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Research Article

The Eigenfunction Expansion for a Dirichlet Problem with Explosive Factor

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We prove the eigenfunction expansion formula for a Dirichlet problem with explosive factor by two ways, first by standard method and second by proving a convergence in some metric space $L_2(0, \pi; \rho(x))$.

1. Introduction

The solutions of many problems of mathematical physics are reduced to the spectral investigation of a differential operator or finding the expansion of arbitrary function, in series or integration, in terms of the eigenfunctions of a differential operator. The differential operator is called regular if it its domain is finite and its coefficients are continuous; otherwise it is called singular differential operator. The Sturm-Liouville theory occupies a central position of the spectral theory of regular operator. During the development of quantum mechanics, there was an increase in the interest of spectral theory of singular operator on which we will restrict our attention. The first basic role in the development of spectral theory of singular operator dates back to E. Ch. Tetchmarsh [1]. He gave a new approach in the spectral theory of singular differential operator of the second order by using contour integration. Also Levitan [2] gave a new method, he obtained the eigenfunction expansion in infinite interval by taking limit of a regular case. In the last time about twenty five or so years, due to the needs of mathematical physics, in particular, quantum mechanics, the question of solving various spectral problems with explosive factor has been arisen. These appeared also in the study of geophysics and electromagnetic field, see Alemov [3]. The spectral theory of differential operators with explosive factor is studied by A. N. Tekhanov, M. G. Krien, M. G. Gasimov, and others. In this paper, we find the eigenfunction expansion formula and prove its convergence for following version of a Dirichlet problem (1.2), (1.3). The introduction of the weight function $\rho(x)$ (1.4) as $\pm$ signs causes many analytical difficulties, see [4], because the problem is to be treated as two separated problems and so the formula of eigenfunctions expansion is written as two sums. In [5] the author considered the weight function of the form

$$\rho(x) = \begin{cases} \alpha^2; & \operatorname{Im}\alpha \neq 0,\ 0 \leq x \leq a < \pi, \\ 1; & a < x \leq \pi, \end{cases} \tag{1.1}$$

and the spectrum was both continuous and discreet so that the formula of eigenfunctions expansion obtained there was written as a summation and integration. We must notice that the result of this paper is a starting point in solving the inverse spectral problem which will be investigated later on.

Consider the following Dirichlet problem:

$$-y'' + q(x)y = \lambda\rho(x)y, \quad 0 \leq x \leq \pi, \tag{1.2}$$

$$y(0) = 0, \qquad y(\pi) = 0, \tag{1.3}$$

where the nonnegative real function $q(x)$ has a second piecewise integrable derivatives on $[0,\pi]$, λ is a spectral parameter, and the weight function or the explosive factor $\rho(x)$ is of the form

$$\rho(x) = \begin{cases} 1, & 0 \leq x \leq a < \pi, \\ -1, & a < x \leq \pi. \end{cases} \tag{1.4}$$

In [4] the author proved that the eigenvalues $\lambda_n^{\pm}, n = 0, 1, 2, \ldots$, of problem (1.2)-(1.3) are real and the corresponding eigenfunctions $\varphi(x, \lambda_n^{\pm})$, $\psi(x, \lambda_n^{\pm})$ are orthogonal with weight function $\rho(x)$. We prove, here, the reality of these eigenfunctions under the condition that $q(x)$ is real. Indeed, let $\varphi(x,\lambda)$ be the solution of the differential equation (1.2), $x \in (0, a)$ which satisfies the conditions

$$\varphi(0,\lambda) = 0, \qquad \varphi'(0,\lambda) = 1 \tag{1.5}$$

so that

$$-\varphi''(x,\lambda) + q(x)\varphi(x,\lambda) = \lambda\varphi(x,\lambda), \tag{1.6}$$

taking the complex conjugate we have

$$\overline{-\varphi''(x,\lambda)} + q(x)\overline{\varphi(x,\lambda)} = \lambda\overline{\varphi(x,\lambda)}. \tag{1.7}$$

By the aid of the uniqueness theorem, we have $\varphi(x,\lambda) = \overline{\varphi(x,\lambda)}$. In a similar way, we can see that $\psi(x,\lambda) = \overline{\psi(x,\lambda)}$ where $\psi(x,\lambda)$ is the solution of (1.2), $x \in (a,\pi)$, $\psi(\pi,\lambda) = 0$,

$\psi'(\pi,\lambda) = 1$, that is, the eigenfunctions of the problem (1.2)-(1.3) are real. As we know from [4], the eigenvalues of problem (1.2)-(1.3) coincide with the roots of the function $\Psi(\lambda) = 0$, where $\Psi(\lambda)$ is the Wronskian of the two solutions $\varphi(x,\lambda)$, $\psi(x,\lambda)$ of (1.2)-(1.3)

$$\Psi(\lambda) = W\left[\varphi(a,\lambda), \psi(a,\lambda)\right] = 0. \tag{1.8}$$

In the following lemma, under the reality of $q(x)$, we prove the simplicity of the eigenvalues, that is, we prove that the roots of (1.8) are simple, in other cases for $q(x)$ is complex the roots of (1.8) may not be simple.

