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Research Article

On p-Convergence in Measure of a Sequence of Measurable Functions

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In the study by Papanastassiou and Papachristodoulos, 2009 the notion of p-convergence in measure was introduced. In a natural way p-convergence in measure induces an equivalence relation on the space M of all sequences of measurable functions converging in measure to zero. We show that the quotient space $\mathcal{M}$ is a complete but not compact metric space.

1. Introduction

Convergence in measure plays a fundamental role in several branches of Mathematics, for example in integration theory and in stochastic processes. In [1] a "Bochner-type" integration theory was developed in the context of Riesz spaces with respect to a convergence introduced axiomatically, and in particular some Vitali convergence theorems and Lebesgue dominated convergence theorems were proved. Similar subjects were investigated by Haluška and Hutník [2, 3] in the setting of operator theory for Bochner- and Dobrakov-type integrals (see also [4, 5]) and in [6–8] for the Kurzweil-Henstock integral in Riesz spaces.

In several contexts of integration theory it could be advisable to extend the concept of convergence in measure in order to get applications, for example, in the study of the stochastic integral and stochastic differential equations (see, e.g., [9]).

This paper is a continuation of [10], where the notion of p-convergence in measure was introduced. In this paper we investigate a structure related to the vector space M of all converging sequences of measurable functions.

Let (Γ, Σ, μ) be an arbitrary measure space, where μ is a $[0, \infty]$-valued measure, and let $f_n, f : \Gamma \to \mathbb{R}$, $n = 1, 2, \ldots$, be measurable functions.

We adopt the following usual terminology. By the notation $f_n \xrightarrow{\mu} f$ we denote that the sequence of measurable functions $(f_n)_n$ converges in measure to f. Also for a pair $((f_n)_n, f)$ and $\varepsilon \geq 0$ we set

$$A_n^{\varepsilon} = \{\gamma \in \Gamma : |f_n(\gamma) - f(\gamma)| \geq \varepsilon\} = \{|f_n - f| \geq \varepsilon\}, \quad n = 1, 2, \ldots. \tag{1.1}$$

We denote by $\mathbb{N}$ the set of all positive integers and c_0^+ the set of all real-valued nonnegative sequences $(\varepsilon_n)_n$ converging to 0. Also for $p > 0$ we set

$$\ell_p^+ = \left\{ (\varepsilon_n)_n : \varepsilon_n \geq 0 \text{ for } n = 1, 2, \ldots, \sum_{n=1}^{\infty} \varepsilon_n^p < \infty \right\}. \tag{1.2}$$

Convergence in measure is characterized by elements $(\varepsilon_n)_n$ of c_0^+ as follows:

$$\begin{gathered} f_n \xrightarrow{\mu} f \text{ iff there exists } (\varepsilon_n)_n \in c_0^+ \\ \text{such that } \lim_{n \to \infty} \mu(A_n^{\varepsilon_n}) = 0. \end{gathered} \tag{1.3}$$

Taking into account that the sequence $(\varepsilon_n)_n$ above expresses the quality of approximation of $(f_n)_n$ to f, in [10] the authors introduced the following notion of convergence which we call p-convergence in measure.

More precisely we say that, given $p > 0$, $(f_n)_n$ *p-converges in measure to f* (and we write $f_n \xrightarrow{p-\mu} f$) if and only if there exists an element $(\varepsilon_n)_n \in \ell_p^+$ such that

$$\lim_{n \to \infty} \mu(A_n^{\varepsilon_n}) = 0. \tag{1.4}$$

Obviously p-convergence in measure implies convergence in measure. It is proved (see [10, Preposition 2.3]) that if the measure μ is not trivial and $0 < p < q$, then p-convergence in measure implies q-convergence in measure, while the converse implication in general fails. So p-convergence in measure is strictly stronger than convergence in measure. As a consequence of the above result we have that

$$M_0 \subsetneqq M_p \subsetneqq M_q \subsetneqq M_\infty \subsetneqq M, \quad 0 < p < q, \tag{1.5}$$

where

$$\begin{gathered} M = \left\{ (f_n)_n : f_n \xrightarrow{\mu} 0 \right\}, \\ M_p = \left\{ (f_n)_n : f_n \xrightarrow{p-\mu} 0 \right\}, \quad p > 0, \\ M_0 = \bigcap_{p>0} M_p, \qquad M_\infty = \bigcup_{p>0} M_p. \end{gathered} \tag{1.6}$$

