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Research Article

Sharp Bounds for Power Mean in Terms of Generalized Heronian Mean

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For $1 < r < +\infty$, we find the least value α and the greatest value β such that the inequality $H_\alpha(a,b) < A_r(a,b) < H_\beta(a,b)$ holds for all $a, b > 0$ with $a \neq b$. Here, $H_\omega(a,b)$ and $A_r(a,b)$ are the generalized Heronian and the power means of two positive numbers a and b, respectively.

1. Introduction and Statement of Result

For $a, b > 0$ with $a \neq b$, the generalized Heronian mean of a and b is defined by Janous [1] as

$$H_\omega(a,b) = \begin{cases} \dfrac{a + \omega\sqrt{ab} + b}{\omega + 2}, & 0 \le \omega < +\infty, \\ \sqrt{ab}, & \omega = +\infty. \end{cases} \tag{1.1}$$

If we take $\omega = 1$ in (1.1), then we arrive at the classical Heronian mean

$$H_e(a,b) = \frac{a + \sqrt{ab} + b}{3}. \tag{1.2}$$

The domain of definition for the function $\omega \mapsto H_\omega(a,b)$ can be extended to all ω with $\omega \in (-2, +\infty)$, that is,

$$H_\omega(a,b) = \begin{cases} \dfrac{a + \omega\sqrt{ab} + b}{\omega + 2}, & -2 < \omega < +\infty, \\ \sqrt{ab}, & \omega = +\infty. \end{cases} \tag{1.3}$$

For all fixed $a, b > 0$, it is easy to derive that $\omega \mapsto H_\omega(a,b)$, $-2 < \omega < +\infty$ is monotonically decreasing, and

$$\lim_{\omega \to -2^+} H_\omega(a,b) = +\infty. \tag{1.4}$$

Let

$$A_r(a,b) = \begin{cases} \left(\dfrac{a^r + b^r}{2}\right)^{1/r}, & r \neq 0, \\ \sqrt{ab}, & r = 0, \\ \max\{a,b\}, & r = +\infty, \\ \min\{a,b\}, & r = -\infty, \end{cases} \tag{1.5}$$

denote the power mean of order r. In particular, the harmonic, geometric, square-root, arithmetic, and root-square means of a and b are

$$\begin{aligned} H(a,b) &= A_{-1}(a,b) = \frac{2a}{a+b}, \\ G(a,b) &= A_0(a,b) = \sqrt{ab}, \\ N_1(a,b) &= A_{1/2}(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2, \\ A(a,b) &= A_1(a,b) = \frac{a+b}{2}, \\ S(a,b) &= A_2(a,b) = \sqrt{\frac{a^2+b^2}{2}}. \end{aligned} \tag{1.6}$$

It is well known that the power mean of order r given in (1.5) is monotonically increasing in r, then we can write

$$\min\{a,b\} < H(a,b) < G(a,b) < N_1(a,b) < A(a,b) < S(a,b) < \max\{a,b\}. \tag{1.7}$$

Recently, the inequalities for means have been the subject of intensive research [1–15]. In particular, many remarkable inequalities for the generalized Heronian and power means can be found in the literature [4–9].

In [4], the authors established two sharp inequalities

$$\begin{aligned} \frac{2}{3}G(a,b) + \frac{1}{3}H(a,b) &\geq A_{-1/3}(a,b), \\ \frac{1}{3}G(a,b) + \frac{2}{3}H(a,b) &\geq A_{-2/3}(a,b). \end{aligned} \tag{1.8}$$

In [5], Long and Chu found the greatest value p and the least value q such that the double inequality

$$A_p(a,b) \le A(a,b)^{\alpha} G(a,b)^{\beta} H(a,b)^{1-\alpha-\beta} \le A_q(a,b) \tag{1.9}$$

holds for all $a, b > 0$ and $\alpha, \beta > 0$ with $\alpha + \beta < 1$.

In [6], Shi et al. gave two optimal inequalities

$$\begin{aligned} A^{\alpha}(a,b) L^{1-\alpha}(a,b) &\le A_{(1+2\alpha)/3}(a,b), \\ G^{\alpha}(a,b) L^{1-\alpha}(a,b) &\le A_{(1-\alpha)/3}(a,b), \end{aligned} \tag{1.10}$$

for $0 < \alpha < 1$, where

$$L(a,b) = \frac{a-b}{\log a - \log b}, \quad a \neq b, \tag{1.11}$$

is the logarithmic mean for $a, b > 0$.