Lemma 1.1. *Under the conditions stated in the introduction with respect to the problem* (1.2)-(1.3), *the eigenvalues of the problem* (1.2)-(1.3) *are simple.*

Proof. We prove that $\dot{\Psi}(\lambda) \neq 0$ where the dot means differentiation with respect to λ. Let $\varphi(x,\lambda)$ be the solution of the problem

$$\begin{gathered} -y'' + q(x)y = \lambda y, \quad (0 \le x \le a), \\ y(0) = 0, \qquad y'(0) = 1, \end{gathered} \tag{1.9}$$

and let $\psi(x,\lambda)$ be the solution of the problem

$$\begin{gathered} -y'' + q(x)y = -\lambda y, \quad (a < x \le \pi), \\ y(\pi) = 0, \qquad y'(\pi) = 1. \end{gathered} \tag{1.10}$$

From (1.9), we have

$$-\varphi''(x,\lambda) + q(x)\varphi(x,\lambda) = \lambda\varphi(x,\lambda). \tag{1.11}$$

Differentiating (1.11) with respect to λ, we have

$$-\dot{\varphi}''(x,\lambda) + q(x)\dot{\varphi}(x,\lambda) = \lambda\dot{\varphi}(x,\lambda) + \varphi(x,\lambda). \tag{1.12}$$

Multiplying (1.11) by $\dot{\varphi}(x,\lambda)$ and (1.12) by $\varphi(x,\lambda)$ and then subtracting the two results we have $(d/dx)[\dot{\varphi}\varphi' - \dot{\varphi}'\varphi] = \varphi^2(x,\lambda)$ from which, by integrating with respect to x, from 0 to a and using the conditions in (1.10), we obtain

$$\varphi'(a,\lambda_n)\dot{\varphi}(a,\lambda_n) - \dot{\varphi}'(a,\lambda_n)\varphi(a,\lambda_n) = \int_0^a \varphi^2(x,\lambda_n)dx. \tag{1.13}$$

In a similar way, from (1.10), we can write

$$-\psi'(a,\lambda_n)\dot{\psi}(a,\lambda_n) + \dot{\psi}'(a,\lambda_n)\psi(a,\lambda_n) = -\int_a^\pi \psi^2(x,\lambda_n)dx. \tag{1.14}$$

From (1.8), we have

$$\varphi(a,\lambda_n)\psi'(a,\lambda_n) = \varphi'(a,\lambda_n)\psi(a,\lambda_n). \tag{1.15}$$

From which we have

$$\begin{aligned} \varphi'(a,\lambda_n) &= \frac{\varphi(a,\lambda_n)\psi'(a,\lambda_n)}{\psi(a,\lambda_n)}, \\ \psi'(a,\lambda_n) &= \frac{\varphi'(a,\lambda_n)\psi(a,\lambda_n)}{\varphi(a,\lambda_n)}. \end{aligned} \tag{1.16}$$

Substituting from (1.16) into (1.13) and (1.14), we can see by adding that

$$\dot{\Psi}(\lambda_n) = r_n a_n, \tag{1.17}$$

where

$$a_n = \int_0^a \varphi^2(x,\lambda_n)dx - \frac{1}{r_n^2}\int_a^\pi \psi^2(x,\lambda_n)dx, \qquad r_n = \frac{\psi(a,\lambda_n)}{\varphi(a,\lambda_n)}, \tag{1.18}$$

and the numbers a_n are the normalization numbers of the eigenfunctions of problem (1.2)-(1.3). Following [4], we have $r_n \neq 0$ and $a_n \neq 0$ which complete the proof of lemma. □

2. The Function $R(x,\xi,\lambda)$

We introduce the function $R(x,\xi,\lambda)$ by

$$R(x,\xi,\lambda) = \frac{-1}{\Psi(\lambda)}\begin{cases} \varphi(x,\lambda)\psi(\xi,\lambda), & x \leq \xi, \\ \varphi(\xi,\lambda)\psi(x,\lambda), & \xi \leq x, \end{cases} \tag{2.1}$$

which is called the Green's function of the nonhomogenous Dirichlet problem

$$\begin{gathered} -y'' + q(x)y = \lambda\rho(x)y + \rho(x)f(x), \quad 0 \leq x \leq \pi, \\ y(0) = y(\pi) = 0, \end{gathered} \tag{2.2}$$

where $f(x) \in L_1(0,\pi)$ and $\rho(x)$ is defined by (1.4). The function $R(x,\xi,\lambda)$ is, also, called the kernel of the resolvent $R_\lambda = (A - \lambda I)^{-1}$, where $A \equiv -(d^2/dx^2) + q(x)$, $D(A) = \{y(x) : \exists y'', y(0) = y(\pi) = 0\}$. In the following lemmas, we prove some essential properties of $R(x,\xi,\lambda)$ which are useful in the forthcoming study of the eigenfunction expansion of the problem (1.2)-(1.3)

Lemma 2.1. *Let $f(x)$ be any function belonging to $L_1(0,\pi)$, then the function*

$$y(x,\lambda) = \int_0^\pi R(x,\xi,\lambda) f(\xi)\rho(\xi)d\xi \tag{2.3}$$

is the solution of the nonhomogenous Dirichlet problem (2.2).

Proof. First we show that (2.3) satisfies the boundary conditions of (2.2). From (2.2) by using (1.9) and (1.10), we, respectively, have

$$\begin{aligned} y(0) &= \frac{-1}{\Psi(\lambda)} \int_0^\pi \varphi(0,\lambda)\psi(\xi,\lambda)\rho(\xi)f(\xi)d\xi = 0, \\ y(\pi) &= \frac{-1}{\Psi(\lambda)} \int_0^\pi \varphi(\xi,\lambda)\psi(\pi,\lambda)\rho(\xi)f(\xi)d\xi = 0. \end{aligned} \tag{2.4}$$

Secondly, we calculate the solution of (2.2) by the method of variation of parameters. We seek a solution of the nonhomogenous problem (2.2) in the form

$$y(x,\lambda) = C_1\varphi(x,\lambda) + C_2\psi(x,\lambda), \tag{2.5}$$

where $\varphi(x,\lambda)$, $\psi(x,\lambda)$ are given together with their asymptotic formulas in [4]. By using the standard calculation, we find

$$\begin{aligned} C_1(x,\lambda) &= \frac{-1}{\Psi(\lambda)} \int_x^\pi f(\xi)\rho(\xi)\psi(\xi,\lambda)d\xi, \\ C_2(x,\lambda) &= \frac{-1}{\Psi(\lambda)} \int_0^x f(\xi)\rho(\xi)\varphi(\xi,\lambda)d\xi. \end{aligned} \tag{2.6}$$

Substituting from (2.6) into (2.5) and keeping in mind (2.1), we get the required formula (2.3). □

Lemma 2.2. *Under the conditions of Lemma 2.1, the function $R(x,\xi,\lambda)$ satisfies the following formula:*

$$R(x,\xi,\lambda) = \frac{-1}{\lambda-\lambda_n} \frac{a_n}{\varphi(x,\lambda_n)\varphi(\xi,\lambda_n)} + R_1(x,\xi,\lambda), \tag{2.7}$$

where $R_1(x,\xi,\lambda)$ is regular in the neighborhood of $\lambda = \lambda_n$ and $a_n = \int_0^\pi \rho(\xi)\varphi^2(x,\lambda_n)dx$.

