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Research Article

Conjugacy of Self-Adjoint Difference Equations of Even Order

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We study oscillation properties of $2n$-order Sturm-Liouville difference equations. For these equations, we show a conjugacy criterion using the p-criticality (the existence of linear dependent recessive solutions at ∞ and $-\infty$). We also show the equivalent condition of p-criticality for one term $2n$-order equations.

1. Introduction

In this paper, we deal with $2n$-order Sturm-Liouville difference equations and operators

$$L(y)_k = \sum_{\nu=0}^{n} (-\Delta)^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k+n-\nu} \right) = 0, \quad r_k^{[n]} > 0,\ k \in \mathbb{Z}, \tag{1.1}$$

where Δ is the forward difference operator, that is, $\Delta y_k = y_{k+1} - y_k$, and $r^{[\nu]}, \nu = 0, \dots, n$, are real-valued sequences. The main result is the conjugacy criterion which we formulate for the equation $L(y)_k + q_k y_{k+n} = 0$, that is viewed as a perturbation of (1.1), and we suppose that (1.1) is at least p-critical for some $p \in \{1, \dots, n\}$. The concept of p-criticality (a disconjugate equation is said to be p-critical if and only if it possesses p solutions that are recessive both at ∞ and $-\infty$, see Section 3) was introduced for second-order difference equations in [1], and later in [2] for (1.1). For the continuous counterpart of the used techniques, see [3–5] from where we get an inspiration for our research.

The paper is organized as follows. In Section 2, we recall necessary preliminaries. In Section 3, we recall the concept of p-criticality, as introduced in [2], and show the first result—the equivalent condition of p-criticality for the one term difference equation

$$\Delta^n \left(r_k \Delta^n y_k \right) = 0 \tag{1.2}$$

(Theorem 3.4). In Section 4 we show the conjugacy criterion for equation

$$(-\Delta)^n \left(r_k \Delta^n y_k \right) + q_k y_{k+n} = 0, \tag{1.3}$$

and Section 5 is devoted to the generalization of this criterion to the equation with the middle terms

$$\sum_{\nu=0}^{n} (-\Delta)^{\nu} \left(r_k^{[\nu]} \Delta^{\nu} y_{k+n-\nu} \right) + q_k y_{k+n} = 0. \tag{1.4}$$

2. Preliminaries

The proof of our main result is based on equivalency of (1.1) and the linear Hamiltonian difference systems

$$\Delta x_k = A x_{k+1} + B_k u_k, \qquad \Delta u_k = C_k x_{k+1} - A^T u_k, \tag{2.1}$$

where A, B_k, and C_k are $n \times n$ matrices of which B_k and C_k are symmetric. Therefore, we start this section recalling the properties of (2.1), which we will need later. For more details, see the papers [6–11] and the books [12, 13].

The substitution

$$x_k^{[y]} = \begin{pmatrix} y_{k+n-1} \\ \Delta y_{k+n-2} \\ \vdots \\ \Delta^{n-1} y_k \end{pmatrix}, \qquad u_k^{[y]} = \begin{pmatrix} \sum_{\nu=1}^{n} (-\Delta)^{\nu-1} \left(r_k^{[\nu]} \Delta^{\nu} y_{k+n-\nu} \right) \\ \vdots \\ -\Delta \left(r_k^{[n]} \Delta^n y_k \right) + r_k^{[n-1]} \Delta^{n-1} y_{k+1} \\ r_k^{[n]} \Delta^n y_k \end{pmatrix} \tag{2.2}$$

transforms (1.1) to linear Hamiltonian system (2.1) with the $n \times n$ matrices A, B_k, and C_k given by

$$\begin{gathered} A = \left(a_{ij} \right)_{i,j=1}^{n}, \qquad a_{ij} = \begin{cases} 1, & \text{if } j = i+1,\ i = 1, \dots, n-1, \\ 0, & \text{elsewhere}, \end{cases} \\ B_k = \operatorname{diag}\left\{ 0, \dots, 0, \frac{1}{r_k^{[n]}} \right\}, \qquad C_k = \operatorname{diag}\left\{ r_k^{[0]}, \dots, r_k^{[n-1]} \right\}. \end{gathered} \tag{2.3}$$

Then, we say that the solution (x, u) of (2.1) is generated by the solution y of (1.1).

Let us consider, together with system (2.1), the matrix linear Hamiltonian system

$$\Delta X_k = AX_{k+1} + B_k U_k, \qquad \Delta U_k = C_k X_{k+1} - A^T U_k, \tag{2.4}$$

where the matrices A, B_k, and C_k are also given by (2.3). We say that the matrix solution (X, U) of (2.4) is generated by the solutions $y^{[1]}, \ldots, y^{[n]}$ of (1.1) if and only if its columns are generated by $y^{[1]}, \ldots, y^{[n]}$, respectively, that is, $(X, U) = (x^{[y_1]}, \ldots, x^{[y_n]}, u^{[y_1]}, \ldots, u^{[y_n]})$. Reversely, if we have the solution (X, U) of (2.4), the elements from the first line of the matrix X are exactly the solutions $y^{[1]}, \ldots, y^{[n]}$ of (1.1). Now, we can define the oscillatory properties of (1.1) via the corresponding properties of the associated Hamiltonian system (2.1) with matrices A, B_k, and C_k given by (2.3), for example, (1.1) is disconjugate if and only if the associated system (2.1) is disconjugate, the system of solutions $y^{[1]}, \ldots, y^{[n]}$ is said to be recessive if and only if it generates the recessive solution X of (2.4), and so forth. Therefore, we define the following properties just for linear Hamiltonian systems.

