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Research Article

Estimates of Exponential Stability for Solutions of Stochastic Control Systems with Delay

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A nonlinear stochastic differential-difference control system with delay of neutral type is considered. Sufficient conditions for the exponential stability are derived by using Lyapunov-Krasovskii functionals of quadratic form with exponential factors. Upper bound estimates for the exponential rate of decay are derived.

1. Introduction

The theory and applications of functional differential equations form an important part of modern nonlinear dynamics. Such equations are natural mathematical models for various real life phenomena where the aftereffects are intrinsic features of their functioning. In recent years, functional differential equations have been used to model processes in different areas such as population dynamics and ecology, physiology and medicine, economics, and other natural sciences [1–3]. In many of the models the initial data and parameters are subjected to random perturbations, or the dynamical systems themselves represent stochastic processes. For this reason, stochastic functional differential equations are widely studied [4, 5].

One of the principal problems of the corresponding mathematical analysis of equations is a comprehensive study of their global dynamics and the related prediction of long-term behaviors in applied models. Of course, the problem of stability of a particular solution plays a significant role. Therefore, the study of stability of linear equations is the first natural and important step in the analysis of more complex nonlinear systems.

When applying the mathematical theory to real-world problems a mere statement of the stability in the system is hardly sufficient. In addition to stability as such, it is of significant importance to obtain constructive and verifiable estimates of the rate of convergence of solutions in time. One of the principal tools used in the related studies is the second Lyapunov method [6–8]. For functional differential equations, this method has been developing in two main directions in recent years. The first one is the method of finite Lyapunov functions with the additional assumption of Razumikhin type [9, 10]. The second one is the method of Lyapunov-Krasovskii functionals [11, 12]. For stochastic functional differential equations, some aspects of these two lines of research have been developed, for example, in [11, 13–19] and [11, 18, 20–25], respectively. In the present paper, by using the method of Lyapunov-Krasovskii functionals, we derive sufficient conditions for stability together with the rate of convergence to zero of solutions for a class of linear stochastic functional differential equation of a neutral type.

2. Preliminaries

In solving control problems for linear systems, very often, a scalar function $u = u(x)$ needs to be found such that the system

$$\dot{x}(t) = Ax(t) + bu(x(t)) \tag{2.1}$$

is asymptotically stable. Frequently, such a function depends on a scalar argument which is a linear combination of phase coordinates and its graph lies in the first and the third quadrants of the plane. An investigation of the asymptotic stability of systems with a control function

$$u(x(t)) = f(\sigma(t)), \quad \sigma(t) = c^T x(t), \tag{2.2}$$

that is, an investigation of systems

$$\dot{x}(t) = Ax(t) + bf(\sigma(t)), \quad \sigma(t) = c^T x(t), \tag{2.3}$$

with a function f satisfying $f(0) = 0$, $f(\sigma)(k\sigma - f(\sigma)) > 0$ for $\sigma \neq 0$ and a $k > 0$ is called an analysis of the absolute stability of control systems [26]. One of the fundamental methods (called a frequency method) was developed by Gelig et al. (see, e.g., the book [27]). Another basic method is the method of Lyapunov's functions and Lyapunov-Krasovskii functionals. Very often, the appropriate Lyapunov functions and Lyapunov-Krasovskii functionals are constructed as quadratic forms with integral terms containing a given nonlinearity [28, 29]. An overview of the present state can be found, for example, in [30, 31]. Problems of absolute stability of stochastic equations are treated, for example, in [11, 14, 15, 24].