In [7], Guan and Zhu obtained sharp bounds for the generalized Heronian mean in terms of the power mean with $\omega > 0$. The optimal values α and β such that

$$A_{\alpha}(a,b) \le H_{\omega}(a,b) \le A_{\beta}(a,b) \tag{1.12}$$

holds in general are

(1) in case of $\omega \in (0, 2]$, $\alpha_{\max} = \log 2 / \log(\omega + 2)$ and $\beta_{\min} = 2/(\omega + 2)$,

(2) in case of $\omega \in [2, +\infty)$, $\alpha_{\max} = 2/(\omega + 2)$ and $\beta_{\min} = \log 2 / \log(\omega + 2)$.

In this paper, we find the least value α and the greatest value β, such that for any fixed $1 < r < +\infty$, the inequality

$$H_{\alpha}(a,b) < A_r(a,b) < H_{\beta}(a,b) \tag{1.13}$$

holds for all $a, b > 0$ with $a \neq b$.

Theorem 1.1. *For $1 < r < +\infty$, the optimal numbers α and β such that*

$$H_{\alpha}(a,b) < A_r(a,b) < H_{\beta}(a,b) \tag{1.14}$$

is valid for all $a, b > 0$ with $a \neq b$, are $\alpha_{\min} = 2^{1/r} - 2$ and $\beta_{\max} = 2(1-r)/r$.

Notice that in our case $r > 1$; the two numbers $\alpha_{\min}$ and $\beta_{\max}$ are all negative see Corollary 2.2 below. Thus, the result in this paper is different from [7, Theorem A].

2. Preliminary Lemmas

The following lemma will be repeatedly used in the proof of Theorem 1.1.

Lemma 2.1. *For* $1 < r < +\infty$*, one has*

$$r2^{1/r-1} > 1. \tag{2.1}$$

Proof. We show that

$$m(r) = (1-r)\log 2 + r\log r > 0, \tag{2.2}$$

which is clearly equivalent to the claim. Equation (2.2) follows from the facts

$$\lim_{r \to 1^+} m(r) = 0, \qquad m'(r) = -\log 2 + \log r + 1 > 0. \tag{2.3}$$

□

Corollary 2.2. *If* $1 < r < +\infty$*, then*

$$-2 < \frac{2(1-r)}{r} < 2^{1/r} - 2 < 0. \tag{2.4}$$

Proof. Since for $1 < r < +\infty$, the two functions

$$\varphi_1(r) = \frac{2(1-r)}{r}, \qquad \varphi_2(r) = 2^{1/r} - 2 \tag{2.5}$$

are strictly decreasing, then one has

$$-2 = \lim_{r \to +\infty} \varphi_1(r) < \varphi_1(r), \qquad \varphi_2(r) < \lim_{r \to 1^+} \varphi_2(r) = 0. \tag{2.6}$$

It suffices to show that

$$2 - 2r < r2^{1/r} - 2r, \tag{2.7}$$

which is equivalent to (2.1). □

Lemma 2.3. *For* $x > 1$ *and* $r > 1$*, let*

$$\ell(x) = \left(x^{2r} + 1\right)^{1/r-2} x^{2(r-1)} \left(x^{2r} + 2r - 1\right). \tag{2.8}$$

Then, $\ell(x)$ *is strictly decreasing for* $x > 1$*, and*

$$\lim_{x \to 1^+} \ell(x) = r2^{1/r-1}, \qquad \lim_{x \to +\infty} \ell(x) = 1. \tag{2.9}$$

Proof. The fact $\ell(x) > 0$ for $x > 1$ and $r > 1$ is obvious, which allows us to take the logarithmic function of $\ell(x)$,

$$\log \ell(x) = \left(\frac{1}{r} - 2\right) \log\left(x^{2r} + 1\right) + 2(r-1)\log x + \log\left(x^{2r} + 2r - 1\right). \tag{2.10}$$

Some tedious, but not difficult calculations lead to

$$\begin{aligned} [\log \ell(x)]' &= \left(\frac{1}{r} - 2\right)\frac{2rx^{2r-1}}{x^{2r}+1} + \frac{2(r-1)}{x} + \frac{2rx^{2r-1}}{x^{2r}+2r-1} \\ &= \frac{m(x)}{x(x^{2r}+1)(x^{2r}+2r-1)}, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} m(x) &= 2(1-2r)x^{2r}\left(x^{2r}+2r-1\right) + (2r-1)\left(x^{2r}+1\right)\left(x^{2r}+2r-1\right) + 2rx^{2r}\left(x^{2r}+1\right) \\ &= 2(r-1)(2r-1)\left(1-x^{2r}\right). \end{aligned} \tag{2.12}$$

