\right) \end{aligned} \tag{2.32}$$

holds for all $\{t_n\} \in l_1$, where

$$\begin{aligned} L\left(\left\{\varepsilon_n^{\Theta}\right\},\{t_n\},\mathfrak{S}_{\Theta}\right) &= \left(\sup_{n<l,(n,l)\in\mathfrak{S}_{\Theta}} \left(1+\varepsilon_{l-1}^{\Theta}\right)^{-1}\left|\sum_{k=n}^{l-1} t_k\right|\right), \\ L\left(\left\{\varepsilon_n^{\Theta}\right\},\{t_n\},\mathfrak{D}_{\Theta}\right) &= \left(\sup_{n<l,(n,l)\in\mathfrak{D}_{\Theta}} \left(1+\varepsilon_{l-1}^{\Theta}\right)^{-1}\left|\sum_{k=n}^{l-1} t_k + 2\sum_{k=l}^{\infty} t_k\right|\right), \\ R\left(\left\{\varepsilon_n^{\Theta}\right\},\{t_n\},\mathfrak{S}_{\Theta}\right) &= \left(\sup_{n<l,(n,l)\in\mathfrak{S}_{\Theta}} \left(1+\varepsilon_{l-1}^{\Theta}\right)\left|\sum_{k=n}^{l-1} t_k\right|\right), \\ R\left(\left\{\varepsilon_n^{\Theta}\right\},\{t_n\},\mathfrak{D}_{\Theta}\right) &= \left(\sup_{n<l,(n,l)\in\mathfrak{D}_{\Theta}} \left(1+\varepsilon_{l-1}^{\Theta}\right)\left|\sum_{k=n}^{l-1} t_k + 2\sum_{k=l}^{\infty} t_k\right|\right). \end{aligned} \tag{2.33}$$

We are interested in $ai\Theta bc_{0D}$ sequences for which the numbers ε_n^{Θ} of Definition 10 are small. We are taking $\{\varepsilon_n^{\Theta}\} \subset (0,(1/2))$.

We know that the set C of Proposition 3 does not have $aisbc_0$ sequences. Now we also prove that C does not contain $ai\Theta bc_{0D}$ sequences with the norm $\|\cdot\|_D$ if $\Theta \neq \pm\Theta_0$.

Proposition 11. *Let $\Theta \neq \pm\Theta_0$. The set $C = \{\sum_{n=1}^{\infty} t_n \xi_n : t_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = 1\}$ does not contain $ai\Theta bc_{0D}$ sequences with the norm $\|\cdot\|_D$.*

Proof. Let $\{y_k\} \subset C$. Then $y_k = \sum_{n=1}^{\infty} \lambda_n^k \xi_n$ for some $\lambda_n^k \geq 0$ with $\sum_{n=1}^{\infty} \lambda_n^k = 1$. Suppose that $\{y_k\}$ is an $ai\Theta bc_{0D}$ with $\|\cdot\|_D$. Since $\Theta \neq \pm\Theta_0$, there exist $m \in \mathbb{N}$ and $\{n_k\} \subset \mathbb{N}$ with $n_1 < n_2 < \cdots$, such that for all $k \in \mathbb{N}$, $m < n_k$ and $\theta_{n_k} \neq \theta_m$. Let $t_n = 0$ for $n \neq m, n_k$ and $t_m = t_{n_k} = 1$. Thus

$$\begin{aligned} \left(1+\varepsilon_{n_k-1}^{\Theta}\right)^{-1} 3 &\leq L\left(\left\{\varepsilon_n^{\Theta}\right\},\{t_n\},\mathfrak{S}_{\Theta}\right) \vee L\left(\left\{\varepsilon_n^{\Theta}\right\},\{t_n\},\mathfrak{D}_{\Theta}\right) \\ &\leq \left\|\sum_{n=1}^{\infty} t_n y_n\right\|_D = \left\|y_m + y_{n_k}\right\|_D = 2. \end{aligned} \tag{2.34}$$