For system (2.4), we have an analog of the continuous *Wronskian identity*. Let (X, U) and $(\widetilde{X}, \widetilde{U})$ be two solutions of (2.4). Then,

$$X_k^T \widetilde{U}_k - U_k^T \widetilde{X}_k \equiv W \tag{2.5}$$

holds with a constant matrix W. We say that the solution (X, U) of (2.4) is a *conjoined basis*, if

$$X_k^T U_k \equiv U_k^T X_k, \qquad \operatorname{rank}\begin{pmatrix} X \\ U \end{pmatrix} = n. \tag{2.6}$$

Two conjoined bases $(X, U), (\widetilde{X}, \widetilde{U})$ of (2.4) are called *normalized* conjoined bases of (2.4) if $W = I$ in (2.5) (where I denotes the identity operator).

System (2.1) is said to be *right disconjugate* in a discrete interval $[l, m]$, $l, m \in \mathbb{Z}$, if the solution $\binom{X}{U}$ of (2.4) given by the initial condition $X_l = 0, U_l = I$ satisfies

$$\ker X_{k+1} \subseteq \ker X_k, \qquad X_k X_{k+1}^{\dagger} (I - A)^{-1} B_k \geq 0, \tag{2.7}$$

for $k = l, \ldots, m-1$, see [6]. Here ker, †, and ≥ stand for kernel, Moore-Penrose generalized inverse, and nonnegative definiteness of the matrix indicated, respectively. Similarly, (2.1) is said to be *left disconjugate* on $[l, m]$, if the solution given by the initial condition $X_m = 0$, $U_m = -I$ satisfies

$$\ker X_k \subseteq \ker X_{k+1}, \quad X_{k+1} X_k^{\dagger} B_k (I - A)^{T-1} \geq 0, \quad k = l, \ldots, m-1. \tag{2.8}$$

System (2.1) is disconjugate on $\mathbb{Z}$, if it is right disconjugate, which is the same as left disconjugate, see [14, Theorem 1], on $[l, m]$ for every $l, m \in \mathbb{Z}$, $l < m$. System (2.1) is said to be *nonoscillatory* at ∞ (*nonoscillatory* at $-\infty$), if there exists $l \in \mathbb{Z}$ such that it is right disconjugate on $[l, m]$ for every $m > l$ (there exists $m \in \mathbb{Z}$ such that (2.1) is left disconjugate on $[l, m]$ for every $l < m$).

We call a conjoined basis $\binom{\tilde{X}}{\tilde{U}}$ of (2.4) the *recessive solution* at ∞, if the matrices $\tilde{X}_k$ are nonsingular, $\tilde{X}_k\tilde{X}_{k+1}^{-1}(I-A_k)^{-1}B_k \geq 0$ (both for large k), and for any other conjoined basis $\binom{X}{U}$, for which the (constant) matrix $X^T\tilde{U} - U^T\tilde{X}$ is nonsingular, we have

$$\lim_{k\to\infty} X_k^{-1}\tilde{X}_k = 0. \tag{2.9}$$

The solution (X, U) is called the *dominant solution* at ∞. The recessive solution at ∞ is determined uniquely up to a right multiple by a nonsingular constant matrix and exists whenever (2.4) is nonoscillatory and eventually controllable. (System is said to be *eventually controllable* if there exist $N, \kappa \in \mathbb{N}$ such that for any $m \geq N$ the trivial solution $\binom{x}{u} = \binom{0}{0}$ of (2.1) is the only solution for which $x_m = x_{m+1} = \cdots = x_{m+\kappa} = 0$.) The equivalent characterization of the recessive solution $\binom{\tilde{X}}{\tilde{U}}$ of eventually controllable Hamiltonian difference systems (2.1) is

$$\lim_{k\to\infty}\left(\sum k\tilde{X}_{j+1}^{-1}(I-A)^{-1}B_j\tilde{X}_j^{T-1}\right)^{-1} = 0, \tag{2.10}$$

see [12]. Similarly, we can introduce the recessive and the dominant solutions at $-\infty$. For related notions and results for second-order dynamic equations, see, for example, [15, 16].

We say that a pair (x, u) is *admissible* for system (2.1) if and only if the first equation in (2.1) holds.

The energy functional of (1.1) is given by

$$\mathcal{F}(y) := \sum_{k=-\infty}^{\infty}\sum_{\nu=0}^{n} r_k^{[\nu]}\left(\Delta^{\nu} y_{k+n-\nu}\right)^2. \tag{2.11}$$

Then, for admissible (x, u), we have

$$\begin{aligned}\mathcal{F}(y) &= \sum_{k=-\infty}^{\infty}\sum_{\nu=0}^{n} r_k^{[\nu]}\left(\Delta^{\nu} y_{k+n-\nu}\right)^2 \\ &= \sum_{k=-\infty}^{\infty}\left[\sum_{\nu=0}^{n-1} r_k^{[\nu]}\left(\Delta^{\nu} y_{k+n-\nu}\right)^2 + \frac{1}{r_k^{[n]}}\left(r_k^{[n]}\Delta^n y_k\right)^2\right] \\ &= \sum_{k=-\infty}^{\infty}\left[x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\right] =: \mathcal{F}(x, u).\end{aligned} \tag{2.12}$$

To prove our main result, we use a variational approach, that is, the equivalency of disconjugacy of (1.1) and positivity of $\mathcal{F}(x, u)$; see [6].

