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Research Article

Approximation of Analytic Functions by Bessel's Functions of Fractional Order

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We will solve the inhomogeneous Bessel's differential equation $x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = \sum_{m=0}^{\infty} a_m x^m$, where ν is a positive nonintegral number and apply this result for approximating analytic functions of a special type by the Bessel functions of fractional order.

1. Introduction

The stability problem for functional equations starts from the famous talk of Ulam and the partial solution of Hyers to the Ulam problem (see [1, 2]). Thereafter, Rassias [3] attempted to solve the stability problem of the Cauchy additive functional equation in a more general setting.

The stability concept introduced by Rassias's theorem significantly influenced a number of mathematicians to investigate the stability problems for various functional equations (see [4–14] and the references therein).

Assume that Y is a normed space and I is an open subset of $\mathbb{R}$. If for any function $f : I \to Y$ satisfying the differential inequality

$$\left\| a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) \right\| \leq \varepsilon, \tag{1.1}$$

for all $x \in I$ and for some $\varepsilon \geq 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) = 0, \tag{1.2}$$

such that $\|f(x) - f_0(x)\| \leq K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ depends on ε only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space $\mathbb{R}$). We may apply these terminologies for other differential equations. For more detailed definition of the Hyers-Ulam stability, refer to [2, 3, 6, 8, 10–12, 14].

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [15, 16]). Here, we will introduce a result of Alsina and Ger (see [17]): If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(x) - y(x)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a constant c such that $|f(x) - ce^x| \leq 3\varepsilon$ for any $x \in I$.

This result of Alsina and Ger has been generalized by Takahasi et al. They proved in [18] that the Hyers-Ulam stability holds for the Banach space-valued differential equation $y'(x) = \lambda y(x)$ (see also [19]).

Using the conventional power series method, the author has investigated the general solution of the inhomogeneous Legendre differential equation under some specific condition, and this result was applied to prove the Hyers-Ulam stability of the Legendre differential equation (see [20]). In a recent paper, he has also investigated an approximation property of analytic functions by the Legendre functions (see [21]). This study has been continued to various special functions including the Airy functions, the exponential functions, the Hermite functions, and the power functions (see [22–25]).

Recently, the author and Kim tried to prove the Hyers-Ulam stability of the Bessel differential equation

$$x^2 y''(x) + xy'(x) + \left(x^2 - \nu^2\right) y(x) = 0. \tag{1.3}$$

However, the obtained theorem unfortunately does not describe the Hyers-Ulam stability of the Bessel differential equation in a strict sense (see [26]).

In Section 2 of this paper, by using the ideas from [21], we will determine the general solution of the inhomogeneous Bessel differential equation

$$x^2 y''(x) + xy'(x) + \left(x^2 - \nu^2\right) y(x) = \sum_{m=0}^{\infty} a_m x^m, \tag{1.4}$$

where the parameter ν is a positive nonintegral number. Section 3 will be devoted to the investigation of an approximation property of the Bessel functions.

Throughout this paper, we denote by $[x]$ the largest integer not exceeding x for any $x \in \mathbb{R}$, and we define $I_\rho = (-\rho, 0) \cup (0, \rho)$ for any $\rho > 0$.

2. Inhomogeneous Bessel's Differential Equation

A function is called a Bessel function (of fractional order) if it is a solution of the Bessel differential equation (1.3), where ν is a positive nonintegral number. The Bessel differential equation plays a great role in physics and engineering. In particular, this equation is most useful for treating the boundary value problems exhibiting cylindrical symmetries.

The convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ seems not to guarantee the existence of solutions to the inhomogeneous Bessel differential equation (1.4). Thus, we adopt an additional condition to ensure the existence of solutions to the equation.

