

Research Article

A Third-Order Differential Equation and Starlikeness of a Double Integral Operator

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Functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in the unit disk and satisfy the differential equation $f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z)$ are considered, where g is subordinated to a normalized convex univalent function h . These functions f are given by a double integral operator of the form $f(z) = \int_0^1 \int_0^1 G(z t^\mu s^\nu) t^{-\mu} s^{-\nu} ds dt$ with G' subordinated to h . The best dominant to all solutions of the differential equation is obtained. Starlikeness properties and various sharp estimates of these solutions are investigated for particular cases of the convex function h .

1. Introduction

Let \mathcal{A} denote the class of all analytic functions f defined in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$. Further, let \mathcal{S} be the subclass of \mathcal{A} consisting of univalent functions, and let \mathcal{S}^* be its subclass of starlike functions. A starlike function f is characterized analytically by the condition $\operatorname{Re}(z f'(z) / f(z)) > 0$ in U , that is, the domain $f(U)$ is starlike with respect to origin. For two functions $f(z) = z + a_2 z^2 + \dots$ and $g(z) = z + b_2 z^2 + \dots$ in \mathcal{A} , the Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \quad (1.1)$$

For f and g in \mathcal{A} , a function f is subordinate to g , written as $f(z) \prec g(z)$, if there is an analytic function w satisfying $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in U$.

When g is univalent in U , then f is subordinated to g which is equivalent to $f(U) \subset g(U)$ and $f(0) = g(0)$.

In a recent paper, Miller and Mocanu [1] investigated starlikeness properties of functions f defined by double integral operators of the form

$$f(z) = \int_0^1 \int_0^1 W(s, t, z) ds dt. \quad (1.2)$$

In this paper, conditions on a different kernel W are investigated from the perspective of starlikeness. Specifically, we consider functions $f \in \mathcal{A}$ given by the double integral operator of the form

$$f(z) = \int_0^1 \int_0^1 G(zt^\mu s^\nu) t^{-\mu} s^{-\nu} ds dt. \quad (1.3)$$

In this case, it follows that

$$f'(z) = \int_0^1 \int_0^1 g(zt^\mu s^\nu) ds dt, \quad (1.4)$$

where $G'(z) = g(z)$. Further, the function f satisfies a third-order differential equation of the form

$$f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z) \quad (1.5)$$

for appropriate parameters α and γ . The investigation of such functions f can be seen as an extension to the study of the class

$$R(\alpha, h) = \{f \in \mathcal{A} : f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) < h(z), z \in U\}. \quad (1.6)$$

The class $R(\alpha, h)$ or its variations for an appropriate function h have been investigated in several works; see, for example, [2–10] and more recently [11, 12].

2. Results on Differential Subordination

We first recall the definition of best dominant solution of a differential subordination.

Definition 2.1 ((dominant and best dominant) [13]). Let $\Psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$, and let h be univalent in U . If p is analytic in U and satisfies the differential subordination

$$\Psi(p(z), zp'(z), z^2 p''(z)) < h(z), \quad (2.1)$$

then p is called a solution of the differential subordination. A univalent function q is called a dominant if $p < q$ for all p satisfying (2.1). A dominant \tilde{q} that satisfies $\tilde{q} < q$ for all dominants q of (2.1) is said to be the best dominant of (2.1).

In the following sequel, we will assume that h is an analytic convex function in U with $h(0) = 1$. For $\alpha \geq \gamma \geq 0$, consider the third-order differential equation

$$f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z), \quad g(z) < h(z). \tag{2.2}$$

We will denote the class consisting of all solutions $f \in \mathcal{A}$ as $R(\alpha, \gamma, h)$, that is,

$$R(\alpha, \gamma, h) = \left\{ f \in \mathcal{A} : f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) < h(z), \quad z \in U \right\}. \tag{2.3}$$

Let

$$\mu = \frac{(\alpha - \gamma) - \sqrt{(\alpha - \gamma)^2 - 4\gamma}}{2}, \quad \nu + \mu = \alpha - \gamma, \quad \mu\nu = \gamma. \tag{2.4}$$