Proof. So long as, from Lemma 1.1, the roots of the function $\Psi(\lambda)$ are simple; it follows that the poles of the function $R(x,\xi,\lambda)$ are simple. So that $R(x,\xi,\lambda)$ can be represented in the form

$$R(x,\xi,\lambda) = \frac{\operatorname{Res}[R(x,\xi,\lambda)]}{\lambda-\lambda_n} + R_1(x,\xi,\lambda) \tag{2.8}$$

from (2.1); for $x \leq \xi$, we have

$$\operatorname{Res}_{\lambda=\lambda_n}[R(x,\xi,\lambda)] = -\frac{\varphi(x,\lambda_n)\psi(\xi,\lambda_n)}{\dot{\Psi}(\lambda_n)}. \tag{2.9}$$

From (1.17) and (1.18), the relation (2.9) takes the form

$$\operatorname{Res}_{\lambda=\lambda_n}[R(x,\xi,\lambda)] = -\frac{\varphi(x,\lambda_n)\varphi(\xi,\lambda_n)}{a_n}. \tag{2.10}$$

Formula (2.7) is obtained by substituting from (2.10) into (2.8) we deduce the formula (2.7). We notice that in case of $\xi \leq x$, in a similar way. in order to prove the convergence of the eigenfunction expansion of the Dirichlet problem (1.2)-(1.3), we must write an equality for the function $R(x,\xi,\lambda)$ and this, in turn, needs to extend the asymptotic formulas of $\varphi(x,\lambda)$, $\psi(x,\lambda)$ over all the interval $[0,\pi]$. In [4], the asymptotic formulas for $\varphi(x,\lambda)$, $\psi(x,\lambda)$ were deduced for $x \in (0,a)$ and (a,π), respectively. In the following lemma, we write this asymptotic formulas over all $[0,\pi]$ for both $\varphi(x,\lambda)$ and $\psi(x,\lambda)$. □

Lemma 2.3. *The solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ of the Dirichlet problem* (1.2)-(1.3) *have the following asymptotic formula:*

$$\varphi(x,\lambda) = \begin{cases} \dfrac{\sin sx}{s} + \bigcirc\left(\dfrac{e^{|\operatorname{Im} s|x}}{|s^2|}\right), & 0 \leq x \leq a, \\ \dfrac{\beta(x)}{s\beta(a)}[\sin sa \cosh s(a-x) - \cos sa \sinh s(a-x)], & \\ \quad + \bigcirc\left(\dfrac{e^{|\operatorname{Im} s|a + |\Re s|(a-x)}}{|s^2|}\right), & a < x \leq \pi, \end{cases} \tag{2.11}$$

$$\psi(x,\lambda) = \begin{cases} \dfrac{\alpha(x)}{s\alpha(a)}[\cos s(x-a) \sinh s(\pi-a) - \sin s(x-a) \cosh s(\pi-a)], & \\ \quad + \bigcirc\left(\dfrac{e^{|\operatorname{Im} s|(x-a) + |\Re s|(\pi-a)}}{|s^2|}\right), & 0 \leq x \leq a, \\ \dfrac{\sinh s(\pi-x)}{s} + \bigcirc\left(\dfrac{e^{|\Re s|(\pi-x)}}{|s^2|}\right), & a \leq x \leq \pi, \end{cases} \tag{2.12}$$

where

$$\alpha(x) = \frac{1}{2}\int_0^x q(t)dt, \qquad \beta(x) = \frac{1}{2}\int_x^\pi q(t)dt, \qquad \lambda = s^2. \tag{2.13}$$

Proof. Following [4], the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ of the Dirichlet problem (1.2)-(1.3) have the representation

$$\varphi(x,\lambda) = \frac{\sin sx}{s} + \bigcirc\left(\frac{e^{|\Im s|x}}{|s^2|}\right), \quad x \in [0,a], \tag{2.14}$$

$$\psi(x,\lambda) = \frac{\sinh s(\pi - x)}{s} + \bigcirc\left(\frac{e^{|\Re s|(\pi-x)}}{|s^2|}\right), \quad x \in [a,\pi]. \tag{2.15}$$

We can see, from [6], that the solution $y(x,s)$ of the equation $-y'' + q(x)y = s^2 y\ 0 \le x \le a,\ y(0) = 0$ has the representation

$$y(x,s) = e^{isx}\left[\frac{\alpha(x)}{s} + \bigcirc\left(\frac{1}{s^2}\right)\right], \qquad y'(x,s) = e^{isx}\left[i\alpha(x) + \bigcirc\left(\frac{1}{s}\right)\right], \tag{2.16}$$

and the solution $z(x,s)$ of the equation $-y'' + q(x)y = -s^2 y, a < x \le \pi,\ z(\pi) = 0$ has the representation

$$z(x,s) = e^{s(\pi-x)}\left[\frac{\beta(x)}{s} + \bigcirc\left(\frac{1}{s^2}\right)\right], \qquad z'(x,s) = e^{s(\pi-x)}\left[-\beta(x) + \bigcirc\left(\frac{1}{s}\right)\right], \tag{2.17}$$