Theorem 2.1. *Let ν be a positive nonintegral number, and let ρ be a positive constant. Assume that the radius of convergence of power series $\sum_{m=0}^{\infty} a_m x^m$ is at least ρ and there exists a constant $\sigma > 0$ satisfying the condition*

$$|a_{m+2}| \leq \frac{m^2}{\sigma^2}|c_m|, \tag{2.1}$$

for all sufficiently large integers m, where

$$c_m = \begin{cases} -\displaystyle\sum_{i=0}^{[m/2]} a_{2i} \prod_{j=i}^{[m/2]} \frac{1}{\nu^2 - (2j)^2} & (\text{for even } m), \\ -\displaystyle\sum_{i=0}^{[m/2]} a_{2i+1} \prod_{j=i}^{[m/2]} \frac{1}{\nu^2 - (2j+1)^2} & (\text{for odd } m), \end{cases} \tag{2.2}$$

for all $m \in \mathbb{N}_0$. Let $\rho_0 = \min\{\rho, \sigma\}$. Then every solution $y : I_{\rho_0} \to \mathbb{C}$ of the Bessel's differential equation (1.4) *can be expressed by*

$$y(x) = y_h(x) + \sum_{m=0}^{\infty} c_m x^m, \tag{2.3}$$

for all $x \in I_{\rho_0}$, where $y_h(x)$ is a solution of the homogeneous Bessel equation (1.3).

Proof. We assume that $y : I_{\rho_0} \to \mathbb{C}$ is a function given in the form (2.3), and we define $y_p(x) = y(x) - y_h(x) = \sum_{m=0}^{\infty} c_m x^m$. Then, it follows from (2.1) and (2.2) that

$$\lim_{m\to\infty} \left|\frac{c_{m+2}}{c_m}\right| = \lim_{m\to\infty} \frac{1}{(m+2)^2 - \nu^2} \left|\frac{a_{m+2}}{c_m} - 1\right| \leq \frac{1}{\sigma^2}, \tag{2.4}$$

since we can deduce the relation $c_{m+2} = (a_{m+2} - c_m)/((m+2)^2 - \nu^2)$ from (2.2) by some manipulations. That is, the power series for $y_p(x)$ converges for all $x \in I_{\rho_0}$. Hence, we see that the domain of $y(x)$ is well defined.

We now prove that the function $y_p(x)$ satisfies the inhomogeneous equation (1.4). Indeed, it follows from (2.2) that

$$\begin{aligned} &x^2 y_p''(x) + x y_p'(x) + \left(x^2 - \nu^2\right) y_p(x) \\ &\quad = \sum_{m=2}^{\infty} m(m-1) c_m x^m + \sum_{m=1}^{\infty} m c_m x^m + \sum_{m=0}^{\infty} c_m x^{m+2} - \sum_{m=0}^{\infty} \nu^2 c_m x^m \\ &\quad = c_1 x - \nu^2 c_0 - \nu^2 c_1 x + \sum_{m=2}^{\infty} \left[c_{m-2} + \left(m^2 - \nu^2\right) c_m\right] x^m \\ &\quad = a_0 + a_1 x + \sum_{m=2}^{\infty} a_m x^m, \end{aligned} \tag{2.5}$$

since we obtain

$$c_0 = -\frac{1}{\nu^2} a_0, \qquad c_1 = \frac{1}{1-\nu^2} a_1, \qquad c_{m-2} + \left(m^2 - \nu^2\right) c_m = a_m, \quad \text{for } m \geq 2, \tag{2.6}$$

which proves that $y_p(x)$ is a particular solution of the inhomogeneous equation (1.4).

On the other hand, since every solution to (1.4) can be expressed as a sum of a solution $y_h(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the inhomogeneous equation, every solution of (1.4) is certainly of the form (2.3). □

3. Approximate Bessel's Differential Equation

In this section, assume that ν is a positive nonintegral number and ρ is a positive constant. For a given $K \geq 0$, we denote by $\mathcal{C}_K$ the set of all functions $y : I_\rho \to \mathbb{C}$ with the properties (a) and (b):

(a) $y(x)$ is expressible by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least ρ;

(b) $\sum_{m=0}^{\infty} |a_m x^m| \leq K |\sum_{m=0}^{\infty} a_m x^m|$ for any $x \in I_\rho$, where $a_m = b_{m-2} + (m^2 - \nu^2) b_m$ for all $m \in \mathbb{N}_0$ and set $b_{-2} = b_{-1} = 0$.