Since (2.34) holds for all $k \in \mathbb{N}$, making $k \to \infty$ in (2.34), we get that $3 \leq 2$, which is a contradiction. □

Proposition 12. *Let $\Theta_1 = \{\theta_i^1\}_i$ and $\Theta_2 = \{\theta_i^2\}_i$ such that $\Theta_1 \neq \pm\Theta_2$ and $\Theta_1, \Theta_2 \neq \pm\Theta_0$. Let $\{\zeta_n^{\Theta_1}\}$ be the Θ_1-basis of c_0 considered in $(c_0, \|\cdot\|_D)$ and let*

$$C(\Theta_1) = \left\{ \sum_{n=1}^{\infty} t_n \zeta_n^{\Theta_1} : t_n \geq 0,\ \sum_{n=1}^{\infty} t_n = 1 \right\}. \tag{2.35}$$

The set $C(\Theta_1)$ does not contain $\boldsymbol{ai}\Theta_2\boldsymbol{bc}_{0D}$ sequences with the norm $\|\cdot\|_D$.

Proof. Let $\{y_k\} \subset C$. Then $y_k = \sum_{n=1}^{\infty} \lambda_n^k \zeta_n^{\Theta_1}$ for some $\lambda_n^k \geq 0$ with $\sum_{n=1}^{\infty} \lambda_n^k = 1$. Suppose that $\{y_k\}$ is an $ai\Theta_2 bc_{0D}$ with the norm $\|\cdot\|_D$.

Suppose first $\theta_1^1 = \theta_1^2$; since $\Theta_1 \neq \Theta_2$, there exists $m > 1$ such that $\theta_m^1 \neq \theta_m^2$.

There are two cases.

Case 1. $(1, m) \in \mathfrak{S}_{\Theta_1}$. In this case $(1, m) \in \mathfrak{D}_{\Theta_2}$. Let $t_n = 0$ for $n \neq 1, m$ and $t_1 = t_m = 1$. Thus

$$\begin{aligned} \left(1 + \varepsilon_{m-1}^{\Theta_2}\right)^{-1} 3 &\leq L\left(\left\{\varepsilon_n^{\Theta_2}\right\}, \{t_n\}, \mathfrak{S}_{\Theta_2}\right) \vee L\left(\left\{\varepsilon_n^{\Theta_2}\right\}, \{t_n\}, \mathfrak{D}_{\Theta_2}\right) \\ &\leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\|_D = \|y_1 + y_m\|_D \leq 2. \end{aligned} \tag{2.36}$$

Since $\varepsilon_{m-1}^{\Theta_2} < 1/2$, we get a contradiction.

Case 2. $(1, m) \in \mathfrak{D}_{\Theta_1}$. In this case $(1, m) \in \mathfrak{S}_{\Theta_2}$. Let $t_n = 0$ for $n \neq 1, m$ and $t_1 = t_m = 1$. Thus

$$\begin{aligned} 2 &\leq \sum_{n=1}^{\infty} \lambda_n^1 + \sum_{n=1}^{\infty} \lambda_n^m + \sum_{n=m}^{\infty} \lambda_n^1 + \sum_{n=m}^{\infty} \lambda_n^m \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\|_D = \|y_1 + y_m\|_D \\ &\leq R\left(\left\{\varepsilon_n^{\Theta_2}\right\}, \{t_n\}, \mathfrak{S}_{\Theta_2}\right) \vee R\left(\left\{\varepsilon_n^{\Theta_2}\right\}, \{t_n\}, \mathfrak{D}_{\Theta_2}\right) \leq \left(1 + \varepsilon_{m-1}^{\Theta_2}\right). \end{aligned} \tag{2.37}$$

Since $\varepsilon_{m-1}^{\Theta_2} < 1/2$, we get a contradiction.

Suppose now $\theta_1^1 \neq \theta_1^2$; since $\Theta_1 \neq -\Theta_2$, there exists m such that $\theta_m^1 = \theta_m^2$.

There are two cases.