3. Main Results

Consider the following control system of stochastic differential-difference equations of a neutral type

$$\begin{aligned} d[x(t) - Dx(t-\tau)] = {} & \left[A_0 x(t) + A_1 x(t-\tau) + a_2 f(\sigma(t))\right] dt \\ & + \left[B_0 x(t) + B_1 x(t-\tau) + b_2 f(\sigma(t))\right] dw(t), \end{aligned} \tag{3.1}$$

where

$$\sigma(t) := c^T [x(t) - Dx(t-\tau)], \tag{3.2}$$

$x : [0, \infty) \to \mathbb{R}^n$ is an n-dimensional column vector, A_0, A_1, B_0, B_1, and D are real $n \times n$ constant matrices, a_2, b_2, and c are $n \times 1$ constant vectors, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\tau > 0$ is a constant delay, and $w(t)$ is a standard scalar Wiener process with

$$M\{dw(t)\} = 0, \qquad M\left\{dw^2(t)\right\} = dt, \qquad M\{dw(t_1)\, dw(t_2), t_1 \neq t_2\} = 0. \tag{3.3}$$

An F_t-measurable random process $\{x(t) \equiv x(t, \omega)\}$ is called a solution of (3.1) if it satisfies, with a probability one, the following integral equation

$$\begin{aligned} x(t) = {} & Dx(t-\tau) + [x(0) - Dx(-\tau)] \\ & + \int_0^t \left[A_0 x(s) + A_1 x(s-\tau) + a_2 f(\sigma(s))\right] ds \\ & + \int_0^t \left[B_0 x(s) + B_1 x(s-\tau) + b_2 f(\sigma(s))\right] dw(s), \quad t \geq 0 \end{aligned} \tag{3.4}$$

and the initial conditions

$$x(t) = \varphi(t), \quad x'(t) = \psi(t), \quad t \in [-\tau, 0], \tag{3.5}$$

where $\varphi, \psi : [-\tau, 0] \to \mathbb{R}^n$ are continuous functions. Here and in the remaining part of the paper, we will assume that the initial functions φ and ψ are continuous random processes. Under those assumptions, a solution to the initial value problem (3.1), (3.5) exists and is unique for all $t \geq 0$ up to its stochastic equivalent solution on the space (Ω, F, P) [4].

We will use the following norms of matrices and vectors

$$\begin{aligned}
\|A\| &:= \sqrt{\lambda_{\max}\left(A^T A\right)}, \\
\|x(t)\| &:= \sqrt{\sum_{i=1}^{n} x_i^2(t)}, \\
\|x(t)\|_\tau &:= \max_{-\tau \le s \le 0} \{\|x(t+s)\|\}, \\
\|x(t)\|_{\tau,\gamma}^2 &:= \int_{t-\tau}^{t} e^{-\gamma(t-s)} \|x(s)\|^2 ds,
\end{aligned} \tag{3.6}$$

where $\lambda_{\max}(*)$ is the largest eigenvalue of the given symmetric matrix (similarly, the symbol $\lambda_{\min}(*)$ denotes the smallest eigenvalue of the given symmetric matrix), and γ is a positive parameter.

Throughout this paper, we assume that the function f satisfies the inequality

$$0 \le f(\sigma)\sigma \le k\sigma^2 \tag{3.7}$$

if $\sigma \in \mathbb{R}$ where k is a positive constant.

For the reader's convenience, we recall that the zero solution of (3.1) is called stable in the square mean if, for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that every solution $x = x(t)$ of (3.1) satisfies $M\{\|x(t)\|^2\} < \varepsilon$ provided that the initial conditions (3.5) are such that $\|\varphi(0)\|_\tau < \delta$ and $\|\psi(0)\|_\tau < \delta$. If the zero solution is stable in the square mean and, moreover,

$$\lim_{t \to +\infty} M\left\{\|x(t)\|^2\right\} = 0, \tag{3.8}$$

then it is called asymptotically stable in the square mean.

Definition 3.1. If there exist positive constants N, γ, and θ such that the inequality

$$M\left\{\|x(t)\|_{\tau,\gamma}^2\right\} \le N \, \|x(0)\|_\tau^2 \, e^{-\theta t} \tag{3.9}$$

holds on $[0, \infty)$, then the zero solution of (3.1) is called exponentially γ-integrally stable in the square mean.