For a positive nonintegral number ν, define

$$\begin{aligned} M_e(x) &= \max\left\{ \prod_{j=i}^{k} \frac{x^2}{\left|\nu^2 - (2j)^2\right|} : 0 \leq i \leq k \leq \mu \right\}, \\ M_o(x) &= \max\left\{ \prod_{j=i}^{k} \frac{x^2}{\left|\nu^2 - (2j+1)^2\right|} : 0 \leq i \leq k \leq \mu \right\}, \\ M(x) &= \max\{M_e(x), M_o(x), 1\}, \end{aligned} \tag{3.1}$$

where $\mu = [\sqrt{\nu^2 + x^2}/2]$ and

$$L_\nu = \sum_{m=0}^{\infty} \frac{1}{(m-\nu)^2} < \infty. \tag{3.2}$$

We remark that $M(x) \to 1$ as $|x| \to 0$.

We will now solve the approximate Bessel differential equations in a special class of analytic functions, $\mathcal{C}_K$.

Theorem 3.1. *Let ν be a positive nonintegral number, and let p be a nonnegative integer with $p < \nu < p+1$. Assume that a function $y \in \mathcal{C}_K$ satisfies the differential inequality*

$$\left| x^2 y''(x) + x y'(x) + \left(x^2 - \nu^2\right) y(x) \right| \le \varepsilon, \tag{3.3}$$

for all $x \in I_\rho$ and for some $\varepsilon \ge 0$. If the sequence $\{b_m\}$ satisfies the condition

$$b_{m+2} = O(b_m) \quad \text{as } m \longrightarrow \infty \tag{3.4}$$

with a Landau constant $C \ge 0$, then there exists a solution $y_h(x)$ of the Bessel differential equation (1.3) *such that*

$$\left| y(x) - y_h(x) \right| \le K L_\nu M(x) \varepsilon, \tag{3.5}$$

for any $x \in I_{\rho_0}$, where $\rho_0 = \min\{\rho, 1/\sqrt{C^*}\}$ and C^* is a positive number larger than C. If C and ρ are sufficiently small and large; respectively, then

$$M(x) \le \max\left\{ \frac{|x|^{|x|+2}}{|\nu^2 - p^2|^{|x|/2+1}}, \frac{|x|^{|x|+2}}{\left|\nu^2 - (p+1)^2\right|^{|x|/2+1}} \right\} \tag{3.6}$$

for all sufficiently large $|x|$.

Proof. Since y belongs to $\mathcal{C}_K$, it follows from (a) and (b) that

$$x^2 y''(x) + x y'(x) + \left(x^2 - \nu^2\right) y(x) = \sum_{m=0}^{\infty} \left[b_{m-2} + \left(m^2 - \nu^2\right) b_m \right] x^m = \sum_{m=0}^{\infty} a_m x^m, \tag{3.7}$$

for all $x \in I_\rho$. By considering (3.3) and (3.7), we get

$$\left| \sum_{m=0}^{\infty} a_m x^m \right| \le \varepsilon, \tag{3.8}$$

for any $x \in I_\rho$. This inequality, together with (b), yields

$$\sum_{m=0}^{\infty} |a_m x^m| \le K \left| \sum_{m=0}^{\infty} a_m x^m \right| \le K\varepsilon, \tag{3.9}$$

for each $x \in I_\rho$.

Now, it follows from (b) that

$$\begin{aligned}
\sum_{i=0}^{n} a_{2i} \prod_{j=i}^{n} \frac{1}{\nu^2 - (2j)^2} &= \sum_{i=0}^{n} b_{2i-2} \prod_{j=i}^{n} \frac{1}{\nu^2 - (2j)^2} - \sum_{i=0}^{n} b_{2i} \prod_{j=i+1}^{n} \frac{1}{\nu^2 - (2j)^2} \\
&= \sum_{i=-1}^{n-1} b_{2i} \prod_{j=i+1}^{n} \frac{1}{\nu^2 - (2j)^2} - \sum_{i=0}^{n} b_{2i} \prod_{j=i+1}^{n} \frac{1}{\nu^2 - (2j)^2} \\
&= b_{-2} \prod_{j=0}^{n} \frac{1}{\nu^2 - (2j)^2} - b_{2n} \\
&= -b_{2n},
\end{aligned} \tag{3.10}$$

since $b_{-2} = 0$. Similarly, we obtain

$$\sum_{i=0}^{n} a_{2i+1} \prod_{j=i}^{n} \frac{1}{\nu^2 - (2j+1)^2} = -b_{2n+1}, \tag{3.11}$$

for all $n \in \mathbb{N}_0$, that is,

$$b_m = \begin{cases} -\displaystyle\sum_{i=0}^{[m/2]} a_{2i} \prod_{j=i}^{[m/2]} \frac{1}{\nu^2 - (2j)^2} & (\text{for even } m), \\[2ex] -\displaystyle\sum_{i=0}^{[m/2]} a_{2i+1} \prod_{j=i}^{[m/2]} \frac{1}{\nu^2 - (2j+1)^2} & (\text{for odd } m), \end{cases} \tag{3.12}$$

for all $m \in \mathbb{N}_0$.