The discriminant is denoted by $\Delta := (\alpha - \gamma)^2 - 4\gamma$. Note that $\operatorname{Re} \mu \geq 0$ and $\operatorname{Re} \nu \geq 0$. We will rewrite the solution of

$$f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z) \tag{2.5}$$

in its equivalent integral form

$$f'(z) = \int_0^1 \int_0^1 g(zt^\mu s^\nu) ds dt. \tag{2.6}$$

It follows from relations (2.4) that

$$\begin{aligned} g(z) &= f'(z) + (\mu(1 + \nu) + \nu)z f''(z) + \mu\nu z^2 f'''(z) \\ &= \nu z^{1-1/\nu} \left(\mu z^{1+1/\nu} f''(z) + z^{1/\nu} f'(z) \right)' \\ &= \nu z^{1-1/\nu} \left(\mu z^{1+1/\nu-1/\mu} \left(z^{1/\mu} f'(z) \right)' \right)'. \end{aligned} \tag{2.7}$$

Thus,

$$\mu z^{1+1/\nu-1/\mu} \left(z^{1/\mu} f'(z) \right)' = \frac{1}{\nu} \int_0^z w^{1/\nu-1} g(w) dw. \tag{2.8}$$

Making the substitution $w = zs^\nu$ in the above integral and integrating again, a change of variables yields

$$f'(z) = \int_0^1 \int_0^1 g(zt^\mu s^\nu) ds dt. \tag{2.9}$$

We will use the notation ϕ_λ for

$$\phi_\lambda(z) = \int_0^1 \frac{dt}{1-zt^\lambda} = \sum_{n=0}^{\infty} \frac{z^n}{1+\lambda n}. \quad (2.10)$$

From [14] it is known that ϕ_λ is convex in U provided $\operatorname{Re} \lambda \geq 0$.

Theorem 2.2. *Let μ and ν be given by (2.4), and*

$$q(z) = \int_0^1 \int_0^1 h(zt^\mu s^\nu) dt ds. \quad (2.11)$$

*Then the function $q(z) = (\phi_\nu * \phi_\mu) * h(z)$ is convex. If $f \in R(\alpha, \gamma, h)$, then*

$$f'(z) < q(z) < h(z), \quad (2.12)$$

and q is the best dominant.

Proof. It follows from (2.10) that

$$h(z) * \phi_\mu(z) = \int_0^1 \frac{1}{1-zt^\mu} dt * h(z) = \int_0^1 h(zt^\mu) dt := k(z). \quad (2.13)$$

Thus,

$$h(z) * (\phi_\mu(z) * \phi_\nu(z)) = k(z) * \phi_\nu(z) = \int_0^1 k(zs^\nu) ds = \int_0^1 \int_0^1 h(zt^\mu s^\nu) dt ds = q(z). \quad (2.14)$$

Since the convolution of two convex functions is convex [15], the function q is convex. Let

$$p(z) = f'(z) + \nu z f''(z). \quad (2.15)$$

Then,

$$p(z) + \mu z p'(z) = f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) < h(z). \quad (2.16)$$

It is known from [16] that

$$p(z) < \frac{1}{\mu z^{1/\mu}} \int_0^z \xi^{1/\mu-1} h(\xi) d\xi = (\phi_\mu * h)(z) < h(z). \quad (2.17)$$

Similarly,

$$p(z) = f'(z) + \nu z f''(z) < (\phi_\mu * h)(z) \quad (2.18)$$

implies

$$\begin{aligned}
 f'(z) &< (\phi_\nu * \phi_\mu * h)(z) \\
 &= \sum_{n=0}^{\infty} \frac{z^n}{(1 + \nu n)(1 + \mu n)} * h(z) \\
 &= \left(\int_0^1 \int_0^1 \frac{dt ds}{1 - zt^\mu s^\nu} \right) * h(z) \\
 &= \int_0^1 \int_0^1 h(zt^\mu s^\nu) dt ds = q(z).
 \end{aligned}
 \tag{2.19}$$

The differential chain

$$f' < q < \phi_\mu * h < h \tag{2.20}$$

shows that $q < h$. Since $q(z) + \alpha z q'(z) + \gamma z^2 q''(z) = h(z)$, the function

$$Q(z) = \int_0^z q(w) dw \tag{2.21}$$

is a solution of the differential subordination $f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) < h(z)$, and thus $q < \tilde{q}$ for all dominants \tilde{q} . Hence, q is the best dominant. \square

Remark 2.3. (1) When $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha$, and the above subordination reduces to the result of [16], that is,

$$f'(z) + \alpha z f''(z) < h(z) \implies f'(z) < \int_0^1 h(zt^\alpha) dt. \tag{2.22}$$

(2) The above proof also reveals that

$$f \in R(\alpha, \gamma, h) \implies f \in R(0, 0, h), \tag{2.23}$$

that is, $f'(z) < h(z)$.