Case 1. $(1, m) \in \mathfrak{S}_{\Theta_1}$; in this case $(1, m) \in \mathfrak{D}_{\Theta_2}$. Let $t_n = 0$ for $n \neq 1, m$ and $t_1 = t_m = 1$. Thus

$$\begin{aligned} \left(1 + \varepsilon_{m-1}^{\Theta_2}\right)^{-1} 3 &\leq L\left(\left\{\varepsilon_n^{\Theta_2}\right\}, \{t_n\}, \mathfrak{S}_{\Theta_2}\right) \vee L\left(\left\{\varepsilon_n^{\Theta_2}\right\}, \{t_n\}, \mathfrak{D}_{\Theta_2}\right) \\ &\leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\|_D = \|y_1 + y_m\|_D \leq 2. \end{aligned} \tag{2.38}$$

Since $\varepsilon_{m-1}^{\Theta_2} < 1/2$, we get a contradiction.

Case 2. $(1,m) \in \mathfrak{D}_{\Theta_1}$. In this case $(1,m) \in \mathfrak{S}_{\Theta_2}$. Let $t_n = 0$ for $n \neq 1, m$ and $t_1 = t_m = 1$. Thus

$$\begin{aligned} 2 &\leq \sum_{n=1}^{\infty}\lambda_n^1 + \sum_{n=1}^{\infty}\lambda_n^m + \sum_{n=m}^{\infty}\lambda_n^1 + \sum_{n=m}^{\infty}\lambda_n^m \leq \left\|\sum_{n=1}^{\infty}t_n y_n\right\|_D = \|y_1 + y_m\|_D \\ &\leq R\left(\left\{\varepsilon_n^{\Theta_2}\right\},\{t_n\},\mathfrak{S}_{\Theta_2}\right) \vee R\left(\left\{\varepsilon_n^{\Theta_2}\right\},\{t_n\},\mathfrak{D}_{\Theta_2}\right) \leq \left(1+\varepsilon_{m-1}^{\Theta_2}\right). \end{aligned} \tag{2.39}$$

Since $\varepsilon_{m-1}^{\Theta_2} < 1/2$, we get a contradiction. □

Propositions 3, 7, and 11 show that, in contrast with Theorem 4 of the Dowling et al. paper [2] for $aisbc_0$ sequences in c_0, in the space $(c_0, \|\cdot\|_D)$ we need an infinite number of sequences (at least $aisbc_0$ and $ai\Theta bc_{0D}$ sequences) to have a similar result.

3. The Space $(c_0, \|\cdot\|_D)^*$

It is known that the dual of the Bynum space c_{01} is the Bynum space $l_{1\infty}$. Below we prove that the dual space of $(c_0, \|\cdot\|_D)$ when the scalar field is the set of real numbers is also the Bynum space $l_{1\infty}$. Let us suppose then that $\mathbb{K} = \mathbb{R}$. First we calculate the extreme points of the unit ball of $(c_0, \|\cdot\|_D)$.

Lemma 13. *Let* $X = (c_0, \|\cdot\|_D)$. *Then we have*

$$\mathcal{E}(B_X) = \{\{x_n\} \in S_X : x_n \in \{1,0\},\ n \in \mathbb{N}\} \cup \{\{x_n\} \in S_X : x_n \in \{-1,0\},\ n \in \mathbb{N}\}. \tag{3.1}$$

Proof. First note that if $\{x_n\} \in S_X$ then $|x_n - x_m| \leq 1,\ n,m \in \mathbb{N}$ and $|x_n| \leq 1,\ n \in \mathbb{N}$. Consequently, if $\{x_n\} \in S_X$ with $x_{n_0} = 1$ for some $n_0 \in \mathbb{N}$, then $0 \leq x_n \leq 1$, for all $n \in \mathbb{N}$. Analogously if $\{x_n\} \in S_X$ with $x_{n_0} = -1$ for some $n_0 \in \mathbb{N}$, then $-1 \leq x_n \leq 0$, for all $n \in \mathbb{N}$. Let $A = \{\{x_n\} \in S_X : x_n \in \{1,0\},\ n \in \mathbb{N}\}$ and $B = \{\{x_n\} \in S_X : x_n \in \{-1,0\},\ n \in \mathbb{N}\}$. Thus $A, B \subset S_X$.