In this paper, we prove the exponential γ-integral stability in the square mean of the differential-difference equation with constant delay (3.1). We employ the method of stochastic Lyapunov-Krasovskii functionals. In [11, 18, 22, 24] the Lyapunov-Krasovskii functional is chosen to be of the form

$$V[x(t), t] = h[x(t) - cx(t-\tau)]^2 + g\int_{-\tau}^{0} x^2(t+s)ds, \tag{3.10}$$

where constants $h > 0$ and $g > 0$ are such that the total stochastic differential of the functional along solutions is negative definite.

In the present paper, we consider the Lyapunov-Krasovskii functional in the following form:

$$\begin{aligned} V[x(t), t] &= [x(t) - Dx(t-\tau)]^T H[x(t) - Dx(t-\tau)] \\ &\quad + \int_{t-\tau}^{t} e^{-\gamma(t-s)} x^T(s) G x(s) ds + \beta \int_0^{\sigma(t)} f(\xi) d\xi, \end{aligned} \tag{3.11}$$

where constants $\gamma > 0$, $\beta > 0$ and $n \times n$ positive definite symmetric matrices G, H are to be restricted later on. This allows us not only to derive sufficient conditions for the stability of the zero solution but also to obtain coefficient estimates of the rate of the exponential decay of solutions.

We set

$$P := \begin{pmatrix} H & -HD \\ -D^T H & D^T H D \end{pmatrix}. \tag{3.12}$$

Then, by using introduced norms, the functional (3.11) yields two-sided estimates

$$\begin{aligned} \lambda_{\min}(G)\|x(t)\|^2_{\tau,\gamma} \le V[x(t), t] &\le \left[\lambda_{\max}(P) + 0.5\beta k\|c\|^2\right]\|x(t)\|^2 \\ &\quad + \left[\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2\right]\|x(t-\tau)\|^2 + \lambda_{\max}(G)\|x(t)\|^2_{\tau,\gamma}, \end{aligned} \tag{3.13}$$

where $t \in [0, \infty)$.

We will use an auxiliary $(2n+1) \times (2n+1)$-dimensional matrix:

$$S = S(\beta, \gamma, \nu, G, H) := \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \tag{3.14}$$

where

$$
\begin{aligned}
s_{11} &:= -A_0 H - H A_0 - B_0^T H B_0 - G,\\
s_{12} &:= A_0^T H D - H A_1 - B_0^T H B_1,\\
s_{13} &:= -H a_2 - B_0^T H b_2 - \frac{1}{2}(\beta A_0 + \nu I)^T c,\\
s_{21} &:= s_{12}^T,\\
s_{22} &:= D^T H A_1 + A_1^T H D - B_1^T H B_1 + e^{-\gamma\tau} G,\\
s_{23} &:= D^T H a_2 - B_1^T H b_2 - \frac{1}{2}\beta A_1 c,\\
s_{31} &:= s_{13}^T,\\
s_{32} &:= s_{23}^T,\\
s_{33} &:= \frac{\nu}{k} - b_2^T H b_2 - \beta c^T a_2,
\end{aligned}
\tag{3.15}
$$

where ν is a parameter.

Now we establish our main result on the exponential γ-integral stability of a trivial solution in the square mean of system (3.1) when $t \to \infty$.

Theorem 3.2. *Let $\|D\| < 1$. Let there exist positive constants β, γ, ν and positive definite symmetric matrices G, H such that the matrix S is positively definite as well. Then the zero solution of the system* (3.1) *is exponentially γ-integrally stable in the square mean on $[0,\infty)$. Moreover, every solution $x(t)$ of* (3.1) *satisfies the inequality*

$$
M\left\{\|x(t)\|_{\tau,\gamma}^2\right\} \le N\|x(0)\|_\tau^2\, e^{-\theta t}
\tag{3.16}
$$

for all $t \ge 0$ where

$$
\begin{aligned}
N &:= \frac{1}{\lambda_{\min}(G)} \cdot \left(2\lambda_{\max}(P) + 0.5\beta k\|c\|^2 + 0.5\beta k\left\|c^T D\right\|^2 + \frac{1}{\gamma}\lambda_{\max}(G)\right),\\
\theta &:= \min\left\{\frac{\gamma\lambda_{\min}(G)}{\lambda_{\max}(G)}, \frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2}\right\}.
\end{aligned}
\tag{3.17}
$$