Theorem 2.4. Let μ, ν , and q be as given in Theorem 2.2. If $f \in R(\alpha, \gamma, h)$, then

$$\begin{aligned}
 \frac{f(z)}{z} &< \int_0^1 q(tz) dt \\
 &= \int_0^1 \int_0^1 \int_0^1 h(zrs^\mu t^\nu) dr ds dt.
 \end{aligned}
 \tag{2.24}$$

Proof. Let $p(z) = f(z)/z$. Then

$$p(z) + zp'(z) = f'(z) < q(z). \quad (2.25)$$

With ϕ_1 given by (2.10), this subordination implies

$$p(z) = (\phi_1 * (p + zp'))(z) < (\phi_1 * q)(z) = \int_0^1 q(tz) dt. \quad (2.26)$$

□

In this paper, starlikeness properties will be investigated for functions f given by a double integral operator of the form (1.3).

3. Applications

First, we consider a class of convex univalent functions h so that $h(U)$ is symmetric with respect to the real axis. Denote by $R(\alpha, \gamma, A, B)$ the class

$$R(\alpha, \gamma, A, B) = \left\{ f \in \mathcal{A} : f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) < \frac{1 + Az}{1 + Bz}, z \in U \right\}, \quad (3.1)$$

where $-1 \leq B < A \leq 1$, and let $h(z; A, B) = (1 + Az)/(1 + Bz)$. When $A = 1 - 2\beta$ and $B = -1$, let $h_\beta(z) := h(z; 1 - 2\beta, -1)$. The class of $R(\alpha, \gamma, h_\beta)$ is of particular significance, and we will simply denote it by

$$\begin{aligned} R(\alpha, \gamma, h_\beta) &:= R(\alpha, \gamma, \beta) \\ &= \left\{ f \in \mathcal{A} : f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) < \frac{1 + (1 - 2\beta)z}{1 - z}, z \in U \right\}. \end{aligned} \quad (3.2)$$

Equivalently,

$$R(\alpha, \gamma, \beta) = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \right) > \beta \right\}. \quad (3.3)$$

The following result is an immediate consequence of Theorems 2.2 and 2.4.

Theorem 3.1. *Under the assumptions of Theorem 2.2, if*

$$f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) < \frac{1 + Az}{1 + Bz}, \quad (3.4)$$

then

$$f'(z) < \begin{cases} q(z; A, B) < \frac{1 + Az}{1 + Bz}, & \text{if } B \neq 0, \\ q(z; A) < 1 + Az, & \text{if } B = 0, \end{cases} \quad (3.5)$$

where

$$\begin{aligned}
 q(z; A, B) &:= 1 + (A - B) \sum_{n=1}^{\infty} \frac{(-B)^{n-1} z^n}{(1 + \mu n)(1 + \nu n)}, \\
 q(z; A) &:= 1 + \frac{Az}{(1 + \alpha)}
 \end{aligned}
 \tag{3.6}$$

is the best dominant. Further,

$$\begin{aligned}
 \frac{f(z)}{z} &< \frac{A}{B} - \frac{A - B}{B} \int_0^1 \int_0^1 \int_0^1 \frac{ds dt du}{1 + Bzut^\mu s^\nu} \\
 &= 1 + (A - B) \sum_{n=1}^{\infty} \frac{(-B)^{n-1} z^n}{(1 + n)(1 + \mu n)(1 + \nu n)}
 \end{aligned}
 \tag{3.7}$$

if $B \neq 0$, and

$$\frac{f(z)}{z} < 1 + \frac{Az}{2(1 + \alpha)}
 \tag{3.8}$$

if $B = 0$.

4. Starlikeness Property

Starlikeness properties of functions given by a double integral operator are investigated in this section. The following result will be required.

Lemma 4.1 (see [5]). *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}(f'(z) + \alpha z f''(z)) > \frac{(-1/\alpha) \int_0^1 t^{1/\alpha-1} ((1-t)/(1+t)) dt}{1 - 1/\alpha \int_0^1 t^{1/\alpha-1} ((1-t)/(1+t)) dt}, \quad z \in \mathcal{U},
 \tag{4.1}$$

for $\alpha \geq 1/3$, then $f \in S^*$. This result is sharp.