We note that M is considered as a vector space under usual operations and the notation $N \subsetneqq M$ means that N is a proper vector space of M.

2. Metric Spaces of Sequences of Measurable Functions

In a natural way p-convergence in measure induces an equivalence relation on the vector space $M = \{(f_n)_n : f_n \xrightarrow{\mu} 0\}$. We consider M as a subspace of $L^0(\Gamma)^{\mathbb{N}}$, $\aleph_0$ copies of the vector space $L^0(\Gamma)$ of all real-valued measurable functions with the usual operations.

Definition 2.1. Let $(f_n)_n, (g_n)_n$ be elements of M. We say that $(f_n)_n$, $(g_n)_n$ are equivalent $((f_n)_n \sim (g_n)_n)$ if and only if for each positive real number p there exists an element $(\varepsilon_n)_n$ of ℓ_p^+ such that

$$\lim_{n\to\infty} \mu\left(\left[\left|f_n - g_n\right| > \varepsilon_n\right]\right) = 0 \tag{2.1}$$

or equivalently

$$f_n - g_n \xrightarrow{p-\mu} 0, \quad \forall p > 0 \Longleftrightarrow \left(f_n - g_n\right)_n \in M_0. \tag{2.2}$$

Since M_0 is a vector subspace of M the relation $\sim$ is an equivalence one. We set $\mathcal{M} = M \sim = M/M_0$.

In the sequel we will define a metric d on $\mathcal{M}$ under which $\mathcal{M}$ turns to be a complete metric space, similarly as a Fréchet space.

Definition 2.2. Let $(f_n)_n \in \mathcal{M}$. We define

$$\left\|(f_n)_n\right\| = \arctan\left(\inf A\left(f_n\right)\right), \tag{2.3}$$

where

$$A\left(f_n\right) = \left\{p > 0 : f_n \xrightarrow{p-\mu} 0\right\} = \left\{p > 0 : \left(f_n\right)_n \in M_p\right\}. \tag{2.4}$$

By (1.5) it follows that $A(f_n)$ is an interval in $\mathbb{R}$.

Remarks 2.3. (i) We note that the above set $A(f_n)$ could be empty. In this case we set $\|(f_n)_n\| = \pi/2$.

(ii) If $(f_n)_n \sim (g_n)_n$, then $\|(f_n)_n\| = \|(g_n)_n\|$.
(Indeed, as $(g_n)_n = (g_n - f_n)_n + (f_n)_n$ and $(f_n)_n \in M_p$, it follows that $(f_n)_n \in M_p$ if and only if $(g_n)_n \in M_p$.)

(iii) We set $\mathcal{M}_p = M_p/\sim, p > 0$, and hence $\mathcal{M}_p$ is a proper vector subspace of $\mathcal{M}$ and the following strict inclusion holds:

$$\mathcal{M}_{p_1} \subsetneqq \mathcal{M}_{p_2} \quad \text{if } 0 < p_1 < p_2 \text{ (see (1.5))}. \tag{2.5}$$

Proposition 2.4. *The function* $\| \ \| : \mathcal{M} \to \mathbb{R}$ *satisfies the following properties:*

(i) $\|(f_n)_n\| \geq 0$

(ii) $\|(f_n)_n\| = 0$ *iff* $(f_n)_n \sim (0)_n$

(iii) $\|a(f_n)_n\| = \|(f_n)_n\|$ *for* $a \neq 0$

(iv) $\|(f_n + g_n)_n\| \leq \|(f_n)_n\| + \|(g_n)_n\|$.