Proof. We will apply the method of Lyapunov-Krasovskii functionals using functional (3.11). Using the Itô formula, we compute the stochastic differential of (3.11) as follows

$$\begin{aligned}
dV[x(t),t] = {}& \Big(\big[A_0x(t) + A_1x(t-\tau) + a_2f(\sigma(t))\big]^T dt \\
&+ \big[B_0x(t) + B_1x(t-\tau) + b_2f(\sigma(t))\big]^T dw(t) \Big) \\
&\times H[x(t) - Dx(t-\tau)] + [x(t) - Dx(t-\tau)]^T \\
&\times H\Big(\big[A_0x(t) + A_1x(t-\tau) + a_2f(\sigma(t))\big] dt \\
&\qquad + \big[B_0x(t) + B_1x(t-\tau) + b_2f(\sigma(t))\big]^T dw(t) \Big) \\
&+ \big[B_0x(t) + B_1x(t-\tau) + b_2f(\sigma(t))\big]^T \\
&\times H\big[B_0x(t) + B_1x(t-\tau) + b_2f(\sigma(t))\big] d\Big(w^2(t)\Big) \\
&+ x^T(t)Gx(t)dt - e^{-\gamma\tau}x^T(t-\tau)Gx(t-\tau)dt + \beta f(\sigma(t))c^T \\
&\times \Big(\big[A_0x(t) + A_1x(t-\tau) + a_2f(\sigma(t))\big] dt \\
&\quad + \big[B_0x(t) + B_1x(t-\tau) + b_2f(\sigma(t))\big]^T dw(t) \Big) \\
&- \gamma \int_{t-\tau}^{t} e^{-\gamma(t-s)}x^T(s)Gx(s)ds\,dt.
\end{aligned} \tag{3.18}$$

Taking the mathematical expectation we obtain (we use properties (3.3))

$$\begin{aligned}
M\{dV[x(t),t]\} = {}& M\Big\{ \big[A_0x(t) + A_1x(t-\tau) + a_2f(\sigma(t))\big]^T \\
&\quad \times H[x(t) - Dx(t-\tau)]dt \Big\} \\
&+ M\Big\{ [x(t) - Dx(t-\tau)]^T \\
&\qquad \times H\big[A_0x(t) + A_1x(t-\tau) + a_2f(\sigma(t))\big]dt \Big\} \\
&+ M\Big\{ \big[B_0x(t) + B_1x(t-\tau) + b_2f(\sigma(t))\big]^T \\
&\qquad \times H\big[B_0x(t) + B_1x(t-\tau) + b_2f(\sigma(t))\big] d\Big(w^2(t)\Big) \Big\} \\
&+ M\Big\{ \Big[x^T(t)Gx(t)dt - e^{-\gamma\tau}x^T(t-\tau)Gx(t-\tau)dt\Big] \Big\} \\
&+ \beta M\Big\{ f(\sigma(t))c^T\big[A_0x(t) + A_1x(t-\tau) + a_2f(\sigma(t))\big]dt \Big\} \\
&- \gamma M\Big\{ \int_{t-\tau}^{t} e^{-\gamma(t-s)}x^T(s)Gx(s)ds\,dt \Big\}.
\end{aligned} \tag{3.19}$$

Utilizing the matrix S defined by (3.14), the last expression can be rewritten in the following vector matrix form

$$\frac{d}{dt}M\{V[x(t),t]\} = -M\left\{\left(x^T(t), x^T(t-\tau), f(\sigma(t))\right) \times S \times \left(x^T(t), x^T(t-\tau), f(\sigma(t))\right)^T\right\} - \nu\left[\sigma(t) - \frac{f(\sigma(t))}{k}\right] f(\sigma(t)) - \gamma M\left\{\int_{t-\tau}^{t} e^{-\gamma(t-s)} x^T(s) G x(s) ds\right\}. \tag{3.20}$$

We will show next that solutions of (3.1) decay exponentially by calculating the corresponding exponential rate.