It is easy to see that

$$\lim_{x\to 1^+} m(x) = 0, \tag{2.13}$$

$$m'(x) = -4r(r-1)(2r-1)x^{2r-1} < 0. \tag{2.14}$$

Equation (2.14) implies that $m(x)$ is strictly decreasing for $x > 1$, which together with (2.13) implies $m(x) < 0$ for $x > 1$. Thus, by (2.11),

$$[\log \ell(x)]' < 0, \tag{2.15}$$

which implies

$$\ell'(x) = [\log \ell(x)]'\ell(x) < 0. \tag{2.16}$$

Hence, $\ell(x)$ is strictly decreasing.

It remains to show (2.9). The first equality in (2.9) is obvious. The second one follows from

$$\begin{aligned} \lim_{x\to+\infty} \ell(x) &= \lim_{x\to+\infty}\left(x^{2r}+1\right)^{1/r-2} x^{2(r-1)}\left(x^{2r}+2r-1\right) \\ &= \lim_{t\to 0^+} \frac{(2r-1)t^{2r}+1}{(1+t^{2r})^{(2r-1)/r}} \\ &= 1. \end{aligned} \tag{2.17}$$

This ends the proof of Lemma 2.3. □

Lemma 2.4. *For $x > 1$, $r > 1$, and $\omega = 2^{1/r} - 2$, let*

$$f_r(x) = 2^{1/r}\left(x^2 + \omega x + 1\right) - (\omega + 2)\left(x^{2r} + 1\right)^{1/r}. \tag{2.18}$$

Then,

$$\begin{aligned} \lim_{x \to +\infty} f_r(x) &= -\infty, \\ \lim_{x \to +\infty} f_r'(x) &= 2^{1/r}\left(2^{1/r} - 2\right). \end{aligned} \tag{2.19}$$

Proof. Simple calculations lead to

$$\begin{aligned} \lim_{x \to +\infty} f_r(x) &= \lim_{x \to +\infty} 2^{1/r}\left(x^2 + \omega x + 1\right) - (\omega + 2)\left(x^{2r} + 1\right)^{1/r} \\ &= \lim_{t \to 0^+} \frac{2^{1/r}(t^2 + \omega t + 1) - (\omega + 2)(t^{2r} + 1)^{1/r}}{t^2} \\ &= -\infty, \\ \lim_{x \to +\infty} f_r'(x) &= \lim_{x \to +\infty} 2^{1/r}(2x + \omega) - 2(\omega + 2)\left(x^{2r} + 1\right)^{1/r} \\ &= \lim_{t \to 0^+} \frac{2^{1/r}(2 + \omega t) - 2(\omega + 2)(1 + t^{2r})^{(1-r)/r}}{t} \\ &= \lim_{t \to 0^+} \frac{2^{1/r}(2 + \omega t)(1 + t^{2r})^{(r-1)/r} - 2(\omega + 2)}{t(1 + t^{2r})^{(r-1)/r}} \\ &= \lim_{t \to 0^+} \frac{2^{1/r}\omega(1 + t^{2r})^{(r-1)/r} + 2^{(1/r)+1}(r - 1)(2 + \omega t)(1 + t^{2r})^{-1/r} t^{2r-1}}{(1 + t^{2r})^{(r-1)/r} + 2(r - 1)(1 + t^{2r})^{-1/r} t^{2r}} \\ &= 2^{1/r}\omega = 2^{1/r}\left(\omega^{1/r} - 2\right) < 0, \end{aligned} \tag{2.20}$$

where we have used L'Hospital's law. This ends the proof of Lemma 2.4. □

3. Proof of Theorem 1.1

Proof. Firstly, we prove that for $1 < r < +\infty$,

$$H_{2(1-r)/r}(a, b) > A_r(a, b), \tag{3.1}$$

$$H_{2^{1/r}-2}(a, b) < A_r(a, b) \tag{3.2}$$

hold true for all $a, b > 0$ with $a \neq b$. It is no loss of generality to assume that $a > b > 0$. Let $x = \sqrt{b/a} > 1$ and $\omega \in \{2(1-r)/r, 2^{1/r} - 2\}$. In view of Corollary 2.2, $-2 < \omega < 0$. Equations (1.3) and (1.5) lead to