Hence, $\mathcal{M}$ becomes a metric space and the metric $d((f_n)_n, (g_n)_n) = \|(f_n - g_n)_n\|$ is invariant under translations.

Proof. (i) It is obvious.

(ii) If $(f_n)_n \sim (0)_n$, then $\|(f_n)_n\| = 0$.

Conversely, if $\|(f_n)_n\| = 0$, then, $(f_n)_n \xrightarrow{p-\mu} 0$ for each $p > 0$, and hence $(f_n)_n \in \mathcal{M}_0$ and consequently $(f_n) \sim (0)_n$.

(iii) For $a \neq 0$, it holds that

$$A_n^{\varepsilon_n} = \left[|f_n| \geq \varepsilon_n\right] = \left[|af_n| \geq |a|\varepsilon_n\right],$$
$$\sum_{n=1}^{\infty} \varepsilon_n^p < \infty \Longleftrightarrow \sum_{n=1}^{\infty} (|a|\varepsilon_n)^p < \infty, \tag{2.6}$$

for each sequence $(\varepsilon_n)_n$ of positive real numbers. Hence,

$$(f_n)_n \xrightarrow{p-\mu} 0 \quad \text{iff } (af_n)_n \xrightarrow{p-\mu} 0, \tag{2.7}$$

which means that $\|(f_n)_n\| = \|(af_n)_n\|$.

(iv) The inequality is obvious if $\|(f_n)_n\| = \pi/2$ or $\|(g_n)_n\| = \pi/2$.

Suppose $\|(f_n)_n\| \leq \|(g_n)_n\| < \pi/2$. Then we conclude that $A(g_n) \subset A(f_n)$. Hence, $(g_n)_n \xrightarrow{p-\mu} 0$ implies $(f_n)_n + (g_n)_n \xrightarrow{p-\mu} 0$. So $A(g_n) \subseteq A(f_n + g_n)$, which implies that

$$\|(f_n + g_n)_n\| \leq \|(g_n)_n\| \leq \|(f_n)_n\| + \|(g_n)_n\|. \tag{2.8}$$

□

Theorem 2.5. *The space $(\mathcal{M}, d)$ is a complete metric space.*

Proof. Let $(F_n)_n$ be a Cauchy sequence in $\mathcal{M}$, where $F_n = (f_{n,i})_i$, $n = 1, 2, \ldots$. Hence, there exists an increasing sequence of positive integers $(n_k)_k$ such that

$$\|F_n - F_m\| < \arctan \frac{1}{k}, \quad \text{for } n, m \geq n_k,\ k = 1, 2, \ldots. \tag{2.9}$$

This means that, for each $n, m \geq n_k$, there exists a sequence $(\varepsilon_{n,m,i})_i$ of positive real numbers with $\sum_{i=1}^{\infty} \varepsilon_{n,m,i}^{1/k} < \infty$ such that

$$\mu(A_{n,m,i}) \longrightarrow 0, \quad i \longrightarrow \infty, \text{ where } A_{n,m,i} = \left[|f_{n,i} - f_{m,i}| \geq \varepsilon_{n,m,i}\right]. \tag{2.10}$$

Then,

$$\sum_{i=1}^{\infty} \left(\max_{n_\ell \leq n \leq n_{k+1}} \varepsilon_{n,n_{k+1},i} \right)^{1/\ell} < \infty, \quad \text{for } \ell = 1, 2, \ldots, k,\ k \in \mathbb{N}. \tag{2.11}$$

From (2.10) and (2.11), proceeding by induction, it follows that there exists an increasing sequence $(i_k)_k$ of positive integers such that for each k we have

$$\mu(A_{n,n_{k+1},i}) < \frac{1}{k} \quad \text{for } i \geq i_k,\ n_1 \leq n \leq n_{k+1},\ k = 1,2,\ldots, \tag{2.12}$$

$$\sum_{i=i_k}^{\infty} \left(\max_{n_\ell \leq n \leq n_{k+1}} \varepsilon_{n,m,i} \right)^{1/\ell} < \frac{1}{2^k}, \quad \text{for } \ell = 1,2,\ldots,k, \tag{2.13}$$

$$\mu\left[\left| f_{n_{k+1},i} \right| \geq \frac{1}{k} \right] < \frac{1}{k}, \quad \text{for } i \geq i_k, \tag{2.14}$$

which express the uniform convergence to zero of finite number of sequence which converges to zero and a finite number of tails of convergence series.