The full derivative of the mathematical expectation for the Lyapunov-Krasovskii functional (3.11) satisfies

$$\begin{aligned}\frac{d}{dt}M\{V[x(t),t]\} \le& -\lambda_{\min}(S)M\left\{\|x(t)\|^2\right\}\\ &- \lambda_{\min}(S)M\left\{\|x(t-\tau)\|^2\right\}\\ &- \gamma\lambda_{\min}(G)M\left\{\|x(t)\|^2_{\tau,\gamma}\right\}.\end{aligned} \tag{3.21}$$

In the following we will use inequalities being a consequence of (3.13).

$$\begin{aligned}\lambda_{\min}(G)M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} &\le M\{V[x(t)]\}\\ &\le \left[\lambda_{\max}(P) + 0.5\beta k\|c\|^2\right] \times M\left\{\|x(t)\|^2\right\}\\ &\quad + \left[\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2\right] M\left\{\|x(t-\tau)\|^2\right\}\\ &\quad + \lambda_{\max}(G)M\left\{\|x(t)\|^2_{\tau,\gamma}\right\}.\end{aligned} \tag{3.22}$$

Let us derive conditions for the coefficients of (3.1) and parameters of the Lyapunov-Krasovskii functional (3.11) such that the following inequality:

$$\frac{d}{dt}M\{V[x(t),t]\} \le -\theta M\{V[x(t),t]\} \tag{3.23}$$

holds. We use a sequence of the following calculations supposing that either inequality

$$\gamma\lambda_{\min}(G) - \frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k|c|^2}\lambda_{\max}(G) \ge 0 \tag{3.24}$$

holds, or the opposite inequality

$$\gamma\lambda_{\min}(G) - \frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k|c|^2}\lambda_{\max}(G) \leq 0 \tag{3.25}$$

is valid.

(1) Let inequality (3.24) holds. Rewrite the right-hand part of inequality (3.22) in the form

$$\begin{aligned} -M\left\{\|x(t)\|^2\right\} \leq{}& \frac{1}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2} \\ &\times \Big[- M\{V[x(t),t]\} + \lambda_{\max}(G) M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} \\ &\quad + \left[\lambda_{\max}(P) + 0.5\beta k \left\|c^T D\right\|^2\right] M\left\{\|x(t-\tau)\|^2\right\}\Big] \end{aligned} \tag{3.26}$$

and substitute the latter into inequality (3.21). This results in

$$\begin{aligned} \frac{d}{dt}M\{V[x(t),t]\} \leq{}& -\frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2} \\ &\times \Big[- M\{V[x(t),t]\} + \lambda_{\max}(G) M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} \\ &\quad + \left[\lambda_{\max}(P) + 0.5\beta k \left\|c^T D\right\|^2\right] M\left\{\|x(t-\tau)\|^2\right\}\Big] \\ &- \gamma\lambda_{\min}(G) M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} - \lambda_{\min}(S) M\left\{\|x(t-\tau)\|^2\right\}, \end{aligned} \tag{3.27}$$

or, equivalently,

$$\begin{aligned} \frac{d}{dt}M\{V[x(t),t]\} \leq{}& -\frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2} M\{V[x(t),t]\} \\ &- \lambda_{\min}(S)\left(1 - \frac{\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2}\right) M\left\{\|x(t-\tau)\|^2\right\} \\ &- \left(\gamma\lambda_{\min}(G) - \frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2}\lambda_{\max}(G)\right) M\left\{\|x(t)\|^2_{\tau,\gamma}\right\}. \end{aligned} \tag{3.28}$$

The inequality

$$\frac{\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2} \leq 1 \tag{3.29}$$

always holds. Because inequality (3.24) is valid, a differential inequality

$$\begin{aligned}\frac{d}{dt}M\{V[x(t),t]\} &\leq -\frac{\lambda_{\min}(S)}{\lambda_{\max}(P)+0.5\beta k\|c\|^2}M\{V[x(t),t]\}\\ &\leq -\theta M\{V[x(t),t]\}\end{aligned} \tag{3.30}$$

will be true as well.