Now, we formulate some auxiliary results, which are used in the proofs of Theorems 3.4 and 4.1. The following Lemma describes the structure of the solution space of

$$\Delta^n\left(r_k\Delta^n y_k\right) = 0, \quad r_k > 0. \tag{2.13}$$

Lemma 2.1 (see [17, Section 2]). *Equation* (2.13) *is disconjugate on* $\mathbb{Z}$ *and possesses a system of solutions* $y^{[j]}, \tilde{y}^{[j]}$, $j = 1, \dots, n$, *such that*

$$y^{[1]} \prec \cdots \prec y^{[n]} \prec \tilde{y}^{[1]} \prec \cdots \prec \tilde{y}^{[n]} \tag{2.14}$$

as $k \to \infty$, *where* $f \prec g$ *as* $k \to \infty$ *for a pair of sequences* f, g *means that* $\lim_{k\to\infty}(f_k/g_k) = 0$. *If* (2.14) *holds, the solutions* $y^{[j]}$ *form the recessive system of solutions at* ∞, *while* $\tilde{y}^{[j]}$ *form the dominant system,* $j = 1, \dots, n$. *The analogous statement holds for the ordered system of solutions as* $k \to -\infty$.

Now, we recall the transformation lemma.

Lemma 2.2 (see [14, Theorem 4]). *Let* $h_k > 0$, $L(y) = \sum_{\nu=0}^{n}(-\Delta)^\nu (r_k^{[\nu]} \Delta^\nu y_{k+n-\nu})$ *and consider the transformation* $y_k = h_k z_k$. *Then, one has*

$$h_{k+n} L(y) = \sum_{\nu=0}^{n} (-\Delta)^\nu \left(R_k^{[\nu]} \Delta^\nu z_{k+n-\nu} \right), \tag{2.15}$$

where

$$R_k^{[n]} = h_{k+n} h_k r_k^{[n]}, \qquad R_k^{[0]} = h_{k+n} L(h), \tag{2.16}$$

that is, y *solves* $L(y) = 0$ *if and only if* z *solves the equation*

$$\sum_{\nu=0}^{n} (-\Delta)^\nu \left(R_k^{[\nu]} \Delta^\nu z_{k+n-\nu} \right) = 0. \tag{2.17}$$

The next lemma is usually called the second mean value theorem of summation calculus.

Lemma 2.3 (see [17, Lemma 3.2]). *Let* $n \in \mathbb{N}$ *and the sequence* a_k *be monotonic for* $k \in [K+n-1, L+n-1]$ *(i.e.,* Δa_k *does not change its sign for* $k \in [K+n-1, L+n-2]$*). Then, for any sequence* b_k *there exist* $n_1, n_2 \in [K, L-1]$ *such that*

$$\begin{gathered} \sum_{j=K}^{L-1} a_{n+j} b_j \le a_{K+n-1} \sum_{i=K}^{n_1-1} b_i + a_{L+n-1} \sum_{i=n_1}^{L-1} b_i, \\ \sum_{j=K}^{L-1} a_{n+j} b_j \ge a_{K+n-1} \sum_{i=K}^{n_2-1} b_i + a_{L+n-1} \sum_{i=n_2}^{L-1} b_i. \end{gathered} \tag{2.18}$$

Now, let us consider the linear difference equation

$$y_{k+n} + a_k^{[n-1]} y_{k+n-1} + \cdots + a_k^{[0]} y_k = 0, \tag{2.19}$$

where $k \ge n_0$ for some $n_0 \in \mathbb{N}$ and $a_k^{[0]} \ne 0$, and let us recall the main ideas of [18] and [19, Chapter IX].

An integer $m > n_0$ is said to be a *generalized zero* of multiplicity k of a nontrivial solution y of (2.19) if $y_{m-1} \neq 0$, $y_m = y_{m+1} = \cdots = y_{m+k-2} = 0$, and $(-1)^k y_{m-1} y_{m+k-1} \geq 0$. Equation (2.19) is said to be eventually disconjugate if there exists $N \in \mathbb{N}$ such that no non-trivial solution of this equation has n or more generalized zeros (counting multiplicity) on $[N, \infty)$.

A system of sequences $u_k^{[1]}, \ldots, u_k^{[n]}$ is said to form the *D-Markov system* of sequences for $k \in [N, \infty)$ if Casoratians

$$C\left(u^{[1]}, \ldots, u^{[j]}\right)_k = \begin{vmatrix} u_k^{[1]} & \cdots & u_k^{[j]} \\ u_{k+1}^{[1]} & \cdots & u_{k+1}^{[j]} \\ \vdots & & \vdots \\ u_{k+j-1}^{[1]} & \cdots & u_{k+j-1}^{[j]} \end{vmatrix}, \quad j = 1, \ldots, n \tag{2.20}$$

are positive on $(N + j, \infty)$.

Lemma 2.4 (see [19, Theorem 9.4.1]). *Equation* (2.19) *is eventually disconjugate if and only if there exist $N \in \mathbb{N}$ and solutions $y^{[1]}, \ldots, y^{[n]}$ of* (2.19) *which form a D-Markov system of solutions on (N, ∞). Moreover, this system can be chosen in such a way that it satisfies the additional condition*

$$\lim_{k \to \infty} \frac{y_k^{[i]}}{y_k^{[i+1]}} = 0, \quad i = 1, \ldots, n-1. \tag{2.21}$$