We set

$$\begin{aligned} F &= \left(f_{n_1,1}, f_{n_1,2}, \ldots, f_{n_1,i_1-1}; f_{n_2,i_1}, \ldots, f_{n_2,i_2-1}; \ldots; f_{n_{k+1},i_k}, \ldots, f_{n_{k+1},i_{k+1}-1}; \ldots \right) \\ &= (f_i)_i. \end{aligned} \tag{2.15}$$

By (2.14) it follows that $F = (f_i)_i \in \mathcal{M}$.

We have to show that

$$\|F_n - F\| \longrightarrow 0, \quad n \longrightarrow \infty \Longleftrightarrow \forall \ell \in \mathbb{N}\ \ \exists n_0 \in \mathbb{N} : \left(f_{n,i} - f_i \right)_i \overset{(1/\ell)-\mu}{\longrightarrow} 0, \ \text{ for } n \geq n_0. \tag{2.16}$$

This means that we have to find $n_0 \in \mathbb{N}$ and for $n \geq n_0$ a sequence of positive real numbers $(\varepsilon_i)_i$ with $\sum_{i=1}^{\infty} \varepsilon_i^{1/\ell} < \infty$ such that

$$\mu(A_i) \longrightarrow 0, \quad i \longrightarrow \infty, \text{ where } A_i = \left[\left| f_{n,i} - f_i \right| \geq \varepsilon_i \right]. \tag{2.17}$$

Indeed let $\ell \in \mathbb{N}$, $n \geq n_0 = n_\ell$, and $n_\ell \leq n_k < n < n_{k+1}$ for some $k \in \mathbb{N}$.

We set

$$\begin{gathered} \varepsilon_i = 1, \quad \text{if } i = 1,2,\ldots,i_k - 1, \\ \varepsilon_i = \varepsilon_{n,n_{k+1},i}, \quad \text{if } i = i_k,\ldots,i_{k+1} - 1, \\ \varepsilon_i = \varepsilon_{n,n_{k+2},i}, \quad \text{if } i = i_{k+1},\ldots,i_{k+2} - 1, \end{gathered} \tag{2.18}$$

and so on.

It holds that

$$\sum_{i=1}^{\infty} \varepsilon_i^{1/\ell} \leq (i_k - 1) + \frac{1}{2^k} + \frac{1}{2^{k+1}} + \cdots < \infty \quad (\text{by}(2.13)). \tag{2.19}$$

Also, for $i_m \leq i \leq i_{m+1}$, $m \geq k$, we have that

$$A_i = \left[\left| f_{n,i} - f_i \right| \geq \varepsilon_i \right] = \left[\left| f_{n,i} - f_{n_{m+1},i} \right| \geq \varepsilon_{n,n_{m+1},i} \right] \tag{2.20}$$

(by definition of f_i). Hence, by (2.12), we take

$$\mu(A_i) < \frac{1}{m}. \tag{2.21}$$

This implies (2.17), and the proof is complete. □

Remarks 2.6. It is easy to see the following

(a) The addition $+ : ((f_n)_n, (g_n)_n) \mapsto (f_n + g_n)_n$ is continuous.

(b) The translation $T_{(g_n)_n} : (f_n)_n \mapsto (f_n)_n + (g_n)_n = (f_n + g_n)_n$ is a homeomorphism. Hence, the system of neighborhoods of $(0)_n$ determines the topology of $(\mathcal{M}, d)$.