where $\alpha(x)$ and $\beta(x)$ are given by (2.13), so that, $\varphi(x,\lambda)$ can be extended to (a,π) in terms of the two linearly independent solutions $z(x,s),\ z(x,-s)$ as

$$\varphi(x,\lambda) = m_1 z(x,s) + m_2 z(x,-s). \tag{2.18}$$

From the continuity of $\varphi(x,\lambda)$ at $x = a$ and by using the asymptotic relations, (2.14), of $\varphi(x,\lambda)$ and (2.17) of $z(x,s)$, the constants $m_1,\ m_2$ are calculated in the form

$$\begin{aligned} m_1 &= e^{s(a-x)}\frac{\beta(x)}{2\beta(a)}\left[\frac{\sin sa - \cos sa}{s} + \bigcirc\left(\frac{e^{|\operatorname{Im} s|a}}{s^2}\right)\right], \\ m_2 &= e^{s(\pi-a)}\frac{1}{2\beta(a)}\left[-\sin sa - \cos sa + \bigcirc\left(\frac{e^{|\operatorname{Im} s|a}}{s}\right)\right]. \end{aligned} \tag{2.19}$$

Substituting from (2.19) into (2.18), we have, for $x \in (a,\pi]$

$$\varphi(x,\lambda) = \frac{\beta(x)}{s\beta(a)}[\sin sa \cosh s(a-x) - \cos sa \sinh s(a-x)] + \bigcirc\left(\frac{e^{|\operatorname{Im} s|a + |\operatorname{Re} s|(a-x)}}{|s^2|}\right). \tag{2.20}$$

From (2.20) together with (2.14), the relation (2.11) is followed. The proof of the relation (2.12) is quite similar to the proof of (2.11). Indeed $\psi(x,\lambda),\ x \in (a,\pi]$ is a linear combination of the two linearly independent solutions $y(x,s),\ y(x,-s)$ as $\psi(x,\lambda) = k_1 y(x,s) + k_2 y(x,-s)$, where

$$k_1 = \frac{e^{-isa}}{2\alpha(a)}[\sinh s(\pi-a) + i\cosh s(\pi-a)] + \bigcirc\left(\frac{e^{|\operatorname{Re} s|(\pi-a)}}{|s|}\right),$$
$$k_2 = \frac{e^{isa}}{2\alpha(a)}[-\sinh s(\pi-a) + i\cosh s(\pi-a)] + \bigcirc\left(\frac{e^{|\operatorname{Re} s|(\pi-a)}}{|s|}\right), \tag{2.21}$$

and by using the asymptotic formulas (2.16) of $y(x,\pm s)$ and (2.15) of $\psi(x,\lambda)$, we have, for $x \in [0,a]$,

$$\psi(x,\lambda) = \frac{\alpha(x)}{s\alpha(a)}[\cos s(x-a)\sinh s(\pi-a) - \sin s(x-a)\cosh s(\pi-a)]$$
$$+ \bigcirc\left(\frac{e^{|\operatorname{Im} s|(x-a)+|\operatorname{Re} s|(\pi-a)}}{|s^2|}\right). \tag{2.22}$$

The relation (2.20) together with (2.14) and the relation (2.22) together with (2.15) complete the proof of lemma. □

The following inequality proves an inequality satisfied by $R(x,\xi,\lambda)$.

Lemma 2.4. *Under the conditions of Lemma 2.3, the resolvent $R(x,\xi,\lambda)$ satisfies the following inequality:*

$$R(x,\xi,\lambda) = \begin{cases} \bigcirc\left(\dfrac{e^{-|\operatorname{Im} s||2a-x-\xi|}}{|s|}\right), & x,\xi \in [0,a], \\ \bigcirc\left(\dfrac{e^{-|\operatorname{Re} s||x+\xi-2a|}}{|s|}\right), & x,\xi \in [a,\pi], \\ \bigcirc\left(\dfrac{e^{-|\operatorname{Im} s|(a-x)-|\operatorname{Re} s|(\xi-a)}}{|s|}\right), & 0 \le x \le a < \xi \le \pi, \\ \bigcirc\left(\dfrac{e^{-|\operatorname{Im} s|(a-\xi)-|\operatorname{Re} s|(x-a)}}{|s|}\right), & 0 \le \xi \le a < x \le \pi. \end{cases} \tag{2.23}$$

Proof. From (2.11) and (2.12), we have

$$\varphi(x,\lambda) = \begin{cases} \bigcirc\left(\dfrac{e^{-|\operatorname{Im} s|x}}{|s|}\right), & 0 \le x \le a, \\ \bigcirc\left(\dfrac{e^{-|\operatorname{Im} s|a+|\operatorname{Re} s|(a-x)}}{|s|}\right), & a < x \le \pi, \end{cases} \tag{2.24}$$

$$\psi(x,\lambda) = \begin{cases} \bigcirc\left(\dfrac{e^{-|\operatorname{Im} s|(x-a)+|\operatorname{Re} s|(\pi-a)}}{|s|}\right), & 0 \le x \le a, \\ \bigcirc\left(\dfrac{e^{-|\operatorname{Re} s|(\pi-x)}}{|s|}\right), & a < x \le \pi. \end{cases} \tag{2.25}$$

It can be easily seen that, for $s \in \Gamma_n$, we have

$$\Psi(\lambda) \geq C\frac{e^{|\operatorname{Im} s|a+|\operatorname{Re} s|(\pi-a)}}{|s|}, \quad s \in \Gamma_n, \tag{2.26}$$

where $s\Gamma_n$ is the quadratic contour, as defined in [4]

$$\Gamma_n = \left\{|\operatorname{Re} s| \leq \frac{\pi}{a}\left(n-\frac{1}{4}\right)+\frac{\pi}{2a},\ |\operatorname{Im} s| \leq \frac{\pi}{\pi-a}\left(n-\frac{1}{4}\right)+\frac{\pi}{2(\pi-a)}\right\}. \tag{2.27}$$