Take $x = \{x_n\} \in A$ and suppose that $x = (y+z)/2$ with $y, z \in S_X$. Also suppose that $y = \{y_n\}$ and $z = \{z_n\}$. Since $x \in A$, there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = 1$. Since $x_n = (y_n + z_n)/2$ and $y_n, z_n \leq 1$, if $x_n = 1$ for some $n \in \mathbb{N}$, we have that $x_n = y_n = z_n$. Thus $x_{n_0} = y_{n_0} = z_{n_0} = 1$. On the other hand, if $x_n = 0$ for some $n \in \mathbb{N}$, we also have that $x_n = y_n = z_n$, because if $y_n < 0$ we get that $|y_n - y_{n_0}| > 1$, which contradicts that $|y_n - y_m| \leq 1,\ n,m \in \mathbb{N}$ and if $y_n > 0$ then $z_n < 0$ and we also have a contradiction. Therefore, $x = y = z$. Hence $x \in \mathcal{E}(B_X)$. Thus $A \subset \mathcal{E}(B_X)$. Analogously $B \subset \mathcal{E}(B_X)$.

Take now $x = \{x_n\} \in S_X \setminus \{A \cup B\}$. Then there exists $n_0 \in \mathbb{N}$ such that $0 < |x_{n_0}| < 1$. Let $a = \inf_n x_n$ and $b = \sup_n x_n$. If $x_{n_0} \in (a,b)$, define $c = \min(|x_{n_0} - a|, |x_{n_0} - b|)$, $y_n = z_n = x_n,\ n \neq n_0,\ y_{n_0} = x_{n_0} - c$, and $z_{n_0} = x_{n_0} + c$. Thus, $x_n = (y+z)/2$ with $y, z \in S_X$ and $x \neq y,\ x \neq z$. Therefore, $x \in S_X \setminus \mathcal{E}(B_X)$. Suppose now that $x_{n_0} = a$ or $x_{n_0} = b$. Since $0 < |x_{n_0}| < 1$ and $\sup_{n,m\in\mathbb{N}}|x_n - x_m| = 1$, we have that $0 \in (a,b)$. Since $x_n \to 0$, there exists $n_1 > n_0$ such that $x_{n_1} \in (a,b)$, which implies that $x \in S_X \setminus \mathcal{E}(B_X)$. Consequently, $\mathcal{E}(B_X) \subset A \cup B$. □

Theorem 14. *Let $f \in (c_0, \|\cdot\|_D)^*$. There exists a unique sequence $\{c_n\} \in l_1$ such that $f = \sum_{n=1}^{\infty} c_n e_n^*$ and*

$$\|f\|_D = \max\left(\sum_{n=1}^{\infty} c_n^+, \sum_{n=1}^{\infty} c_n^-\right), \tag{3.2}$$

where $c_n^+ = \max(c_n, 0)$ and $c_n^- = -\min(c_n, 0)$.

Proof. Let $f \in (c_0, \|\cdot\|_D)^*$. Since $\{e_n\}$ is a shrinking basis of $(c_0, \|\cdot\|_D)$, there exists a unique sequence $\{c_n\} \subset \mathbb{K}$ such that $f = \sum_{n=1}^{\infty} c_n e_n^*$. As sets $(c_0, \|\cdot\|_D)^* = (c_0)^*$ and hence $f \in (c_0)^*$. Thus $f = R\{a_n\}$ where $R : l_1 \to c_0^*$ is the Riesz representation. Consequently,