3. Criticality of One-Term Equation

Suppose that (1.1) is disconjugate on $\mathbb{Z}$ and let $\hat{y}^{[i]}$ and $\tilde{y}^{[i]}$, $i = 1, \ldots, n$, be the recessive systems of solutions of $L(y) = 0$ at $-\infty$ and ∞, respectively. We introduce the linear space

$$\mathscr{H} = \operatorname{Lin}\left\{\hat{y}^{[1]}, \ldots, \hat{y}^{[n]}\right\} \cap \operatorname{Lin}\left\{\tilde{y}^{[1]}, \ldots, \tilde{y}^{[n]}\right\}. \tag{3.1}$$

Definition 3.1 (see [2]). Let (1.1) be disconjugate on $\mathbb{Z}$ and let $\dim \mathscr{H} = p \in \{1, \ldots, n\}$. Then, we say that the operator L (or (1.1)) is *p-critical* on $\mathbb{Z}$. If $\dim \mathscr{H} = 0$, we say that L is *subcritical* on $\mathbb{Z}$. If (1.1) is not disconjugate on $\mathbb{Z}$, that is, $L \not\geq 0$, we say that L is *supercritical* on $\mathbb{Z}$.

To prove the result in this section, we need the following statements, where we use the generalized power function

$$k^{(0)} = 1, \qquad k^{(i)} = k(k-1)\cdots(k-i+1), \quad i \in \mathbb{N}. \tag{3.2}$$

For reader's convenience, the first statement in the following lemma is slightly more general than the corresponding one used in [2] (it can be verified directly or by induction).

Lemma 3.2 (see [2]). *The following statements hold.*

(i) *Let z_k be any sequence, $m \in \{0, \dots, n\}$, and*

$$y_k := \sum_{j=0}^{k-1} (k-j-1)^{(n-1)} z_j, \tag{3.3}$$

then

$$\Delta^m y_k = \begin{cases} (n-1)^{(m)} \sum_{j=0}^{k-1} (k-j-1)^{(n-1-m)} z_j, & m \le n-1, \\ (n-1)! z_k, & m = n. \end{cases} \tag{3.4}$$

(ii) *The generalized power function has the binomial expansion*

$$(k-j)^{(n)} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} k^{(n-i)} (j+i-1)^{(i)}. \tag{3.5}$$

We distinguish two types of solutions of (2.13). The *polynomial* solutions $k^{(i)}$, $i = 0, \dots, n-1$, for which $\Delta^n y_k = 0$, and *nonpolynomial* solutions

$$\sum_{j=0}^{k-1} (k-j-1)^{(n-1)} j^{(i)} r_j^{-1}, \quad i = 0, \dots, n-1, \tag{3.6}$$

for which $\Delta^n y_k \neq 0$. (Using Lemma 3.2(i) we obtain $\Delta^n y_k = (n-1)! k^{(i)} r_k^{-1}$.)

Now, we formulate one of the results of [20].

Proposition 3.3 (see [20, Theorem 4]). *If for some $m \in \{0, \dots, n-1\}$*

$$\sum_{k=-\infty}^{0} \left[k^{(n-m-1)}\right]^2 r_k^{-1} = \infty = \sum_{k=0}^{\infty} \left[k^{(n-m-1)}\right]^2 r_k^{-1}, \tag{3.7}$$

then

$$\operatorname{Lin}\left\{1, \dots, k^{(m)}\right\} \subseteq \mathscr{H}, \tag{3.8}$$

that is, (2.13) *is at least* $(m+1)$*-critical on* $\mathbb{Z}$.

Now, we show that (3.7) is also sufficient for (2.13) to be at least $(m+1)$-critical.

Theorem 3.4. *Let $m \in \{0, \dots, n-1\}$. Equation* (2.13) *is at least* $(m+1)$*-critical if and only if* (3.7) *holds.*

Proof. Let $\mathcal{U}^{+}$ and $\mathcal{U}^{-}$ denote the subspaces of the solution space of (2.13) generated by the recessive system of solutions at ∞ and $-\infty$, respectively. Necessity of (3.7) follows directly from Proposition 3.3. To prove sufficiency, it suffices to show that if one of the sums in (3.7) is convergent, then $\{1,\dots,k^{(m)}\} \nsubseteq \mathcal{U}^{+}\cap\mathcal{U}^{-}$. We show this statement for the sum $\sum^{\infty}$. The other case is proved similarly, so it will be omitted. Particularly, we show

$$\sum_{k=0}^{\infty}\left[k^{(n-m-1)}\right]^{2} r_k^{-1} < \infty \Longrightarrow k^{(m)} \notin \mathcal{U}^{+}. \tag{3.9}$$

Let us denote $p := n-m-1$, and let us consider the following nonpolynomial solutions of (2.13):

$$y_k^{[\ell]} = \sum_{j=0}^{k-1}(k-j-1)^{(n-1)} j^{(p+\ell-1)} r_j^{-1} - \sum_{i=0}^{p}\left[(-1)^i\binom{n-1}{i}(k-1)^{(n-1-i)}\sum_{j=0}^{\infty} j^{(p+\ell-1)}(j+i-1)^{(i)} r_j^{-1}\right], \tag{3.10}$$

where $\ell = 1-p,\dots,m+1$. By Stolz-Cesàro theorem, since (using Lemma 3.2(i)) $\Delta^n y_k^{[\ell]} = (n-1)!k^{(p+\ell-1)}r_k^{-1}$, these solutions are ordered, that is, $y^{[i]} \prec y^{[i+1]}$, $i = 1-p,\dots,m$, as well as the polynomial solutions, that is, $k^{(i)} \prec k^{(i+1)}$, $i = 0,\dots,n-2$.