(2) Let inequality (3.25) hold. We rewrite the right-hand side of inequality (3.22) in the form

$$\begin{aligned}-M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} \leq \frac{1}{\lambda_{\max}(G)} \times \Bigg(&-M\{V[x(t),t]\}+\left(\lambda_{\max}(P)+0.5\beta k\|c\|^2\right)M\left\{\|x(t)\|^2\right\}\\ &+\left[\lambda_{\max}(P)+0.5\beta k\left\|c^TD\right\|^2\right]M\left\{\|x(t-\tau)\|^2\right\}\Bigg)\end{aligned} \tag{3.31}$$

and substitute the latter again into inequality (3.21). This results in

$$\begin{aligned}\frac{d}{dt}M\{V[x(t),t]\} \leq{}& -\lambda_{\min}(S)M\left\{\|x(t)\|^2\right\}-\lambda_{\min}(S)M\left\{\|x(t-\tau)\|^2\right\}+\gamma\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\\ &\times\Bigg\{-M\{V[x(t),t]\}+\left(\lambda_{\max}(P)+0.5\beta k\|c\|^2\right)M\left\{\|x(t)\|^2\right\}\\ &+\left[\lambda_{\max}(P)+0.5\beta k\left\|c^TD\right\|^2\right]M\left\{\|x(t-\tau)\|^2\right\}\Bigg\}\end{aligned} \tag{3.32}$$

or in

$$\begin{aligned}\frac{d}{dt}M\{V[x(t),t]\} \leq{}& -\gamma\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}M\{V[x(t),t]\}\\ &-\left(\lambda_{\min}(S)-\frac{\lambda_{\max}(P)+0.5\beta k\|c\|^2}{\lambda_{\max}(G)}\gamma\lambda_{\min}(G)\right)M\left\{\|x(t)\|^2\right\}\\ &-\left(\lambda_{\min}(S)-\frac{\gamma\lambda_{\min}(G)\left[\lambda_{\max}(P)+0.5\beta k\left\|c^TD\right\|^2\right]}{\lambda_{\max}(G)}\right)M\left\{\|x(t-\tau)\|^2\right\}.\end{aligned} \tag{3.33}$$

Because inequality (3.25) is valid, a differential inequality

$$\frac{d}{dt}M\{V[x(t),t]\} \leq -\gamma\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}M\{V[x(t),t]\} \leq -\theta M\{V[x(t),t]\} \tag{3.34}$$

will be valid as well.

Analysing inequalities (3.30) and (3.34) we conclude that (3.23) always holds. Solving inequality (3.23) we obtain

$$M\{V[x(t),t]\} \leq M\{V[x(0),0]\}e^{-\theta t}. \tag{3.35}$$

Now we derive estimates of the rate of the exponential decay of solutions. We use inequalities (3.22), (3.35). It is easy to see that

$$\begin{aligned}
\lambda_{\min}(G)M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} &\leq M\{V[x(t),t]\} \leq M\{V[x(0),0]\}e^{-\theta t} \\
&\leq \Big(\left(\lambda_{\max}(P) + 0.5\beta k\|c\|^2\right)\|x(0)\|^2 \\
&\quad + \left[\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2\right]\|x(-\tau)\|^2 + \lambda_{\max}(G)\|x(0)\|^2_{\tau,\gamma}\Big)e^{-\theta t} \\
&\leq \left(2\lambda_{\max}(P) + 0.5\beta k\|c\|^2 + 0.5\beta k\left\|c^T D\right\|^2 + \frac{1}{\gamma}\lambda_{\max}(G)\right)\|x(0)\|^2_{\tau}e^{-\theta t}.
\end{aligned} \tag{3.36}$$