(c) The multiplication operator

$$H_a : (f_n)_n \longmapsto a \cdot (f_n)_n, \quad a \neq 0, \tag{2.22}$$

is a homeomorphism.

(d) The multiplication $(a, (f_n)_n) \mapsto a(f_n)_n$ is not continuous. (If $a_n \to 0$, $a_n \neq 0$, $n = 1, 2, \ldots$, and $F = (f_n)_n \in \mathcal{M}$, $F \neq 0$, and $F_n = F$ for $n = 1, 2, \ldots$, it holds that

$$a_n \longrightarrow a \neq 0, \quad F_n \xrightarrow{d} F, \tag{2.23}$$

but $\|a_n F_n - 0F\| = \|a_n F_n\| = \|F_n\| = \|F\| \nrightarrow 0$).

(e) The family $(\mathcal{M}_p)_{p>0}$ is a system of neighborhoods of $(0)_n$. (Indeed, if $S_r = S((0)_n, r) = \{(f_n)_n : \|(f_n)_n\| < r\}$ for $r > 0$, then for $0 < r_1 < p < r_2$ we have $S_{r_1} \subset \mathcal{M}_p \subset S_{r_2}$.)

Though $(\mathcal{M}, d)$ is not a topological vector space, $(\mathcal{M}, d)$ is complete and the subspaces $\mathcal{M}_p$, $p > 0$ constitute a system of closed and convex neighborhoods of $(0)_n$, as we will see in the sequel (Proposition 2.7). Hence, $(\mathcal{M}, d)$ is something like a Fréchet space. For example, the principle of uniform boundedness holds true, as for this principle only continuity of H_a is needed (see [11]).

Proposition 2.7. *The subspaces $\mathcal{M}_p$ are closed for each $p > 0$.*

Proof. Suppose that $p > 0$ and $(F_n)_n$ is a sequence in $\mathcal{M}_p$, where $F_n = (f_{n,i})_i$, $n = 1, 2, \ldots$, and $F = (f_n)_n \in \mathcal{M}$ such that

$$F_n \xrightarrow{d} F \Longleftrightarrow \|F_n - F\| \longrightarrow 0, \quad n \to \infty. \tag{2.24}$$

Hence, there exist $p' < p$ and n_0 such that

$$F_{n_0} - F \xrightarrow{p'-\mu} 0. \tag{2.25}$$

This implies that $F_{n_0} - F \in \mathcal{M}_{p'} \subset \mathcal{M}_p$, and, since $F_{n_0} \in \mathcal{M}_p$, it follows that $F \in \mathcal{M}_p$. □

Proposition 2.8. *$\mathcal{M}_\infty = \bigcup_{p>0} \mathcal{M}_p$ is a closed subspace of $\mathcal{M}$.*

Proof. Suppose $F_0 = (f_{0,i})_i \notin \mathcal{M}_\infty$, then $F_0 + \mathcal{M}_q,\ q > 0$ is a neighborhood of F_0 and $(F_0 + \mathcal{M}_q) \cap \mathcal{M}_p = \emptyset$ for all $p > 0$.

Indeed, if $F_0 + F_1 = F_2$ for some $F_1 \in \mathcal{M}_q$ and some $F_2 \in \mathcal{M}_p$, then $F_0 \in \mathcal{M}_r$, where $r = \max(p, q)$, which is a contradiction. Hence, $(F_0 + \mathcal{M}_q) \cap \mathcal{M}_\infty = \emptyset$, which implies that $\mathcal{M}_\infty$ is closed. □

Remark 2.9. If $S((0)_n, r)$ denotes the open sphere with center $(0)_n$ and radius r, it is easy to see that the family

$$\{S((0)_n, r)\}_{r>0} \cup \{F + S((0)_n, r)\}_{F \notin \mathcal{M}_\infty}, \quad r > 0 \tag{2.26}$$

is an open covering of $\mathcal{M}$ without a finite subcovering. Hence $\mathcal{M}$ is not compact.