By some simple calculation and by Lemma 3.2 (at first, we use (i), and at the end, we use (ii)), we have

$$\begin{aligned}
&\Delta^m y_k^{[1]} \\
&\quad= \frac{(n-1)!}{(n-m-1)!}\sum_{j=0}^{k-1}(k-j-1)^{(n-m-1)} j^{(p)} r_j^{-1} \\
&\qquad - \sum_{i=0}^{p}\left[(-1)^i\binom{n-1}{i}\frac{(n-1-i)!}{(n-m-1-i)!}(k-1)^{(n-m-1-i)}\sum_{j=0}^{\infty} j^{(p)}(j+i-1)^{(i)} r_j^{-1}\right] \\
&\quad= \frac{(n-1)!}{p!}\sum_{j=0}^{k-1}(k-j-1)^{(p)} j^{(p)} r_j^{-1} \\
&\qquad - \sum_{i=0}^{p}\left[(-1)^i\frac{(n-1)!(n-1-i)!}{(n-1-i)!i!(p-i)!}(k-1)^{(p-i)}\sum_{j=0}^{\infty} j^{(p)}(j+i-1)^{(i)} r_j^{-1}\right] \\
&\quad= \frac{(n-1)!}{p!}\left\{\sum_{j=0}^{k-1}(k-j-1)^{(p)} j^{(p)} r_j^{-1} - \sum_{i=0}^{p}\left[(-1)^i\binom{p}{i}(k-1)^{(p-i)}\sum_{j=0}^{\infty} j^{(p)}(j+i-1)^{(i)} r_j^{-1}\right]\right\} \\
&\quad= \frac{(n-1)!}{p!}\left[\sum_{j=0}^{k-1}(k-j-1)^{(p)} j^{(p)} r_j^{-1} - \sum_{j=0}^{\infty}(k-j-1)^{(p)} j^{(p)} r_j^{-1}\right]
\end{aligned}$$

$$= -\frac{(n-1)!}{p!}\sum_{j=k}^{\infty}(k-j-1)^{(p)} j^{(p)} r_j^{-1}$$

$$= (-1)^{p+1}\frac{(n-1)!}{p!}\sum_{j=k}^{\infty}(j+1-k)^{(p)} j^{(p)} r_j^{-1},$$

$$\sum_{j=k}^{\infty}(j+1-k)^{(p)} j^{(p)} r_j^{-1} \leq \sum_{j=k}^{\infty}\left[j^{(p)}\right]^2 r_j^{-1}. \tag{3.11}$$

Hence, from this and by Stolz-Cesàro theorem, we get

$$\lim_{k\to\infty}\frac{y_k^{[1]}}{k^{(m)}} = \frac{1}{m!}\lim_{k\to\infty}\Delta^m y_k^{[1]} = 0, \tag{3.12}$$

thus $y_k^{[1]} \prec k^{(m)}$. We obtained that $\{1, k, \dots, k^{(m-1)}, y^{[1-p]}, \dots, y^{[1]}\} \prec k^{(m)}$, which means that we have n solutions less than $k^{(m)}$, therefore $k^{(m)} \notin \mathcal{U}^+$ and (2.13) is at most m-critical. □

4. Conjugacy of Two-Term Equation

In this section, we show the conjugacy criterion for two-term equation.

Theorem 4.1. *Let $n > 1$, q_k be a real-valued sequence, and let there exist an integer $m \in \{0, \dots, n-1\}$ and real constants $c_0, \dots, c_m$ such that* (2.13) *is at least* $(m+1)$*-critical and the sequence $h_k := c_0 + c_1 k + \cdots + c_m k^{(m)}$ satisfies*

$$\limsup_{K\downarrow-\infty,\, L\uparrow\infty}\sum_{k=K}^{L} q_k h_{k+n}^2 \leq 0. \tag{4.1}$$

If $q \not\equiv 0$, then

$$(-\Delta)^n\left(r_k\Delta^n y_k\right) + q_k y_{k+n} = 0 \tag{4.2}$$

is conjugate on $\mathbb{Z}$.

Proof. We prove this theorem using the variational principle; that is, we find a sequence $y \in \ell_0^2(\mathbb{Z})$ such that the energy functional $F(y) = \sum_{k=-\infty}^{\infty}[r_k(\Delta^n y_k)^2 + q_k y_{k+n}^2] < 0$.

At first, we estimate the first term of $F(y)$. To do this, we use the fact that this term is an energy functional of (2.13). Let us denote it by $\widetilde{F}$ that is,

$$\widetilde{F}(y) = \sum_{k=-\infty}^{\infty} r_k\left(\Delta^n y_k\right)^2. \tag{4.3}$$

Using the substitution (2.2), we find out that (2.13) is equivalent to the linear Hamiltonian system (2.1) with the matrix $C_k \equiv 0$; that is,

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \qquad \Delta u_k = -A^T u_k, \tag{4.4}$$

and to the matrix system

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \qquad \Delta U_k = -A^T U_k. \tag{4.5}$$