On the other hand, by (b) and (3.4), we have

$$\begin{aligned}
|a_{m+2}| &= \left| b_m + \left[(m+2)^2 - \nu^2 \right] b_{m+2} \right| \\
&\le |b_m| + (m+2)^2 |b_{m+2}| \\
&\le \frac{m^2}{\left(1/\sqrt{C^*}\right)^2} |b_m|,
\end{aligned} \tag{3.13}$$

for all sufficiently large integers m, where C^* is a positive number larger than C. Hence, in view of (3.12) and (3.13), the condition (2.1) is satisfied with $\sigma = 1/\sqrt{C^*}$. Moreover, we know that the radius of convergence of power series $\sum_{m=0}^{\infty} a_m x^m$ is at least ρ because the convergence radius of the power series expression for $y(x)$ is at least ρ (see (a) and (3.7)).

According to (3.7) and Theorem 2.1, there exists a solution $y_h(x)$ of the homogeneous Bessel differential equation (1.3) satisfying (2.3) for all $x \in I_{\rho_0}$. Thus, it follows from (2.2) and (3.12) that

$$\begin{aligned}
\left|y(x) - y_h(x)\right| &= \left|\sum_{n=0}^{\infty} c_{2n}x^{2n} + \sum_{n=0}^{\infty} c_{2n+1}x^{2n+1}\right| \\
&= \left|-\sum_{n=0}^{\infty} x^{2n} \sum_{i=0}^{n} a_{2i} \prod_{j=i}^{n} \frac{1}{\nu^2 - (2j)^2} - \sum_{n=0}^{\infty} x^{2n+1} \sum_{i=0}^{n} a_{2i+1} \prod_{j=i}^{n} \frac{1}{\nu^2 - (2j+1)^2}\right| \\
&= \left|\sum_{n=0}^{\infty} \frac{1}{\nu^2 - (2n)^2} \sum_{i=0}^{n} a_{2i}x^{2i} \prod_{j=i}^{n-1} \frac{x^2}{\nu^2 - (2j)^2}\right. \\
&\qquad \left. + \sum_{n=0}^{\infty} \frac{1}{\nu^2 - (2n+1)^2} \sum_{i=0}^{n} a_{2i+1}x^{2i+1} \prod_{j=i}^{n-1} \frac{x^2}{\nu^2 - (2j+1)^2}\right|,
\end{aligned} \tag{3.14}$$

for any $x \in I_{\rho_0}$. Moreover, we have

$$\begin{aligned}
\left|y(x) - y_h(x)\right| \le{}& \sum_{n=0}^{\mu+1} \frac{1}{\left|\nu^2 - (2n)^2\right|} \sum_{i=0}^{n} \left|a_{2i}x^{2i}\right| \prod_{j=i}^{n-1} \frac{x^2}{\left|\nu^2 - (2j)^2\right|} \\
&+ \sum_{n=\mu+2}^{\infty} \frac{1}{\left|\nu^2 - (2n)^2\right|} \sum_{i=0}^{\mu} \left|a_{2i}x^{2i}\right| \prod_{j=i}^{n-1} \frac{x^2}{\left|\nu^2 - (2j)^2\right|} \\
&+ \sum_{n=\mu+2}^{\infty} \frac{1}{\left|\nu^2 - (2n)^2\right|} \sum_{i=\mu+1}^{n} \left|a_{2i}x^{2i}\right| \prod_{j=i}^{n-1} \frac{x^2}{\left|\nu^2 - (2j)^2\right|} \\
&+ \sum_{n=0}^{\mu+1} \frac{1}{\left|\nu^2 - (2n+1)^2\right|} \sum_{i=0}^{n} \left|a_{2i+1}x^{2i+1}\right| \prod_{j=i}^{n-1} \frac{x^2}{\left|\nu^2 - (2j+1)^2\right|} \\
&+ \sum_{n=\mu+2}^{\infty} \frac{1}{\left|\nu^2 - (2n+1)^2\right|} \sum_{i=0}^{\mu} \left|a_{2i+1}x^{2i+1}\right| \prod_{j=i}^{n-1} \frac{x^2}{\left|\nu^2 - (2j+1)^2\right|} \\
&+ \sum_{n=\mu+2}^{\infty} \frac{1}{\left|\nu^2 - (2n+1)^2\right|} \sum_{i=\mu+1}^{n} \left|a_{2i+1}x^{2i+1}\right| \prod_{j=i}^{n-1} \frac{x^2}{\left|\nu^2 - (2j+1)^2\right|},
\end{aligned} \tag{3.15}$$