$$\begin{aligned}\frac{1}{a}[H_\omega(a,b) - A_r(a,b)] &= H_\omega\left(x^2,1\right) - A_r\left(x^2,1\right) \\ &= \frac{x^2+\omega x+1}{\omega+2} - \left(\frac{x^{2r}+1}{2}\right)^{1/r} \\ &= \frac{2^{1/r}\left(x^2+\omega x+1\right) - (\omega+2)\left(x^{2r}+1\right)^{1/r}}{2^{1/r}(\omega+2)} \\ &= \frac{f_r(x)}{2^{1/r}(\omega+2)},\end{aligned} \tag{3.3}$$

where $f_r(x)$ is defined by (2.18). It is easy to see that

$$\lim_{x\to 1^+} f_r(x) = 0, \tag{3.4}$$

$$f_r'(x) = 2^{1/r}(2x+\omega) - 2(\omega+2)\left(x^{2r}+1\right)^{1/r-1}x^{2r-1}, \tag{3.5}$$

$$\lim_{x\to 1^+} f_r'(x) = 0. \tag{3.6}$$

By Lemma 2.3,

$$\begin{aligned}f_r''(x) &= 2\left\{2^{1/r} - (\omega+2)\left[2(1-r)\left(x^{2r}+1\right)^{1/r-2}x^{4r-2} + (2r-1)\left(x^{2r}+1\right)^{1/r-1}x^{2(r-1)}\right]\right\} \\ &= 2\left[2^{1/r} - (\omega+2)\ell(x)\right] > 2\left[2^{1/r} - (\omega+2)r2^{1/r-1}\right] = 2^{1/r}[2-(\omega+2)r],\end{aligned} \tag{3.7}$$

$$\lim_{x\to 1^+} f_r''(x) = 2\left\{2^{1/r} - (\omega+2)\left[2(1-r)2^{1/r-2} + (2r-1)2^{1/r-1}\right]\right\} = 2^{1/r}[2-(\omega+2)r]. \tag{3.8}$$

We now distinguish between two cases.

Case 1 ($\omega = 2(1-r)/r$). Since $2-(\omega+2)r = 0$, then by (3.7), $f_r''(x) > 0$. Thus, $f_r'(x)$ is strictly increasing for $x > 1$, which together with (3.6) implies $f_r'(x) > 0$. Hence, $f_r(x)$ is strictly increasing for $x > 1$. Since (3.4), then $f_r(x) > 0$. Equation (3.1) follows from (3.3).

Case 2 ($\omega = 2^{1/r} - 2$). By (3.5) and (2.11),

$$\begin{aligned} f_r'''(x) &= -2(\omega+2)\ell'(x) = -2(\omega+2)\left[\log \ell(x)\right]'\ell(x) \\ &= -2(\omega+2)\frac{m(x)\ell(x)}{x(x^{2r}+1)(x^r+2r-1)} \\ &> 0. \end{aligned} \tag{3.9}$$

Thus, $f_r''(x)$ is strictly increasing. Equations (3.8) and (2.1) imply

$$\lim_{x\to 1^+} f_r''(x) = 2^{1/r}[2-(\omega+2)r] = 2^{1/r+1}\left(1-r2^{1/r-1}\right) < 0. \tag{3.10}$$

Equations (3.7) and (2.9) imply

$$\lim_{x\to +\infty} f_r''(x) = \lim_{x\to +\infty} 2\left[2^{1/r} - (\omega+2)\ell(x)\right] = 2\left(2^{1/r}-1\right) > 0. \tag{3.11}$$

Combining (3.10) with (3.11), we obviously know that there exists $\lambda_1 > 1$ such that $f_r''(x) < 0$ for $x \in (1,\lambda_1)$ and $f_r''(x) > 0$ for $x \in (\lambda_1,+\infty)$. This implies that $f_r'(x)$ is strictly decreasing for $x \in (1,\lambda_1)$ and strictly increasing for $x \in (\lambda_1,+\infty)$. By (3.6) and Lemma 2.4, we know that $f_r'(x) < 0$ for $x > 1$. Therefore, $f_r(x)$ is strictly decreasing. By (3.4) and Lemma 2.4 again, we derive that $f_r(x) < 0$ for $x > 1$. Equation (3.2) follows from (3.3).