Now, let us denote the recessive solutions of (4.5) at $-\infty$ and ∞ by (X^-, U^-) and (X^+, U^+), respectively, such that the first $m+1$ columns of X^+ and X^- are generated by the sequences $1, k, \ldots, k^{(m)}$. Let K, L, M, and N be arbitrary integers such that $N - M > 2n$, $M - L > 2n$, and $L - K > 2n$ (some additional assumptions on the choice of K, L, M, N will be specified later), and let $(x^{[f]}, u^{[f]})$ and $(x^{[g]}, u^{[g]})$ be the solutions of (4.4) given by the formulas

$$\begin{gathered}
x_k^{[f]} = X_k^- \left(\sum_{j=K}^{k-1} \mathcal{B}_j^-\right) \left(\sum_{j=K}^{L-1} \mathcal{B}_j^-\right)^{-1} (X_L^-)^{-1} x_L^{[h]}, \\
u_k^{[f]} = U_k^- \left(\sum_{j=K}^{k-1} \mathcal{B}_j^-\right) \left(\sum_{j=K}^{L-1} \mathcal{B}_j^-\right)^{-1} (X_L^-)^{-1} x_L^{[h]} + (X_k^-)^{T-1} \left(\sum_{j=K}^{L-1} \mathcal{B}_j^-\right)^{-1} (X_L^-)^{-1} x_L^{[h]}, \\
x_k^{[g]} = X_k^+ \left(\sum_{j=k}^{N-1} \mathcal{B}_j^+\right) \left(\sum_{j=M}^{N-1} \mathcal{B}_j^+\right)^{-1} (X_M^+)^{-1} x_M^{[h]}, \\
u_k^{[g]} = U_k^+ \left(\sum_{j=k}^{N-1} \mathcal{B}_j^+\right) \left(\sum_{j=M}^{N-1} \mathcal{B}_j^+\right)^{-1} (X_M^+)^{-1} x_M^{[h]} - (X_k^+)^{T-1} \left(\sum_{j=M}^{N-1} \mathcal{B}_j^+\right)^{-1} (X_M^+)^{-1} x_M^{[h]},
\end{gathered} \tag{4.6}$$

where

$$\begin{aligned}
\mathcal{B}_k^- &= \left(X_{k+1}^-\right)^{-1} (I - A)^{-1} B_k \left(X_k^-\right)^{T-1}, \\
\mathcal{B}_k^+ &= \left(X_{k+1}^+\right)^{-1} (I - A)^{-1} B_k \left(X_k^+\right)^{T-1},
\end{aligned} \tag{4.7}$$

and $(x^{[h]}, u^{[h]})$ is the solution of (4.4) generated by h. By a direct substitution, and using the convention that $\sum_k^{k-1} = 0$, we obtain

$$x_K^{[f]} = 0, \qquad x_L^{[f]} = x_L^{[h]}, \qquad x_M^{[g]} = x_M^{[h]}, \qquad x_N^{[g]} = 0. \tag{4.8}$$

Now, from (4.1), together with the assumption $q \not\equiv 0$, we have that there exist $\widetilde{k} \in \mathbb{Z}$ and $\varepsilon > 0$ such that $q_{\widetilde{k}} \leq -\varepsilon$. Because the numbers K, L, M, and N have been "almost free" so far, we may choose them such that $L < \widetilde{k} < M - n - 1$.

Let us introduce the test sequence

$$y_k := \begin{cases} 0, & k \in (-\infty, K-1], \\ f_k, & k \in [K, L-1], \\ h_k(1+D_k), & k \in [L, M-1], \\ g_k, & k \in [M, N-1], \\ 0, & k \in [N, \infty), \end{cases} \tag{4.9}$$

where

$$D_k = \begin{cases} \delta > 0, & k = \tilde{k} + n, \\ 0, & \text{otherwise}. \end{cases} \tag{4.10}$$

To finish the first part of the proof, we use (4.4) to estimate the contribution of the term

$$\widetilde{F}(y) = \sum_{k=-\infty}^{\infty} r_k (\Delta^n y_k)^2 = \sum_{k=-\infty}^{\infty} u_k^{[y]T} B_k u_k^{[y]} = \sum_{k=K}^{N-1} u_k^{[y]T} B_k u_k^{[y]}. \tag{4.11}$$

Using the definition of the test sequence y, we can split $\widetilde{F}$ into three terms. Now, we estimate two of them as follows. Using (4.4), we obtain

$$\begin{aligned} \sum_{k=K}^{L-1} u_k^{[f]T} B_k u_k^{[f]} &= \sum_{k=K}^{L-1} \left[u_k^{[f]T} \left(\Delta x_k^{[f]} - A x_{k+1}^{[f]} \right) \right] = \sum_{k=K}^{L-1} \left[u_k^{[f]T} \Delta x_k^{[f]} - u_k^{[f]T} A x_{k+1}^{[f]} \right] \\ &= \sum_{k=K}^{L-1} \left[\Delta \left(u_k^{[f]T} x_k^{[f]} \right) - \Delta u_k^{[f]T} x_{k+1}^{[f]} - u_k^{[f]T} A x_{k+1}^{[f]} \right] \\ &= \sum_{k=K}^{L-1} \left[\Delta \left(u_k^{[f]T} x_k^{[f]} \right) - x_{k+1}^{[f]T} \left(\Delta u_k^{[f]} + A^T u_k^{[f]} \right) \right] = u_k^{[f]T} x_k^{[f]} \Big|_K^L = x_L^{[f]T} u_L^{[f]} \\ &= x_L^{[h]T} \left[U_L^- (X_L^-)^{-1} x_L^{[h]} + (X_L^-)^{T-1} \left(\sum_{j=K}^{L-1} \mathcal{B}_j^- \right)^{-1} (X_L^-)^{-1} x_L^{[h]} \right] \\ &= x_L^{[h]T} (X_L^-)^{T-1} \left(\sum_{j=K}^{L-1} \mathcal{B}_j^- \right)^{-1} (X_L^-)^{-1} x_L^{[h]} =: \mathcal{G}, \end{aligned} \tag{4.12}$$

where we used the fact that $x_L^{[h]T} U_L^{-} (X_L^{-})^{-1} x_L^{[h]} \equiv 0$ (recall that the last $n-m-1$ entries of $x_L^{[h]}$ are zeros and that the first $m+1$ columns of X^{-} and U^{-} are generated by the solutions $1, \ldots, k^{(m)}$). Similarly,

$$\sum_{k=M}^{N-1} u_k^{[g]T} B_k u_k^{[g]} = -x_M^{[g]T} u_M^{[g]} = x_M^{[h]T} (X_M^{+})^{T-1} \left(\sum_{j=M}^{N-1} \mathcal{B}_j^{+} \right)^{-1} (X_M^{+})^{-1} x_M^{[h]} =: \mathcal{H}. \tag{4.13}$$