From (2.1) we have six possibilities, three of which for $x \leq \xi$ and the other three for $\xi \leq x$. Now for $\xi \leq x$ we have the following situation: (i) $0 \leq x \leq \xi \leq a$, (ii) $a < x \leq \xi \leq \pi$, and (iii) $0 \leq x \leq a \leq \xi \leq \pi$. In cases (i), (ii), and (iii) by direct substitution from (2.24), (2.25), (2.26) into the first branch of (2.1), we obtain

$$\text{(i)} \quad R(x,\xi,\lambda) = \bigcirc\left(\frac{e^{-|\operatorname{Im} s||2a-x-\xi|}}{|s|}\right), \quad 0 \leq x \leq \xi \leq a, \tag{2.28}$$

$$\text{(ii)} \quad R(x,\xi,\lambda) = \bigcirc\left(\frac{e^{-|\operatorname{Re} s||x+\xi-2a|}}{|s|}\right), \quad a \leq x \leq \xi \leq \pi, \tag{2.29}$$

$$\text{(iii)} \quad R(x,\xi,\lambda) = \bigcirc\left(\frac{e^{-|\operatorname{Im} s|(a-x)-|\operatorname{Re} s|(\xi-a)}}{|s|}\right), \quad 0 \leq x \leq a < \xi \leq \pi. \tag{2.30}$$

In the case of $\xi \geq x$, we discuss (i*) $0 \leq \xi \leq x \leq a$, (ii*) $a \leq \xi \leq x \leq \pi$, and (iii*) $0 \leq \xi \leq a \leq x \leq \pi$.

Again by substituting (2.24), (2.25), and (2.26) into the second branch of (2.1), we get

$$\text{(i}^*\text{)} \quad R(x,\xi,\lambda) = \bigcirc\left(\frac{e^{-|\operatorname{Im} s||2a-x-\xi|}}{|s|}\right), \quad 0 \leq \xi \leq x \leq a, \tag{2.31}$$

$$\text{(ii}^*\text{)} \quad R(x,\xi,\lambda) = \bigcirc\left(\frac{e^{-|\operatorname{Re} s||x+\xi-2a|}}{|s|}\right), \quad a \leq \xi \leq x \leq \pi, \tag{2.32}$$

$$\text{(iii}^*\text{)} \quad R(x,\xi,\lambda) = \bigcirc\left(\frac{e^{-|\operatorname{Im} s|(a-\xi)-|\operatorname{Re} s|(x-a)}}{|s|}\right), \quad 0 \leq \xi \leq a < x \leq \pi. \tag{2.33}$$

From (2.28) and (2.31), we have

$$R(x,\xi,\lambda) = \bigcirc\left(\frac{e^{-|\mathrm{Ims}||2a-x-\xi|}}{|s|}\right), \quad x,\xi \in [0,a], \tag{2.34}$$

and from (2.29) and (2.32), we have

$$R(x,\xi,\lambda) = \bigcirc\left(\frac{e^{-|\mathrm{Re}s||x+\xi-2a|}}{|s|}\right), \quad x,\xi \in [a,\pi]. \tag{2.35}$$

From (2.30) and (2.33) together with (2.34) and (2.35), the lemma is proved. In the following lemma, we prove an integral formula which is satisfied by $R(x,\xi,\lambda)$ and help in proving the eigenfunction expansion formula □

Lemma 2.5. *If the function $f(x)$ on $[0,\pi]$ has a second-order integrable derivatives and satisfies the Dirichlet condition $f(0) = f(\pi) = 0$, then the following integral formula is true:*

$$\int_0^\pi R(x,\xi,\lambda)\rho(\xi)f(\xi)d\xi = \frac{-f(x)}{\lambda} + \int_0^\pi \frac{R(x,\xi,\lambda)}{\lambda}\left[-f''(x) + q(x)f(\xi)\right]d\xi, \tag{2.36}$$

where $R(x,\xi,\lambda)$ is the kernel of the resolvent of the nonhomogenous Dirichlet problem (2.2)*.*

Proof. By the aid of (2.1), we have

$$\int_0^\pi R(x,\xi,\lambda)\rho(\xi)f(\xi)d\xi = \frac{-1}{\Psi(\lambda)}\left\{\psi(x,\lambda)\int_0^x \varphi(\xi,\lambda)\rho(\xi)f(\xi)d\xi + \varphi(x,\lambda)\int_x^\pi \psi(\xi,\lambda)\rho(\xi)f(\xi)d\xi\right\}, \tag{2.37}$$

where the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are the solutions of the homogenous Dirichlet problem (1.2)-(1.3), so that

$$\begin{aligned}\int_0^\pi R(x,\xi,\lambda)\rho(\xi)f(\xi)d\xi &= \frac{-1}{\Psi(\lambda)}\left\{\frac{\psi(x,\lambda)}{\lambda}\int_0^x \left[\varphi''(\xi,\lambda) + q(\xi)\varphi(\xi,\lambda)\right]f(\xi)d\xi\right.\\ &\quad\left.+\frac{\varphi(x,\lambda)}{\lambda}\int_x^\pi \left[\psi''(\xi,\lambda) + q(\xi)\psi(\xi,\lambda)\right]f(\xi)d\xi\right\}\end{aligned} \tag{2.38}$$

from which we have

$$\begin{aligned}\int_0^\pi R(x,\xi,\lambda)\rho(\xi)f(\xi)d\xi &= \frac{1}{\Psi(\lambda)}\left\{\frac{\psi(x,\lambda)}{\lambda}\int_0^x \varphi''(\xi,\lambda)f(\xi)d\xi + \frac{\varphi(x,\lambda)}{\lambda}\int_x^\pi \psi''(\xi,\lambda)f(\xi)d\xi\right\}\\ &\quad+\frac{1}{\lambda}\int_0^\pi R(x,\xi,\lambda)q(\xi)f(\xi)d\xi.\end{aligned} \tag{2.39}$$