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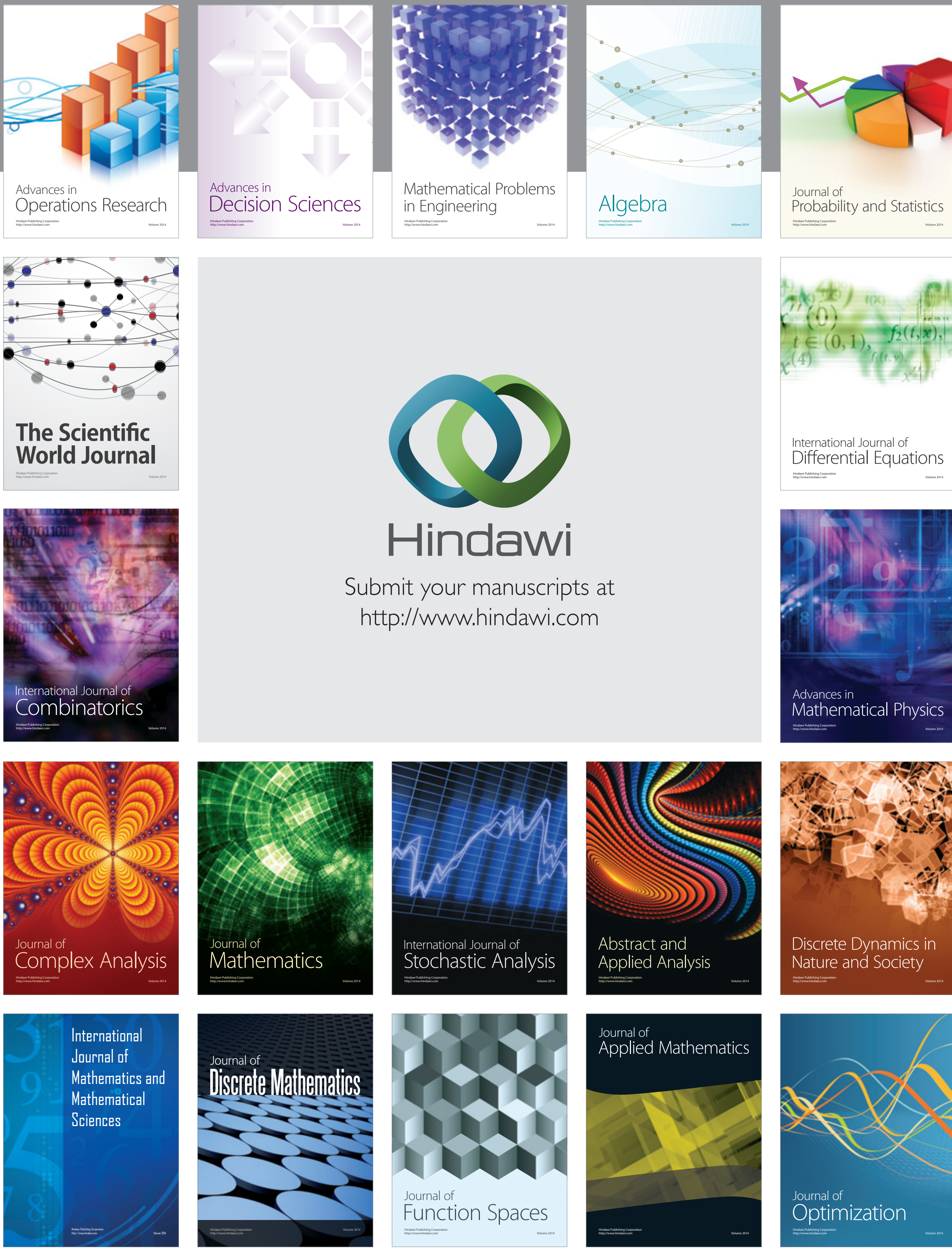
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