for all $x \in I_{\rho_0}$, where $\mu = [\sqrt{\nu^2 + x^2}/2]$.

We know that $x^2/|\nu^2-(2j)^2|<1$ and $x^2/|\nu^2-(2j+1)^2|<1$ for $j\ge\mu+1$ and $x^2/|\nu^2-(2j)^2|$ or $x^2/|\nu^2-(2j+1)^2|$ is not perhaps less than 1 for $j\le\mu$. Then, we have

$$\prod_{j=i}^{n-1}\frac{x^2}{\left|\nu^2-(2j)^2\right|}=\left(\prod_{j=i}^{\mu}\frac{x^2}{\left|\nu^2-(2j)^2\right|}\right)\left(\prod_{j=\mu+1}^{n-1}\frac{x^2}{\left|\nu^2-(2j)^2\right|}\right)\le M_e(x)\le M(x), \tag{3.16}$$

for all $n\ge\mu+2$ and $i=0,1,\dots,\mu$. Similarly, we get

$$\prod_{j=i}^{n-1}\frac{x^2}{\left|\nu^2-(2j+1)^2\right|}\le M(x), \tag{3.17}$$

for all $n\ge\mu+2$ and $i=0,1,\dots,\mu$.

Thus, it follows from (3.9) and (3.15) that

$$\begin{aligned}
\left|y(x)-y_h(x)\right| &\le \sum_{n=0}^{\mu+1}\frac{M(x)}{\left|\nu^2-(2n)^2\right|}\sum_{i=0}^{n-1}\left|a_{2i}x^{2i}\right|+\sum_{n=0}^{\mu+1}\frac{\left|a_{2n}x^{2n}\right|}{\left|\nu^2-(2n)^2\right|}\\
&\quad+\sum_{n=\mu+2}^{\infty}\frac{M(x)}{\left|\nu^2-(2n)^2\right|}\sum_{i=0}^{\mu}\left|a_{2i}x^{2i}\right|\\
&\quad+\sum_{n=\mu+2}^{\infty}\frac{1}{\left|\nu^2-(2n)^2\right|}\sum_{i=\mu+1}^{n}\left|a_{2i}x^{2i}\right|\\
&\quad+\sum_{n=0}^{\mu+1}\frac{M(x)}{\left|\nu^2-(2n+1)^2\right|}\sum_{i=0}^{n-1}\left|a_{2i+1}x^{2i+1}\right|+\sum_{n=0}^{\mu+1}\frac{\left|a_{2n+1}x^{2n+1}\right|}{\left|\nu^2-(2n+1)^2\right|}\\
&\quad+\sum_{n=\mu+2}^{\infty}\frac{M(x)}{\left|\nu^2-(2n+1)^2\right|}\sum_{i=0}^{\mu}\left|a_{2i+1}x^{2i+1}\right|\\
&\quad+\sum_{n=\mu+2}^{\infty}\frac{1}{\left|\nu^2-(2n+1)^2\right|}\sum_{i=\mu+1}^{n}\left|a_{2i+1}x^{2i+1}\right|\\
&\le M(x)\left(\sum_{n=0}^{\infty}\frac{1}{\left|\nu^2-(2n)^2\right|}\sum_{i=0}^{n}\left|a_{2i}x^{2i}\right|+\sum_{n=0}^{\infty}\frac{1}{\left|\nu^2-(2n+1)^2\right|}\sum_{i=0}^{n}\left|a_{2i+1}x^{2i+1}\right|\right)\\
&\le M(x)K\varepsilon\sum_{m=0}^{\infty}\frac{1}{|\nu^2-m^2|}\\
&\le KL_\nu M(x)\varepsilon,
\end{aligned} \tag{3.18}$$

for $x\in I_{\rho_0}$.