Now, inequality (3.16) is a simple consequence of the latter chain of inequalities. □

4. A Scalar Case

As an example, we will apply Theorem 3.2 to a scalar control stochastic differential-difference equation of a neutral type

$$\begin{aligned}
d[x(t) - d_0x(t-\tau)] &= \left[a_0x(t) + a_1x(t-\tau) + a_2f(\sigma(t))\right]dt \\
&\quad + \left[b_0x(t) + b_1x(t-\tau) + b_2f(\sigma(t))\right]dw(t),
\end{aligned} \tag{4.1}$$

where $\sigma(t) = c[x(t) - d_0x(t-\tau)]$, $x \in \mathbb{R}$, a_0, a_1, a_2, b_0, b_1, d_2, d_0, and c are real constants, $\tau > 0$ is a constant delay, and $w(t)$ is a standard scalar Wiener process satisfying (3.3). An F_t-measurable random process $\{x(t) \equiv x(t,\omega)\}$ is called a solution of (4.1) if it satisfies, with a probability one, the following integral equation:

$$\begin{aligned}
x(t) &= d_0x(t-\tau) + [x(0) - d_0x(-\tau)] \\
&\quad + \int_0^t \left[a_0x(s) + a_1x(s-\tau) + a_2f(\sigma(t))\right]ds \\
&\quad + \int_0^t \left[b_0x(s) + b_1x(s-\tau) + b_2f(\sigma(t))\right]dw(s), \quad t \geq 0.
\end{aligned} \tag{4.2}$$

The Lyapunov-Krasovskii functional V reduces to

$$V[x(t),t] = [x(t) - d_0 x(t-\tau)]^2 + g\int_{t-\tau}^{t} e^{-\gamma(t-s)} x^2(s)ds + \beta\int_0^{\sigma(t)} f(\xi)d\xi, \tag{4.3}$$

where we assume $g > 0$ and $\beta > 0$. The matrix S reduces to (for simplicity we set $H = (1)$)

$$S = S(g,\beta,\gamma,\nu) := \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \tag{4.4}$$

and has entries

$$\begin{aligned} s_{11} &:= -2a_0 - b_0^2 - g, \\ s_{12} &:= a_0 d_0 - a_1 - b_0 b_1, \\ s_{13} &:= -a_2 - b_0 b_2 - \frac{1}{2}(\beta a_0 + \nu)c, \\ s_{21} &:= s_{12}, \\ s_{22} &:= 2a_1 d_0 - b_1^2 + e^{-\gamma\tau} g, \\ s_{23} &:= a_2 d_0 - b_1 b_2 - 0.5\beta a_1 c, \\ s_{31} &:= s_{13}, \\ s_{32} &:= s_{23}, \\ s_{33} &:= \frac{\nu}{k} - b_2^2 - \beta c a_2, \end{aligned} \tag{4.5}$$

where ν is a parameter. Therefore, the above calculation yields the following result.

Theorem 4.1. *Let $|d_0| < 1$. Assume that positive constants β, γ, g, and ν are such that the matrix S is positive definite. Then the zero solution of (4.1) is exponentially γ-integrally stable in the square mean on $[0,\infty)$. Moreover, every solution $x(t)$ satisfies the following convergence estimate:*

$$M\left\{\|x(t)\|_{\tau,\gamma}^2\right\} \le N\|x(0)\|_\tau^2 e^{-\theta t} \tag{4.6}$$

for all $t \ge 0$ where

$$\begin{aligned} N &:= \frac{1}{g}\left(2 + 2d_0^2 + 0.5\beta k c^2 + 0.5\beta k (c d_0)^2\right) + \frac{1}{\gamma}, \\ \theta &:= \min\left\{\gamma, \frac{\lambda_{\min}(S)}{1 + d_0^2 + 0.5\beta k c^2}\right\}. \end{aligned} \tag{4.7}$$