$$c_n = f(e_n) = R\{a_n\}(e_n) = a_n. \tag{3.3}$$

Therefore, $\{c_n\} = \{a_n\} \in l_1$. Thus

$$\begin{aligned}
\|f\|_D &= \sup_{x \in B_X} |f(x)| = \sup_{x \in \mathcal{E}(B_X)} |f(x)| \\
&= \sup\left\{\left|\sum_{n \in F} c_n\right| : F \subset \mathbb{N},\ F \text{ finite}\right\} \\
&= \max\left(\sum_{n=1}^{\infty} c_n^+, \sum_{n=1}^{\infty} c_n^-\right),
\end{aligned} \tag{3.4}$$

where $c_n^+ = \max(c_n, 0)$ and $c_n^- = -\min(c_n, 0)$. □

Corollary 15. *$(c_0, \|\cdot\|_D)^*$ is the Bynum space $l_{1\infty}$ and it has the ω-FPP.*

Remark 16. It is well known that $l_1(c_0)^*$ has the ω^* fixed point property for left reversible semigroups, that is, whenever S is a semigroup such that $aS \cap bS \neq \emptyset$ for any $a, b \in S$, and $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as nonexpansive mappings on a nonempty ω^*-compact convex subset K of l_1, there is a common fixed point in K for $\mathcal{S}$. (see [10–12]). In particular, l_1 has the ω^* fixed point property. Is this the case for $(c_0, \|\cdot\|_D)^*$?

Next we will see that every infinite-dimensional subspace of $l_{1\infty}$ contains an asymptotically isometric copy of l_1 and then, by a result of Dowling and Lennard [13], it does not have the FPP.

First recall that a Banach space $(X, \|\cdot\|)$ contains an asymptotically isometric copy of l_1 if there exists $\{x_n\}_n \subset X$ and $\{\varepsilon_n\} \subset (0,1)$, $\varepsilon_n \to 0$ such that for every $k \in \mathbb{N}$ and every scalars $b_1, \dots, b_k$,

$$\sum_{i=1}^{k} (1 - \varepsilon_i)|b_i| \leq \left\|\sum_{i=1}^{k} b_i x_i\right\| \leq \sum_{i=1}^{k} (1 + \varepsilon_i)|b_i|. \tag{3.5}$$

In this case we say that $\{x_n\}_n$ is an asymptotically isometric l_1-sequence (ail_1-sequence for short).

Observe that if $\{y_n\}_n$ is another sequence in X such that $\|y_n - x_n\| < \delta_n$ for all n, where $\{\varepsilon_n + \delta_n\} \subset (0,1)$ and $\delta_n \to 0$, then for every k and every scalars $b_1, \dots, b_k$,

$$\sum_{i=1}^{k}(1-\varepsilon_i-\delta_i)|b_i| \le \left\| \sum_{i=1}^{k} b_i y_i \right\| \le \sum_{i=1}^{k}(1+\varepsilon_i+\delta_i)|b_i| \tag{3.6}$$

and $\{y_n\}$ is also an ail_1-sequence.

Proposition 17. *Let* $\{u_i\}_i \subset l_{1\infty}$, *and let* $\{n_i\}$ *be a strictly increasing sequence in* $\mathbb{N}$ *such that* $u_i = \sum_{j=n_i+1}^{n_{i+1}} a_j^i e_j$. *If* $\sum_{j=n_i+1}^{n_{i+1}} (a_j^i)^+ = \sum_{j=n_i+1}^{n_{i+1}} (a_j^i)^-$, *then* $\{u_i\}_i$ *is isometrically equivalent to the canonical basis in* l_1, *that is, for every* $k \in \mathbb{N}$ *and every scalars* $b_1, \dots, b_k$, *we have that* $\|\sum_{i=1}^{k} b_i u_i\| = \sum_{i=1}^{k}|b_i|$.