Using property (2.10) of recessive solutions of the linear Hamiltonian difference systems, we can see that $\mathcal{G} \to 0$ as $K \to -\infty$ and $\mathcal{H} \to 0$ as $N \to \infty$. We postpone the estimation of the middle term of $\widetilde{F}$ to the end of the proof.

To estimate the second term of $F(y)$, we estimate at first its terms

$$\sum_{k=K}^{L-1} q_k f_{k+n}^2, \qquad \sum_{k=M}^{N-1} q_k g_{k+n}^2. \tag{4.14}$$

For this estimation, we use Lemma 2.3. To do this, we have to show the monotonicity of the sequences

$$\begin{aligned} &\frac{f_k}{h_k} \quad \text{for } k \in [K+n-1, L+n-1], \\ &\frac{g_k}{h_k} \quad \text{for } k \in [M+n-1, N+n-1]. \end{aligned} \tag{4.15}$$

Let $x^{[1]}, \ldots, x^{[2n]}$ be the ordered system of solutions of (2.13) in the sense of Lemma 2.1. Then, again by Lemma 2.1, there exist real numbers $d_1, \ldots, d_n$ such that $h = d_1 x^{[1]} + \cdots + d_n x^{[n]}$. Because $h \not\equiv 0$, at least one coefficient d_i is nonzero. Therefore, we can denote $p := \max\{i \in [1, n] : d_i \neq 0\}$, and we replace the solution $x^{[p]}$ by h. Let us denote this new system again $x^{[1]}, \ldots, x^{[2n]}$ and note that this new system has the same properties as the original one.

Following Lemma 2.2, we transform (2.13) via the transformation $y_k = h_k z_k$, into

$$\sum_{\nu=0}^{n} (-\Delta)^{\nu} \left(R_k^{[\nu]} \Delta^{\nu} z_{k+n-\nu} \right) = 0, \tag{4.16}$$

that is,

$$(-\Delta)^n \left(r_k h_k h_{k+n} \Delta^{n-1} w_k \right) + \cdots - \Delta \left(R_k^{[1]} w_{k+n-1} \right) = 0 \tag{4.17}$$

possesses the fundamental system of solutions

$$\begin{aligned} w^{[1]} &= -\Delta\left(\frac{x^{[1]}}{h}\right), \dots, w^{[p-1]} = -\Delta\left(\frac{x^{[p-1]}}{h}\right), \\ w^{[p]} &= \Delta\left(\frac{x^{[p+1]}}{h}\right), \dots, w^{[2n-1]} = \Delta\left(\frac{x^{[2n]}}{h}\right). \end{aligned} \tag{4.18}$$

Now, let us compute the Casoratians

$$\begin{aligned} C\left(w^{[1]}\right) &= w^{[1]} = -\Delta\left(\frac{x^{[1]}}{h}\right) = \frac{C(x^{[1]}, h)}{h_k h_{k+1}} > 0, \\ C\left(w^{[1]}, w^{[2]}\right) &= \frac{C(x^{[1]}, x^{[2]}, h)}{h_k h_{k+1} h_{k+2}} > 0, \\ &\vdots \\ C\left(w^{[1]}, \dots, w^{[2n-1]}\right) &= \frac{C(x^{[1]}, \dots, x^{[p-1]}, x^{[p+1]}, \dots, x^{[2n]}, h)}{h_k \cdots h_{k+2n-1}} > 0. \end{aligned} \tag{4.19}$$

Hence, $w^{[1]}, \dots, w^{[2n-1]}$ form the D-Markov system of sequences on $[M, \infty)$, for M sufficiently large. Therefore, by Lemma 2.4, (4.17) is eventually disconjugate; that is, it has at most $2n-2$ generalized zeros (counting multiplicity) on $[M, \infty)$. The sequence $\Delta(g/h)$ is a solution of (4.17), and we have that this sequence has generalized zeros of multiplicity $n-1$ both at M and at N; that is,

$$\Delta\left(\frac{g_{M+i}}{h_{M+i}}\right) = 0 = \Delta\left(\frac{g_{N+i}}{h_{N+i}}\right), \quad i = 0, \dots, n-2. \tag{4.20}$$

Moreover, $g_M/h_M = 1$ and $g_N/h_N = 0$. Hence, $\Delta(g_k/h_k) \le 0, k \in [M, N+n-1]$. We can proceed similarly for the sequence f/h.