Finally, assume that C is sufficiently small and ρ is sufficiently large. Then ρ_0 is sufficiently large. For any $j \geq 0$ and $x \in I_{\rho_0}$, we have

$$\frac{x^2}{\left|\nu^2-(2j)^2\right|} \leq \max\left\{\frac{x^2}{\left|\nu^2-p^2\right|}, \frac{x^2}{\left|\nu^2-(p+1)^2\right|}\right\}. \tag{3.19}$$

If $|x|$ is so large that

$$\mu = \left[\frac{1}{2}\sqrt{\nu^2+x^2}\right] = \left[\frac{|x|}{2}\right] \leq \frac{|x|}{2}, \tag{3.20}$$

then it follows from the definition of $M(x)$ that

$$M(x) \leq \max\left\{\frac{|x|^{|x|+2}}{\left|\nu^2-p^2\right|^{|x|/2+1}}, \frac{|x|^{|x|+2}}{\left|\nu^2-(p+1)^2\right|^{|x|/2+1}}\right\}, \tag{3.21}$$

for all sufficiently large $|x|$. □

If ν is large enough, then we can prove the (local) Hyers-Ulam stability of the Bessel differential equation (1.3) as we see in the following corollary.

Corollary 3.2. *Let ν be a positive nonintegral number and let p be a nonnegative integer with $p < \nu < p+1$. Assume that a function $y \in C_K$ satisfies the differential inequality* (3.3) *for all $x \in I_\rho$ and for some $\varepsilon \geq 0$. Suppose the sequence $\{b_m\}$ satisfies the condition* (3.4) *with a Landau constant $C \geq 0$ and define $\rho_0 = \min\{\rho, 1/\sqrt{C^*}\}$ for a positive number $C^* > C$. If*

$$\frac{x^2}{\left|\nu^2-p^2\right|} \leq 1, \qquad \frac{x^2}{\left|\nu^2-(p+1)^2\right|} \leq 1, \tag{3.22}$$

for all $x \in I_{\rho_0}$, then there exists a solution $y_h(x)$ of the Bessel differential equation (1.3) *such that*

$$\left|y(x)-y_h(x)\right| \leq K L_\nu \varepsilon, \tag{3.23}$$

for any $x \in I_{\rho_0}$.

Proof. For any $j \geq 0$ and $x \in I_{\rho_0}$, we have

$$\max\left\{\frac{x^2}{\left|\nu^2-(2j)^2\right|}, \frac{x^2}{\left|\nu^2-(2j+1)^2\right|}\right\} \leq \max\left\{\frac{x^2}{\left|\nu^2-p^2\right|}, \frac{x^2}{\left|\nu^2-(p+1)^2\right|}\right\} \leq 1. \tag{3.24}$$

Thus, we get

$$M(x) \leq \max\left\{ \frac{x^2}{|\nu^2 - p^2|}, \frac{x^2}{\left|\nu^2 - (p+1)^2\right|}, 1 \right\} = 1, \tag{3.25}$$

and the assertion is true due to Theorem 3.1. □

4. Examples

We will show that there exist functions $y(x)$ which satisfy all the conditions given in Theorem 3.1 and Corollary 3.2. Let us define a function $y : I_{10} \to \mathbb{R}$ by

$$y(x) = J_{100.5}(x) + cx^2 = \sum_{m=0}^{\infty} b_m x^m, \tag{4.1}$$

where $J_{100.5}(x)$ is the Bessel function of the first kind of order 100.5, n is a positive integer, and c is a constant satisfying

$$c = \frac{\varepsilon}{999625}, \tag{4.2}$$

for some $\varepsilon > 0$. It is obvious that the convergence radius of the power series $\sum_{m=0}^{\infty} b_m x^m$ is infinity. (So we can set $\rho = 10$.) In fact, the infinite series $\sum_{m=0}^{\infty} b_m 11^m$ converges. So we have

$$\limsup_{m\to\infty} \left| \frac{11^{m+1} b_{m+1}}{11^m b_m} \right| \leq 1, \qquad \limsup_{m\to\infty} \left| \frac{11^{m+2} b_{m+2}}{11^{m+1} b_{m+1}} \right| \leq 1. \tag{4.3}$$