Integrating by parts twice the terms $\int_0^x$ and $\int_x^\pi$, in (2.39), and then using the boundary conditions $f(0) = f(\pi) = \varphi(0,\lambda) = 0$ and $f(0) = f(\pi) = \psi(\pi,\lambda) = 0$, respectively, and keeping in mind (2.1), we deduce that

$$\begin{aligned}\psi(x,\lambda)\int_0^x \varphi''(\xi,\lambda)f(\xi)d\xi + \varphi(x,\lambda)\int_x^\pi \psi''(\xi,\lambda)f(\xi)d\xi \\ = -\Psi(\lambda)f(x) - \Psi(\lambda)\int_0^\pi R(x,\xi,\lambda)f''(\xi)d\xi.\end{aligned} \tag{2.40}$$

Substituting from (2.40) into (2.39), we get the required result. □

3. The Eigenfunctions Expansion Formula

We now construct and prove the eigenfunction expansion formula for the Dirichlet problem (1.2)-(1.3). Let $\lambda_n^+,\ n = 0,1,2,\dots$ and $\lambda_n^-,\ n = 0,1,2,\dots$ be the nonnegative and the negative eigenvalues of the problem (1.2)-(1.3), and let also

$$a_n^+ = \int_0^\pi \rho(x)\varphi^2(x,\lambda_n^+)dx, \qquad a_n^- = \int_0^\pi \rho(x)\varphi^2(x,\lambda_n^-)dx \tag{3.1}$$

be the normalization numbers of the corresponding eigenfunctions $\varphi(x,\lambda_n^\pm)$. We put

$$v_k^\pm(x) = \frac{\varphi(x,\lambda_k^\pm)}{\sqrt{a_k^\pm}}, \quad k = 0,1,2,\dots. \tag{3.2}$$

The set $\{v_k^\pm(x)\}_{k=0}^\infty$ is an orthonormal system of eigenfunctions of the Dirichlet problem (1.2)-(1.3).

Theorem 3.1. *Let f(x) be a second-order integrable derivatives on $[0,\pi]$ and satisfy the conditions $f(0) = f(\pi) = 0$; then the following formula of eigenfunction expansion is true:*

$$f(x) = \sum_{k=0}^\infty b_k^+ v_k^+(x) + \sum_{k=0}^\infty b_k^- v_k^-(x), \tag{3.3}$$

where $b_k^\pm = \int_0^\pi v_k^\pm(\xi)f(\xi)\rho(\xi)d\xi$ and the series uniformly converges to $f(x),\ x \in [0,\pi]$.

Notice that, the expansion (3.3) can be written, more explicitly, in terms of $\varphi(x,\lambda_k^\pm)$ as

$$f(x) = \sum_{k=0}^\infty \frac{1}{a_k^+}\varphi(x,\lambda_k^+)\int_0^\pi \varphi(\xi,\lambda_k^+)f(\xi)\rho(\xi)d\xi + \sum_{k=0}^\infty \frac{1}{a_k^-}\varphi(x,\lambda_k^-)\int_0^\pi \varphi(\xi,\lambda_k^-)f(\xi)\rho(\xi)d\xi \tag{3.4}$$

or in terms of $\psi(x,\lambda_k^{\pm})$

$$f(x)=\sum_{k=0}^{\infty}\frac{1}{(r_k^{+})^2a_k^{+}}\varphi(x,\lambda_k^{+})\int_0^{\pi}\varphi(\xi,\lambda_k^{+})f(\xi)\rho(\xi)d\xi
+\sum_{k=0}^{\infty}\frac{1}{(r_k^{-})^2a_k^{-}}\varphi(x,\lambda_k^{-})\int_0^{\pi}\varphi(\xi,\lambda_k^{-})f(\xi)\rho(\xi)d\xi, \tag{3.5}$$

where $r_k^{\pm}$ are defined by $\psi(x,\lambda_k^{\pm})=r_k^{\pm}\varphi(x,\lambda_k^{\pm}),\ 0\le x\le\pi$.

Proof. We write (2.36) in the form

$$\int_0^{\pi}R(x,\xi,\lambda)\rho(\xi)f(\xi)d\xi=\frac{-f(x)}{\lambda}+r(x,\lambda), \tag{3.6}$$

where

$$r(x,\lambda)=\int_0^{\pi}\frac{R(x,\xi,\lambda)}{\lambda}\left[-f''(x)+q(x)f(\xi)\right]d\xi. \tag{3.7}$$

By the aid of Lemma 2.3 and the condition of the theorem imposed on $q(x)$, it can be easily seen that

$$|r(x,\lambda)|\le\frac{M_o}{|\lambda|^{3/2}},\qquad s\in\Gamma_n, \tag{3.8}$$

where M_o is constant which is independent of $x,\ \xi,\ \lambda$ and the contour Γ_n, as defined in [4], is given by (2.27). Let $\lambda=s^2$; we denote by Γ_n^{+} the upper half of the Γ_n; let also Ł$_n$ denote the image of the contour Γ_n^{+} under the transformation $\lambda=s^2$. We multiply both sides of (3.6) by $1/2\pi i$ and integrating with respect to λ on the contour L_n:

$$\frac{1}{2\pi i}\oint_{L_n}\left\{\int_0^{\pi}R(x,\xi,\lambda)f(\xi)\rho(\xi)d\xi\right\}=\frac{-f(x)}{2\pi i}\oint_{L_n}\frac{d\lambda}{\lambda}+\frac{1}{2\pi i}\oint_{L_n}r(x,\lambda)d\lambda. \tag{3.9}$$