Proof. Let $b_1, \dots, b_k$ be scalars; then

$$\begin{aligned}\sum_{i=1}^{k}|b_i| &\le \sum_{i=1}^{k} b_i^+ \sum_{j=n_i+1}^{n_{i+1}} \left(a_j^i\right)^+ + \sum_{i=1}^{k} b_i^- \sum_{j=n_i+1}^{n_{i+1}} \left(a_j^i\right)^- \\ &= \sum_{i=1}^{k} \sum_{j=n_i+1}^{n_{i+1}} \left(b_i a_j^i\right)^+ \\ &\le \left\| \sum_{i=1}^{k} b_i u_i \right\|_{1\infty} \le \sum_{i=1}^{k}|b_i|.\end{aligned} \tag{3.7}$$

□

Theorem 18. *Every infinite-dimensional subspace of* $l_{1\infty}$ *contains an asymptotically isometric copy of* l_1 *and hence it does not have the FPP.*

Proof. Let Y be an infinite-dimensional subspace of $l_{1\infty}$, $\{\varepsilon_n\} \subset (0,(1/2))$, $\varepsilon_n \searrow 0$ and $\{x_n\}$ a sequence in S_Y such that $x_i = \sum_{j=m_i+1}^{\infty} a_j^i e_j$, where $0 = m_0 < m_1 < \cdots$ and $\sum_{j=m_{i+1}+1}^{\infty}|a_j^i| < \varepsilon_i/8$. Define

$$\begin{aligned} w_i &= \sum_{j=m_i+1}^{m_{i+1}} a_j^i e_j, \\ c_i^+ &= \frac{1}{\|w_i\|_{1\infty}} \sum_{j=m_i+1}^{m_{i+1}} \left(a_j^i\right)^+ \le 1, \\ c_i^- &= \frac{1}{\|w_i\|_{1\infty}} \sum_{j=m_i+1}^{m_{i+1}} \left(a_j^i\right)^- \le 1. \end{aligned} \tag{3.8}$$

Changing w_i by $-w_i$, if necessary, we can assume that $c_i^+ = 1$, $n \in \mathbb{N}$. If there is a sequence $\{k_i\}$ such that $c_{k_i}^- = 1$, then by Proposition 17, $\{w_{k_i}/\|w_{k_i}\|_{1\infty}\}$ is isometrically equivalent to the canonical basis of l_1. It is straightforward to see that $\|x_{k_i} - (w_{k_i}/\|w_{k_i}\|_{1\infty})\|_{1\infty} < (1/4)\varepsilon_{k_i}$. Then by the above remark, $\{x_{k_i}\}$ is an ail_1-sequence.

Suppose that $c_i^- \neq 1$ for all i and let

$$\alpha_i = \frac{1 - c_{2i}^-}{1 - c_{2i}^- c_{2i-1}^-}, \qquad \beta_i = \frac{1 - c_{2i-1}^-}{1 - c_{2i-1}^- c_{2i}^-}. \tag{3.9}$$

Then $0 \leq \alpha_i < 1$, $0 \leq \beta_i < 1$ and

$$\alpha_i c_{2i-1}^+ + \beta_i c_{2i}^- = \alpha_i c_{2i-1}^- + \beta_i c_{2i}^+ = 1. \tag{3.10}$$

Now let

$$v_i = \alpha_i \frac{w_{2i-1}}{\|w_{2i-1}\|_{1\infty}} - \beta_i \frac{w_{2i}}{\|w_{2i}\|_{1\infty}}. \tag{3.11}$$

Suppose that $v_i = \sum_{j=m_{2i-1}+1}^{m_{2i+1}} b_j^i e_j$. It is easy to check, using (3.10), that

$$\sum_{j=m_{2i-1}+1}^{m_{2i+1}} \left(b_j^i\right)^+ = \sum_{j=m_{2i-1}+1}^{m_{2i+1}} \left(b_j^i\right)^- = 1. \tag{3.12}$$

Hence, by Proposition 17, $\{v_i\}$ is isometrically equivalent to the canonical basis of l_1.

Now, if we define $y_n = \alpha_n x_{2n-1} - \beta_n x_{2n} \in Y$, it is straightforward to see that $\|y_n - v_n\|_{1\infty} < \varepsilon_n$ and by the above remark, $\{y_n\}$ is an ail_1-sequence.