Thus, it holds true that

$$\limsup_{m\to\infty} \left| \frac{b_{m+2}}{b_m} \right| = \limsup_{m\to\infty} \left| \frac{b_{m+2}}{b_{m+1}} \right| \limsup_{m\to\infty} \left| \frac{b_{m+1}}{b_m} \right| \leq \frac{1}{121} \tag{4.4}$$

which implies that the sequence $\{b_m\}$ satisfies the condition (3.4) with a Landau constant $C = 1/121$. If we take $C^* = 1/100$, then $\rho_0 = \min\{\rho, 1/\sqrt{C^*}\} = 10$.

Since $J_{100.5}(x)$ is a particular solution of the Bessel differential equation (1.3) with $\nu = 100.5$, it follows from (4.1) that

$$x^2 y''(x) + xy'(x) + \left(x^2 - \frac{40401}{4} \right) y(x) = -\frac{40385}{4} cx^2 + cx^4, \tag{4.5}$$

for any $x \in I_{10}$.

If we set

$$a_m = \begin{cases} -\dfrac{40385}{4}c & \text{for } m = 2, \\ c & \text{for } m = 4, \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

then we have

$$x^2 y''(x) + xy'(x) + \left(x^2 - \frac{40401}{4}\right)y(x) = \sum_{m=0}^{\infty} a_m x^m,$$

$$\left|x^2 y''(x) + xy'(x) + \left(x^2 - \frac{40401}{4}\right)y(x)\right| = \left|\sum_{m=0}^{\infty} a_m x^m\right| = c\left(\frac{40385}{4}x^2 - x^4\right) < 999625c = \varepsilon, \tag{4.7}$$

for all $x \in I_{10}$.

Moreover, we have

$$\frac{\left|\sum_{m=0}^{\infty} a_m x^m\right|}{\sum_{m=0}^{\infty} |a_m x^m|} = \frac{(40385/4)\ cx^2 - cx^4}{(40385/4)cx^2 + cx^4} > \frac{39985}{40785}, \tag{4.8}$$

and hence, we get

$$\sum_{m=0}^{\infty} |a_m x^m| < \frac{8157}{7997}\left|\sum_{m=0}^{\infty} a_m x^m\right|, \tag{4.9}$$

for all $x \in I_{10}$. That is, $\{a_m\}$ satisfies the property (b) with $K = 8157/7997$.

It holds true that

$$\frac{x^2}{\left|\nu^2 - p^2\right|} < 1, \qquad \frac{x^2}{\left|\nu^2 - (p+1)^2\right|} < 1, \tag{4.10}$$

for all $x \in I_{10}$, and since

$$\begin{aligned}
L_{100.5} &= \sum_{m=0}^{\infty} \frac{1}{(m-100.5)^2} \\
&= \frac{1}{(-100.5)^2} + \frac{1}{(-99.5)^2} + \frac{1}{(-98.5)^2} + \cdots + \frac{1}{(-0.5)^2} \\
&\quad + \frac{1}{0.5^2} + \frac{1}{1.5^2} + \frac{1}{2.5^2} + \frac{1}{3.5^2} + \cdots \\
&\le \frac{1}{100^2} + \frac{1}{99^2} + \frac{1}{98^2} + \cdots + \frac{1}{1^2} + \left(\frac{1}{0.5^2} + \frac{1}{0.5^2}\right) \\
&\quad + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \\
&\le 2\zeta(2) + 8 = \frac{\pi^2}{3} + 8,
\end{aligned} \tag{4.11}$$

it follows from Corollary 3.2 that there exists a solution $y_h(x)$ of the Bessel differential equation (1.3) such that

$$\left| y(x) - y_h(x) \right| \le \frac{8157}{7997} L_{100.5}\varepsilon < \frac{8157}{7997}\left(\frac{\pi^2}{3} + 8\right)\varepsilon, \tag{4.12}$$

for any $x \in I_{10}$.

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