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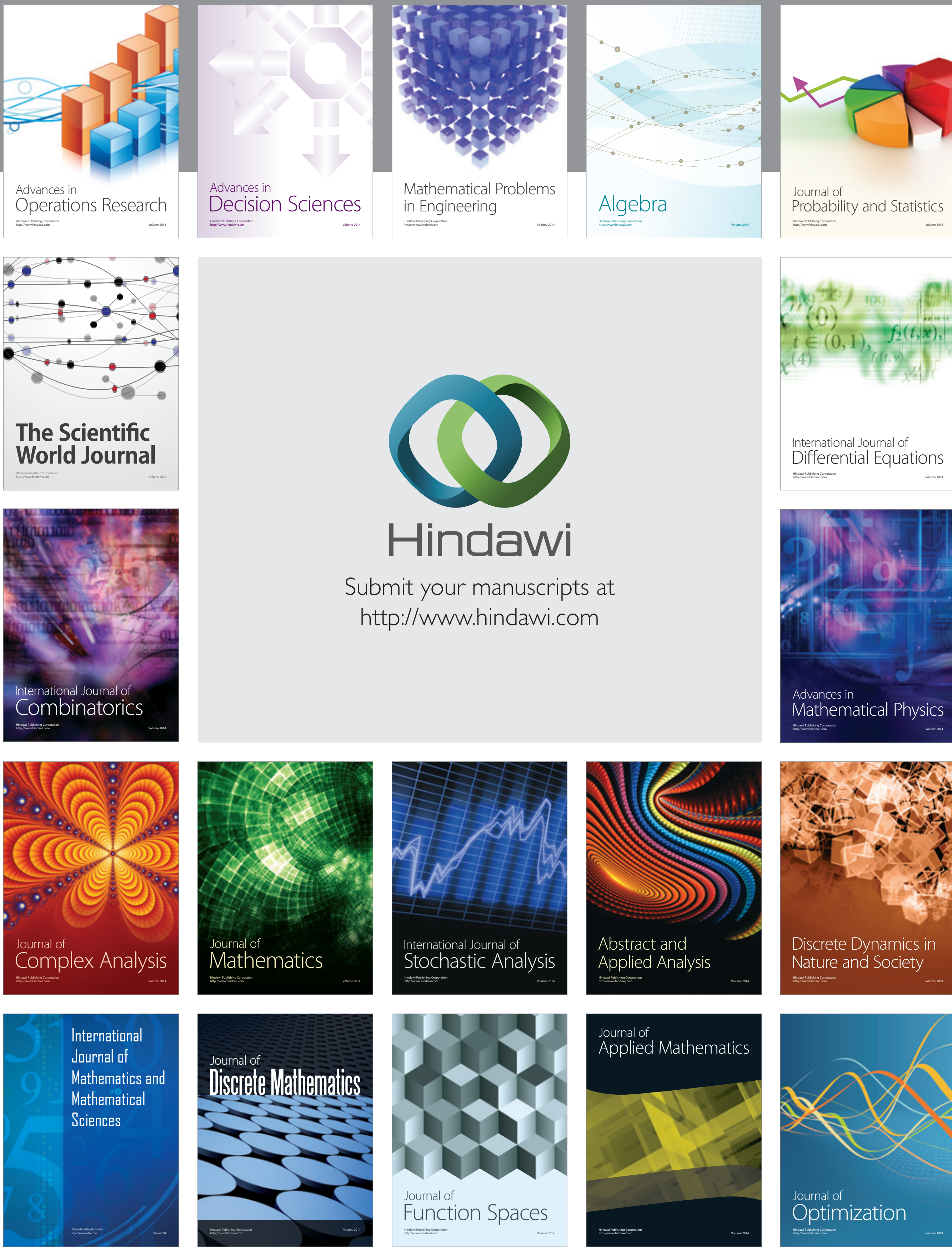
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