Secondly, we prove that $H_{2^{1/r}-2}(a,b)$ is the best lower bound for the power mean $A_r(a,b)$ for $1 < r < +\infty$. For any $\alpha < 2^{1/r} - 2$,

$$\lim_{x\to +\infty} \frac{H_\alpha(x,1)}{A_r(x,1)} = \lim_{x\to +\infty} \frac{2^{1/r}\left(x+\alpha\sqrt{x}+1\right)}{(\alpha+2)(x^r+1)^r} = \frac{2^{1/r}}{\alpha+2} > 1. \tag{3.12}$$

Hence, there exists $X = X(\alpha) > 1$ such that $H_\alpha(x,1) > A_r(x,1)$ for $x \in (X,+\infty)$.

Finally, we prove that $H_{2(1-r)/r}(a,b)$ is the best upper bound for the power mean $A_r(a,b)$ for $1 < r < +\infty$. For any $\beta > 2(1-r)/r$, by (3.7) (with β in place of ω), we have

$$\lim_{x\to 1^+} f_r''(x) = 2^{1/r}\left[2-(\beta+2)r\right] < 0. \tag{3.13}$$

Hence, by the continuity of $f_r''(x)$, there exists $\delta = \delta(\beta) > 0$ such that $f_r''(x) < 0$ for $x \in (1,1+\delta)$. Thus $f_r(x)$ is strictly decreasing for $x \in (1,1+\delta)$. From (3.6), $f_r'(x) < 0$ for $x \in (1,1+\delta)$. This result together with (3.4) implies that $f_r(x) < 0$ for $x \in (1,1+\delta)$. Hence, by (3.3),

$$H_\beta\left(x^2,1\right) < A_r\left(x^2,1\right), \tag{3.14}$$

for $x \in (1,1+\delta)$. □

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References

[1] W. Janous, "A note on generalized Heronian means," *Mathematical Inequalities & Applications*, vol. 4, no. 3, pp. 369–375, 2001.
[2] H. J. Seiffert, "Problem 887," *Nieuw Archief voor Wiskunde*, vol. 11, no. 2, pp. 176–176, 1993.
[3] M. qiji, "Dual mean, logatithmic and Heronian dual mean of two positive numbers," *Journal of Suzhou College of Education*, vol. 16, pp. 82–85, 1999.
[4] Y. M. Chu and W. F. Xia, "Two sharp inequalities for power mean, geometric mean, and harmonic mean," *Journal of Inequalities and Applications*, Article ID 741923, 6 pages, 2009.
[5] B. Y. Long and Y. M. Chu, "Optimal power mean bounds for the weighted geometric mean of classical means," *Journal of Inequalities and Applications*, Article ID 905679, 6 pages, 2010.
[6] M. Y. Shi, Y. M. Chu, and Y. P. Jiang, "Optimal inequalities among various means of two arguments," *Abstract and Applied Analysis*, Article ID 694394, 10 pages, 2009.
[7] K. Z. Guan and H. T. Zhu, "The generalized Heronian mean and its inequalities," *Univerzitet u Beogradu. Publikacije Elektrotehničkog Fakulteta. Serija Matematika*, vol. 17, pp. 60–75, 2006.
[8] P. A. Hästö, "A monotonicity property of ratios of symmetric homogeneous means," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 5, article 71, pp. 1–54, 2002.
[9] J. J. Wen and W. L. Wang, "The optimization for the inequalities of power means," *Journal of Inequalities and Applications*, vol. 2006, Article ID 46782, 25 pages, 2006.
[10] P. A. Hästö, "Optimal inequalities between Seiffert's mean and power means," *Mathematical Inequalities & Applications*, vol. 7, no. 1, pp. 47–53, 2004.
[11] Y. M. Chu and B. Y. Long, "Best possible inequalities between generalized logarithmic mean and classical means," *Abstract and Applied Analysis*, vol. 2010, Article ID 303286, 13 pages, 2010.
[12] Y. M. Chu and W. F. Xia, "Inequalities for generalized logarithmic means," *Journal of Inequalities and Applications*, vol. 2009, Article ID 763252, 7 pages, 2009.
[13] W. F. Xia, Y. M. Chu, and G. D. Wang, "The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means," *Abstract and Applied Analysis*, vol. 2010, Article ID 604804, 9 pages, 2010.
[14] T. Hara, M. Uchiyama, and S. E. Takahasi, "A refinement of various mean inequalities," *Journal of Inequalities and Applications*, vol. 2, no. 4, pp. 387–395, 1998.
[15] E. Neuman and J. Sándor, "On certain means of two arguments and their extensions," *International Journal of Mathematics and Mathematical Sciences*, no. 16, pp. 981–993, 2003.