Finally in [13] Dowling and Lennard proved that if a Banach space contains an ail_1-sequence, then it does not have the FPP. Hence Y does not have the FPP. □

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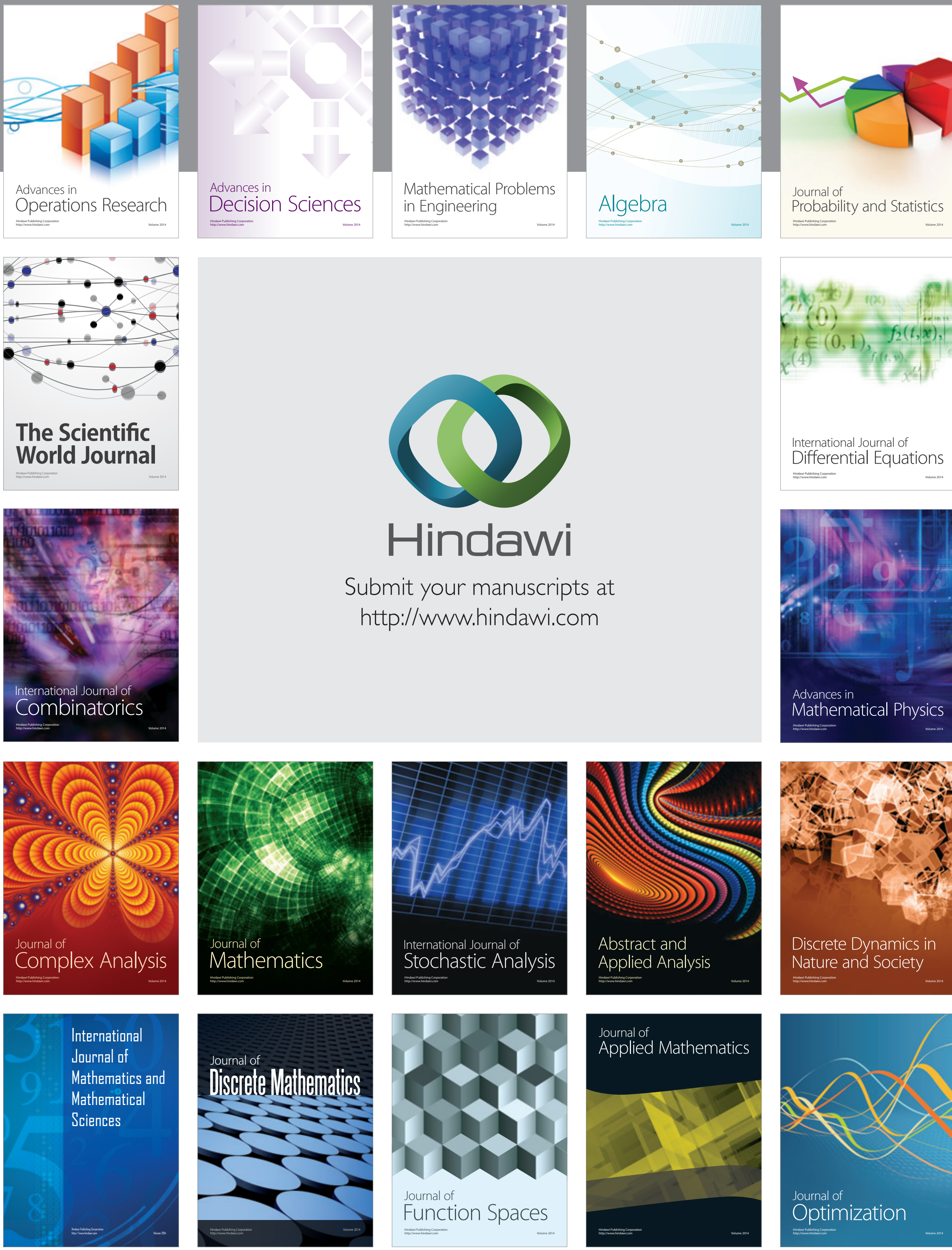
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