Theorem 4.2. *Let μ and ν be given by (2.4) with $\Delta \geq 0$ and $\nu \geq 1/3$. If*

$$f(z) = \int_0^1 \int_0^1 G(z t^\mu s^\nu) t^{-\mu} s^{-\nu} ds dt,
 \tag{4.2}$$

where $G'(z) \prec h_\beta(z) = h(z; 1 - 2\beta, -1)$, and β satisfies

$$\beta = 1 - \frac{1}{2\left(1 - (1/\nu) \int_0^1 t^{1/\nu-1}((1-t)/(1+t))dt\right)\left(1 - \int_0^1 (dt/(1+t^\mu))\right)}, \quad (4.3)$$

then $f \in S^*$.

Proof. The function f satisfies

$$f'(z) = \int_0^1 \int_0^1 g(zt^\mu s^\nu) ds dt, \quad G'(z) = g(z) \prec h_\beta(z), \quad (4.4)$$

and thus

$$f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z) \prec h_\beta(z). \quad (4.5)$$

Now, $\operatorname{Re} h_\beta(z) > \beta$ also implies that $\operatorname{Re} g(z) > \beta$, and so

$$\operatorname{Re}\left(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)\right) > \beta, \quad \beta < 1. \quad (4.6)$$

It follows from the proof of Theorem 2.2 that

$$f'(z) + \nu z f''(z) \prec (\phi_\mu * h_\beta)(z) := q_\mu(z), \quad (4.7)$$

where

$$q_\mu(z) = 2\beta - 1 + 2(1 - \beta) \int_0^1 \frac{dt}{1 - zt^\mu}. \quad (4.8)$$

Since

$$\operatorname{Re} q_\mu(z) > 2\beta - 1 + 2(1 - \beta) \int_0^1 \frac{dt}{1 + t^\mu}, \quad (4.9)$$

an application of Lemma 4.1 yields the result. \square

Corollary 4.3. *Let $\alpha \geq 3$ and*

$$\operatorname{Re}\left(f'(z) + \alpha z f''(z) + \frac{\alpha - 1}{2} z^2 f'''(z)\right) > \beta, \quad \beta < 1. \quad (4.10)$$

If β satisfies

$$\beta = 1 - \frac{1}{2(1 - \log 2) \left(1 - (2/(\alpha - 1)) \int_0^1 t^{2/(\alpha-1)-1} ((1-t)/(1+t)) dt\right)}, \quad (4.11)$$

then $f \in S^*$.

Proof. In this case, $\mu = 1$, $\nu = (\alpha - 1)/2$, and the result now follows from Theorem 4.2. \square

Example 4.4. If

$$\operatorname{Re} \left(f'(z) + 3zf''(z) + z^2 f'''(z) \right) > \beta \quad (4.12)$$

and β satisfies

$$\beta = \frac{4(1 - \log 2)^2 - 1}{4(1 - \log 2)^2} \approx -1.65509, \quad (4.13)$$

then $f \in S^*$.

Theorem 4.5. Let $f, g \in R(\alpha, \gamma, \beta)$ and let μ and ν be given by (2.4) with $\Delta \geq 0$. If β satisfies

$$\beta = 1 - \frac{1}{4 \left(1 - \int_0^1 \int_0^1 \int_0^1 (ds dt du / (1 + ut^\mu s^\nu))\right)}, \quad (4.14)$$

then $f * g \in R(\alpha, \gamma, \beta)$.

Proof. Clearly,

$$(f * g)'(z) + \alpha z(f * g)''(z) + \gamma z^2(f * g)'''(z) = \left((f' + \alpha z f'' + \gamma z^2 f''') * \frac{g}{z} \right)(z). \quad (4.15)$$

Since $f \in R(\alpha, \gamma, \beta)$, substituting $A = 1 - 2\beta$ and $B = -1$ in (3.7) gives

$$\operatorname{Re} \frac{g(z)}{z} > 2\beta - 1 + 2(1 - \beta) \int_0^1 \int_0^1 \int_0^1 \frac{ds dt du}{1 + ut^\mu s^\nu} = \frac{1}{2}. \quad (4.16)$$

Hence, it follows that

$$\operatorname{Re} \left((f * g)'(z) + \alpha z(f * g)''(z) + \gamma z^2(f * g)'''(z) \right) > \beta. \quad (4.17)$$

\square

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