Using Lemma 2.3, we have that there exist integers $\xi_1 \in [K, L-1]$ and $\xi_2 \in [M, N-1]$ such that

$$\begin{aligned} \sum_{k=K}^{L-1} q_k f_{k+n}^2 &= \sum_{k=K}^{L-1}\left[q_k h_{k+n}^2\left(\frac{f_{k+n}}{h_{k+n}}\right)^2\right] \le \sum_{k=\xi_1}^{L-1} q_k h_{k+n}^2, \\ \sum_{k=M}^{N-1} q_k g_{k+n}^2 &= \sum_{k=M}^{N-1}\left[q_k h_{k+n}^2\left(\frac{g_{k+n}}{h_{k+n}}\right)^2\right] \le \sum_{k=M}^{\xi_2-1} q_k h_{k+n}^2. \end{aligned} \tag{4.21}$$

Finally, we estimate the remaining term of $F(y)$. By (4.9), we have

$$\begin{aligned}
\sum_{k=L}^{M-1}&\left[r_k(\Delta^n y_k)^2+q_k y_{k+n}^2\right]\\
&=\sum_{k=L}^{M-1}\left\{r_k[\Delta^n h_k+\Delta^n(h_kD_k)]^2+q_k(h_{k+n}+h_{k+n}D_{k+n})^2\right\}\\
&=\sum_{k=L}^{M-1}\left\{r_k[\Delta^n(h_kD_k)]^2+q_kh_{k+n}^2+2q_kh_{k+n}^2D_{k+n}+q_kh_{k+n}^2D_{k+n}^2\right\}\\
&=\sum_{k=\tilde{k}}^{\tilde{k}+n}\left\{r_k[\Delta^n(h_kD_k)]^2\right\}+\sum_{k=L}^{M-1}\left[q_kh_{k+n}^2\right]+2q_{\tilde{k}}h_{\tilde{k}+n}^2D_{\tilde{k}+n}+q_{\tilde{k}}h_{\tilde{k}+n}^2D_{\tilde{k}+n}^2\\
&=\sum_{k=\tilde{k}}^{\tilde{k}+n}\left\{r_k\left[(-1)^{k-\tilde{k}}\binom{n}{k-\tilde{k}}h_{\tilde{k}+n}\delta\right]^2\right\}+\sum_{k=L}^{M-1}\left[q_kh_{k+n}^2\right]+2\delta q_{\tilde{k}}h_{\tilde{k}+n}^2+\delta^2q_{\tilde{k}}h_{\tilde{k}+n}^2\\
&\le\delta^2h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]+\sum_{k=L}^{M-1}\left[q_kh_{k+n}^2\right]-2\delta\varepsilon h_{\tilde{k}+n}^2-\delta^2\varepsilon h_{\tilde{k}+n}^2\\
&<\delta^2h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]+\sum_{k=L}^{M-1}\left[q_kh_{k+n}^2\right]-2\delta\varepsilon h_{\tilde{k}+n}^2.
\end{aligned}\tag{4.22}$$

Altogether, we have

$$\begin{aligned}
F(y)&<\delta^2h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]+\sum_{k=L}^{M-1}\left[q_kh_{k+n}^2\right]-2\delta\varepsilon h_{\tilde{k}+n}^2+\mathcal{G}+\mathcal{H}+\sum_{k=\xi_1}^{L-1}q_kh_{k+n}^2+\sum_{k=M}^{\xi_2-1}q_kh_{k+n}^2\\
&=\delta^2h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]-2\delta\varepsilon h_{\tilde{k}+n}^2+\mathcal{G}+\mathcal{H}+\sum_{k=\xi_1}^{\xi_2-1}q_kh_{k+n}^2,
\end{aligned}\tag{4.23}$$

where for K sufficiently small is $\mathcal{G}<\delta^2/3$, for N sufficiently large is $\mathcal{H}<\delta^2/3$, and, from (4.1), $\sum_{k=\xi_1}^{\xi_2-1}q_kh_{k+n}^2<\delta^2/3$ for $\xi_1<L$ and $\xi_2>M$. Therefore,

$$\begin{aligned}
F(y)&<\delta^2+\delta^2h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]-2\delta\varepsilon h_{\tilde{k}+n}^2\\
&=\delta\left\{\delta\left[1+h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]\right]-\varepsilon h_{\tilde{k}+n}^2\right\},
\end{aligned}\tag{4.24}$$

which means that $F(y)<0$ for δ sufficiently small, and (4.2) is conjugate on $\mathbb{Z}$. □

5. Equation with the Middle Terms

Under the additional condition $q_k \leq 0$ for large $|k|$, and by combining of the proof of Theorem 4.1 with the proof of [2, Lemma 1], we can establish the following criterion for the full $2n$-order equation.

Theorem 5.1. *Let $n > 1$, q_k be a real-valued sequence, and let there exist an integer $m \in \{0, \dots, n-1\}$ and real constants $c_0, \dots, c_m$ such that (1.1) is at least $(m+1)$-critical and the sequence $h_k := c_0 + c_1 k + \cdots + c_m k^{(m)}$ satisfies*

$$\limsup_{K \downarrow -\infty, L \uparrow \infty} \sum_{k=K}^{L} q_k h_{k+n}^2 \leq 0. \tag{5.1}$$

If $q_k \leq 0$ for large $|k|$ and $q \not\equiv 0$, then

$$L(y)_k + q_k y_{k+n} = \sum_{\nu=0}^{n} (-\Delta)^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k+n-\nu} \right) + q_k y_{k+n} = 0 \tag{5.2}$$

is conjugate on $\mathbb{Z}$.

Remark 5.2. Using Theorem 3.4, we can see that the statement of Theorem 4.1 holds if and only if (3.7) holds. Finding a criterion similar to Theorem 3.4 for (1.1) is still an open question.

Remark 5.3. In the view of the matrix operator associated to (1.1) in the sense of [21], we can see that the perturbations in Theorem 4.1 affect the diagonal elements of the associated matrix operator. A description of behavior of (1.1), with regard to perturbations of limited part of the associated matrix operator (but not only of the diagonal elements), is given in [2].

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