Among the poles of the function $R(x,\xi,\lambda)$, as a function of λ, lie only $\lambda_o^{\pm},\lambda_1^{\pm},\ldots,\lambda_n^{\pm}$ inside L_n. By using the residues formula and (2.10), we have

$$\frac{1}{2\pi i}\oint_{L_n}\left\{\int_0^{\pi}R(x,\xi,\lambda)f(\xi)\rho(\xi)d\xi\right\}=\sum_{k=0}^{n}\operatorname{Res}_{\lambda=\lambda_k^{\pm}}\left\{\int_0^{\pi}R(x,\xi,\lambda)f(\xi)\rho(\xi)d\xi\right\}
=-\sum_{k=0}^{n}\frac{\varphi(x,\lambda_k^{+})}{a_k^{+}}\int_0^{\pi}\varphi(\xi,\lambda_k^{+})\rho(\xi)f(\xi)d\xi-\sum_{k=0}^{n}\frac{\varphi(x,\lambda_k^{-})}{a_k^{-}}\int_0^{\pi}\varphi(\xi,\lambda_k^{-})\rho(\xi)f(\xi)d\xi. \tag{3.10}$$

Further

$$\frac{-f(x)}{2\pi i}\oint_{L_n}\frac{d\lambda}{\lambda}=-f(x). \tag{3.11}$$

By using (3.8), we have

$$\left|\frac{1}{2\pi i}\oint_{L_n} r(x,\lambda)d\lambda\right|\leq\frac{M_o}{2\pi}\oint_{L_n}\frac{|d\lambda|}{|\lambda|^{3/2}}, \tag{3.12}$$

from which, by using the substitution $\lambda = s^2$, we have

$$\left|\frac{1}{2\pi i}\oint_{L_n} r(x,\lambda)d\lambda\right|\leq\frac{M_o}{\pi}\int_{\Gamma_n^+}\frac{|ds|}{s^2}\leq\frac{\text{constant}}{n}. \tag{3.13}$$

By substitution from (3.10), (3.11), and (3.13) into (3.9), we obtain

$$\left|f(x)-\sum_{k=0}^{n}\left(b_k^+v_k^+ + b_k^-v_k^-\right)\right|\leq\frac{\text{constant}}{n},\quad x\in[0,\pi]\ \forall n, \tag{3.14}$$

where

$$b_k^{\pm}=\int_0^{\pi}\frac{\varphi(\xi,\lambda_k^{\pm})}{\sqrt{a_k^{\pm}}}f(\xi)\rho(\xi)d\xi,\qquad v_k^{\pm}=\frac{\varphi(\xi,\lambda_k^{\pm})}{\sqrt{a_k^{\pm}}}, \tag{3.15}$$

which completes the uniform convergence of the series $\sum_{k=0}^{\infty}[b_k^+v_k^+ + b_k^-v_k^-]$ to $f(x)$, $x\in[0,\pi]$. That is,

$$f(x)=\sum_{k=0}^{\infty}\left[b_k^+v_k^+(x)+b_k^-v_k^-(x)\right]. \tag{3.16}$$

It can be proved that the series (3.16) is not only uniformly but also absolutely convergent, to show this we use the asymptotic relations of $v_k^{\pm}$ and $b_k^{\pm}$ for $n\to\infty$. Following [4], we have

$$a_n^+=\frac{d_1}{n^2}+\bigcirc\left(\frac{1}{n^3}\right),\qquad a_n^-=-d_2^2e^{2nd_2}e^{-d_2}\left[\frac{1}{n^2}+\bigcirc\left(\frac{1}{n^3}\right)\right], \tag{3.17}$$

where $d_1 = a^3/2\pi^2$ and $d_2=(\pi-a)/\pi$, from which we can write

$$\sqrt{a_n^{\pm}}=\bigcirc\left(\frac{1}{n}\right). \tag{3.18}$$

Using (3.18), (2.11) and (3.2), we deduce that

$$\left|v_k^{\pm}\right| \leq M^{\pm}, \quad \forall x \in [0,\pi], \text{ and all } k, \tag{3.19}$$

where $M^{\pm}$ are some constants. Further, arguing as in Lemma 2.4 and noticing that $f(0) = f(\pi) = 0$, we have

$$b_k^{\pm} = \int_0^{\pi} v_k^{\pm}(x) f(x)\rho(x)dx = \frac{1}{\lambda_k^{\pm}} \int_0^{\pi} \left[f''(x) + q(x)f(x)\right] v_k^{\pm}(x)dx. \tag{3.20}$$

From [4], we have $\lambda_k^{\pm} = \pm k^2 + \bigcirc(1)$, and using (3.20) we have

$$\left|b_k^{\pm} v_k^{\pm}(x)\right| \leq \frac{\text{costant}}{k^2}, \quad k \longrightarrow \infty, \tag{3.21}$$

which complete the proof of absolute convergence of the series (3.16). It should be noted here that, in the proof of the absolute convergence of the series (3.16) we did not give the sum of the series as in the proof of uniform convergence (Theorem 3.1). In the following lemma, as a consequence of Theorem 3.1, we prove the Parsval's identity which insures the convergence of the series (3.16) and helps in the proof of Theorem 3.3. □

Lemma 3.2. *Let $f(x)$ satisfy the conditions of Theorem 3.1; then the following Parsval's identity holds true*

$$\int_0^{\pi} \rho(x)\left|f(x)\right|^2 dx = \sum_{k=0}^{\infty}\left(\left|b_k^+\right|^2 + \left|b_k^-\right|^2\right), \tag{3.22}$$

where

$$b_k^{\pm} = \int_0^{\pi} v_k^{\pm}(x) f(x)\rho(x)dx. \tag{3.23}$$

Proof. From Theorem 3.1, we have

$$f(x) = \sum_{k=0}^{\infty}\left(b_k^+ v_k^+(x) + b_k^- v_k^-(x)\right), \tag{3.24}$$

where $b_k^{\pm}$ are given by (3.22). Multiplying both sides of (3.24) by $\overline{f(x)}\rho(x)$ and integrating with respect to $x \in [0,\pi]$, we have

$$\int_0^{\pi} \rho(x)\left|f(x)\right|^2 dx = \int_0^{\pi} \sum_{k=0}^{\infty}\left(b_k^+ v_k^+(x) + b_k^- v_k^-(x)\right)\overline{f(x)}\rho(x)dx. \tag{3.25}$$

By the aid of uniform convergence of the series (3.16), the integration and summation can be interchanged and we have

$$\int_0^{\pi} \rho(x)\left|f(x)\right|^2 dx = \sum_{k=0}^{\infty} b_k^+ \int_0^{\pi} \overline{v_k^+(x)}\,\overline{f(x)}\,\overline{\rho(x)}dx + \sum_{k=0}^{\infty} b_k^- \int_0^{\pi} \overline{v_k^-(x)}\,\overline{f(x)}\,\overline{\rho(x)}dx, \tag{3.26}$$

where $\rho(x)$ and $v_k^{\pm}(x)$ are real (see introduction) which complete the proof of the lemma. In the following theorem, the validity of eigenfunction expansion and the Parsval's identity can be extended to any function of $L_2(0,\pi;\rho)$ but the convergence of the expansion will be in some weak sense, that is, in the metric sense of $L_2(0,\pi;\rho)$. □

Theorem 3.3. *Suppose that $f(x)$ is any function from $L_2(0,\pi;\rho)$; then the following Parsval's identity* (3.22) *and the eigenfunction expansion* (3.24) *are true and the convergence of the series* (3.16) *to $f(x)$ is in the metric sense of the space $L_2(0,\pi;\rho)$.*

Proof. Let $f(x)$ be any function that belongs to $L_2(0,\pi;\rho)$. It is known that the set of infinitely differential functions which vanish at the neighbourhood of the points $x = 0$, $x = \pi$ are dense in $L_2(0,\pi;\rho)$, so that there exists a sequence $\{f_n(x)\}$ of finite smooth functions (and consequently, satisfy the conditions of the theorem) which converges to $f(x)$ in the metric of $L_2(0,\pi;\rho)$; in equation notation this is can be written as

$$\left\|f_n(x) - f(x)\right\|_{L_2} = \left(\int_0^{\pi} \rho(x)\left|f_n(x) - f(x)\right|^2 dx\right)^{1/2} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.27}$$

By the last lemma, every function $f_n(x)$ satisfies the parseval's identity

$$\int_0^{\pi} \rho(x)\left|f_n(x)\right|^2 dx = \sum_{k=0}^{\infty}\left(\left|b_k^{(n)+}\right|^2 + \left|b_k^{(n)-}\right|^2\right), \tag{3.28}$$

where $b_k^{(n)\pm} = \int_0^{\pi} \rho(x) f_n(x) v_k^{\pm}(x)dx$, $k = 0,1,2,\ldots$

The identity (3.28) can be written as

$$\left\|f_n(x)\right\|_{L_2}^2 = \left\|b_k^{(n)+}\right\|_{l_2}^2 + \left\|b_k^{(n)-}\right\|_{l_2}^2. \tag{3.29}$$

Consider the difference

$$\left\|f_n(x) - f_m(x)\right\|_{L_2}^2 = \left\|b_k^{(n)+} - b_k^{(m)+}\right\|_{l_2}^2 + \left\|b_k^{(n)-} - b_k^{(m)-}\right\|_{l_2}^2. \tag{3.30}$$

By the aid of (3.27), it follows that $\{f_n(x)\}$ is a fundamental sequence and hence by the completeness of l_2 the sequences $\{b_k^{n\pm}\}$ are fundamentals, so, that there exists a limiting b_k^+ and b_k^- such that $\|b_k^{(n)+} - b_k^+\|_{l_2}^2 \to 0$ and $\|b_k^{(n)-} - b_k^-\|_{l_2}^2 \to 0$; by using the continuity of the norm and passing to the limit as $n \to \infty$ in (3.29), we obtain

$$\left\|f(x)\right\|_{L_2}^2 = \left\|b_k^+\right\|_{l_2}^2 + \left\|b_k^-\right\|_{l_2}^2, \tag{3.31}$$

which is the Parsval's identity. Now we prove the eigenfunction expansion formula by the help of Parsval's identity. For any n, we have

$$\begin{aligned}
&\int_0^{\pi}\rho(x)\left|f(x)-\sum_{k=0}^{n}\left(b_k^{+}v_k^{+}(x)+b_k^{-}v_k^{-}(x)\right)\right|^2dx\\
&\quad=\int_0^{\pi}\rho(x)\left\{\left[f(x)-\sum_{k=0}^{n}\left(b_k^{+}v_k^{+}(x)+b_k^{-}v_k^{-}(x)\right)\right]\right.\\
&\qquad\left.\times\left[\overline{f(x)}-\sum_{k=0}^{n}\left(\overline{b_k^{+}}v_k^{+}(x)+\overline{b_k^{-}}v_k^{-}(x)\right)\right]\right\}dx;
\end{aligned}\tag{3.32}$$

after calculation, we have

$$\int_0^{\pi}\rho(x)\left|f(x)-\sum_{k=0}^{n}\left(b_k^{+}v_k^{+}(x)+b_k^{-}v_k^{-}(x)\right)\right|dx=\int_0^{\pi}\rho(x)\left|f(x)\right|^2dx-\sum_{k=0}^{n}\left(\left|b_k^{+}\right|^2+\left|b_k^{-}\right|^2\right)\tag{3.33}$$

from which, and by using Parseval identity (3.22), we have

$$\lim_{n\to\infty}\int_0^{\pi}\rho(x)\left|f(x)-\sum_{k=0}^{n}\left(b_k^{+}v_k^{+}(x)+b_k^{-}v_k^{-}(x)\right)\right|^2dx\longrightarrow 0.\tag{3.34}$$

So that, $\sum_{k=0}^{\infty}(b_k^{+}v_k^{+}(x)+b_k^{-}v_k^{-}(x)) \rightarrow f(x)$ in the metric of $L_2(0,\pi;\rho)$, which completes the proof. □

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